Print Your Name Here: ______________________________

- **Show all work:** *Answers without work are not sufficient.* We can give credit *only* for what you write! *Indicate clearly if you continue on the back side*, and write your name at the top of the scratch sheet if you will turn it in for grading.
- ***Books, notes (electronic or paper), cell phones, smart phones, and internet-connected devices are prohibited!*** A scientific calculator is allowed—but it is not needed. If you use a calculator, you must write out the operations performed on the calculator to show that you know how to solve the problem. *Please do not replace precise answers with decimal approximations.*
- There are **six (6)** problems: maximum total score = 200.

1. (30) The differential equation $xy'' + y' + y = 0$ has a regular singular point at $x = 0$. The Method of Frobenius guarantees at least one solution of the form $y = \sum_{k=0}^{\infty} a_k x^{k+r}$.

a. (10) Find and solve the indicial equation for the admissible value(s) of r.

b. (10) Using the largest available value for r, write a recursion formula for a_k in terms of a_{k-1}, $k \geq 1$.

c. (5) Letting $a_0 = 1$, find a_k in terms of k for all $k \geq 0$.

d. (5) Use part (c) to write a solution for y as the sum of an infinite series.

2. (40) The differential equation $xy'' + y' + y = 0$ (from the first problem) can be solved by using the change of independent variable to $z = 2\sqrt{x}$ to transform this equation into a Bessel equation of the form $z^2\dfrac{d^2y}{dz^2} + z\dfrac{dy}{dz} + (z^2 - \nu^2)y = 0$. Use the following steps.

a. (10) Use the chain rule to express y' in terms of z and $\dfrac{dy}{dz}$.

b. (10) Express y'' in terms of z and derivatives of y with respect to z.

c. (10) Transform the given equation into a Bessel equation. Find the value of ν.

d. (10) Write the general solution of the original equation for y in terms of x.

3. (30) Let $f(x) = x$ on the domain $[0, 3]$.

a. (10) In the Fourier Cosine expansion $f(x) = a_0 + \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{3}$ find a_0 and the general coefficient $a_n,\ n \geq 1$.

b. (10) In the Fourier Sine expansion $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3}$ find the general coefficient $b_n,\ n \geq 1$.

c. (10) The Fourier series in parts (a) and (b) converge to some functions, $A(x)$ and $B(x)$ respectively, for all $x \in (-\infty, \infty)$, and in particular for all x in $[-3, 3]$. Sketch the graphs of $A(x)$ and $B(x)$ on two sets of axes on $[-3, 3]$, labeling the graphs $A(x)$ and $B(x)$ respectively.

4. (30) Consider the regular Sturm-Liouville problem

$$\frac{d}{dx}(xy') + \frac{\lambda}{x}y = 0;\ \ y(1) = 0 = y(e).$$

a. (10) Make a change of independent variable from x to the new variable $t = \ln x,\ 0 \leq t \leq 1$. Rewrite the differential equation in terms of y and derivatives of y with respect to t and rewrite the boundary conditions on y for $t = 0$ and $t = 1$.

b. (10) Find a complete set of eigenvalues λ_n and eigenfunctions y_n expressed in terms of t

c. (10) Rewrite y_n in terms of the original variable x and write out the orthogonality relation $\langle y_m, y_n\rangle_p(x) = 0,\ m \neq n$ in the form of a suitable integral being zero.

5. (40) Consider the wave equation $u_{tt} = a^2 u_{xx}$ with the boundary conditions $u(0,t) = 0 = u(2,t),\ t \geq 0$, and the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x),\ 0 \leq x \leq 2$.

a. (10) Consider product solutions $u(x,t) = F(x)G(t)$ and use the separation constant $-\lambda$ to write ordinary differential equations for $F(x)$ and for $G(t)$. Include boundary conditions on F.

b. (10) Find the eigenvalues λ_n and the corresponding eigenfunctions $F_n(x)$.

c. (10) Find the general form of the corresponding functions $G_n(t)$ and write the result for $u_n(x,t) = F_n(x)G_n(t)$.

d. (10) Express the final solution $u(x,t)$ as the sum of an infinite series of product solutions with coefficients, and write formulas for the necessary coefficients in terms of integrals involving $f(x)$ and $g(x)$ respectively.

6. (30) Consider the heat equation $u_t = k(u_{xx}+u_{yy})$ on a square plate of side 1 with boundary conditions $u(0,y,t) = 0 = u(1,y,t) = u(x,0,t) = u(x,1,t),\ 0 \le x, y \le 1,\ t \ge 0$. Let $u(x,y,t) = F(x)G(y)H(t)$ be any product solution.

a. (10) Separate the x-terms from the other two variables with a separation constant $-\lambda$. Find the eigenvalues λ_m and the corresponding eigenfunctions $F_m(x)$.

b. (10) Separate the y-terms from the t-terms with a separation constant $= -\mu$. Find the eigenvalues μ_n and the corresponding eigenfunctions $G_n(y)$.

c. (10) For each pair (m,n) of natural numbers, find the corresponding functions $H_{m,n}(t)$

Solutions

1.

a. $r^2 = 0$ is the indicial equation, so $r = 0$ is only available root.

b. $a_k = -\frac{a_{k-1}}{k^2}$, $k \geq 1$.

c. $a_k = \frac{(-1)^k}{(k!)^2}$, $k \geq 0$.

d. $y = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} x^k$.

2.

a. $y' = \dfrac{2}{z}\dfrac{dy}{dz}$.

b. $y'' = \dfrac{4}{z^2}\left(\dfrac{d^2y}{dz^2} - \dfrac{1}{z}\dfrac{dy}{dz}\right)$.

c. $z^2\dfrac{d^2y}{dz^2} + z\dfrac{dy}{dz} + z^2 y = 0$, and $\nu = 0$.

d. $y = c_1 J_0(2\sqrt{x}) + c_2 Y_0(2\sqrt{x})$.

3.

a. $a_0 = \dfrac{3}{2}$ and $a_n = \dfrac{6((-1)^n - 1)}{n^2\pi^2}$, $n \geq 1$.

b. $b_n = \dfrac{6(-1)^{n+1}}{n\pi}$, $n \geq 1$.

c. The graph of $A(x)$ will be the graph of $|x|$ and the graph of $B(x)$ will be the graph of x on $[-3, 3]$. These are the graphs of the even- and odd- periodic extensions of $f(x)$ with period 6.

4.

a. $\dfrac{d^2y}{dt^2} + \lambda y = 0$ with $y(t) = 0$ if $t = 0$ and if $t = 1$.

b. $\lambda_n = n^2\pi^2$ and $y_n = \sin n\pi t$, $n = 1, 2, 3, \ldots$.

c. $y_n = \sin(n\pi \ln x)$, $n = 1, 2, 3, \ldots$ and $\int_1^e \sin(m\pi \ln x)\sin(n\pi \ln x)\,\frac{1}{x}dx = 0$, $m \neq n$.

5.

a. $F''(x) + \lambda F(x) = 0$, $F(0) = 0 = F(2)$, and $G''(t) + \lambda a^2 G(t) = 0$.

b. $\lambda_n = \dfrac{n^2\pi^2}{4}$ and $F_n(x) = \sin\dfrac{n\pi x}{2}$.

c. $G_n(t) = A_n \cos\dfrac{n\pi at}{2} + B_n \sin\dfrac{n\pi at}{2}$ and $u_n(x,t) = \left(A_n \cos\frac{n\pi at}{2} + B_n \sin\frac{n\pi at}{2}\right)\sin\frac{n\pi x}{2}$.

d. $u(x,t) = \displaystyle\sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi at}{2} + B_n \sin\frac{n\pi at}{2}\right)\sin\frac{n\pi x}{2}$ where $A_n = \displaystyle\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx$ and $B_n = \dfrac{2}{n\pi a}\displaystyle\int_0^2 g(x)\sin\frac{n\pi x}{2}\,dx$.

6.

a. $\lambda_m = m^2\pi^2$ and $F_m(x) = \sin m\pi x$.

b. $\mu_n = n^2\pi^2$ and $G_n(y) = \sin n\pi y$.

c. $H_{m,n}(t) = e^{-k(m^2+n^2)\pi^2 t}$.

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	7	2	3	5	6
80-89 (B)	1	2	3	2	1
70-79 (C)	1	2	1	0	1
60-69 (D)	0	1	1	1	0
0-59 (F)	0	2	0	0	0
Test Avg	91.6%	74.8%	85.9%	87.7%	91.12%
HW Avg	96.73%	96.9%	98.1%	98.09%	98.09%