1. (30) The differential equation $y'' + 2xy' - 2y = 0$ has analytic solutions of the form $y = \sum_{n=0}^{\infty} a_n x^n$.

a. (15) Find a recursion formula for $a_{n+2}$ in terms of $a_n$, $n \geq 0$.

b. (10) The general solution has the form $y = c_1 y_1(x) + c_2 y_2(x)$ where $y_1(x)$ is a polynomial of odd degree, and $y_2(x)$ is not a polynomial. Find a suitable polynomial solution $y_1(x)$, showing why all the higher odd-indexed coefficients must be zero.

c. (5) Choosing $a_0 = 1$ write out the first 3 nonzero terms of the non-polynomial power series solution $y_2(x)$. 
2. (30) The equation $2xy'' - y' + 2y = 0$ has a regular singular point at $x = 0$. Using the method of Frobenius, there is a solution of the form $y = x^r \sum_{n=0}^{\infty} a_n x^n$.

a. (10) Write the indicial equation.

b. (10) Find the allowable values of $r$ in the indicial equation.

c. (10) Find the recursion formula for $a_{n+1}$ in terms of $a_n$, $n$ and $r$. You are not asked to find the general solution for this problem.
3. (30) Let \( f(x) = \begin{cases} 
1, & 0 < x \leq \pi \\
0, & x = 0 \\
-1, & -\pi \leq x < 0. 
\end{cases} \)

a. (20) Find all the Fourier coefficients in the expansion

\[
f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)
\]

b. (5) Calculate the (very easy) integral \( \int_{-\pi}^{\pi} f(x)^2 \, dx \).

c. (5) By Eq. (1),

\[
\int_{-\pi}^{\pi} f(x)^2 \, dx = \langle f, f \rangle = \langle a_0 + \sum_{n} (a_n \cos nx + b_n \sin nx), a_0 + \sum_{n} (a_n \cos nx + b_n \sin nx) \rangle.
\]

Use the orthogonality relations and the results of (a) and (b) to find the value of

\[
\sum_{n \text{ odd}} \frac{1}{n^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.
\]
4. (40) The differential equation with boundary conditions

\[ y'' + 4y' + (\lambda + 4)y = 0, \quad y(0) = 0 = y(1) \]  

becomes a Sturm-Liouville problem (in self-adjoint form) with equation \( (r(x)y')' + (q(x))y = 0 \) if you multiply both sides of the equation by \( e^{kx} \) for some \( k \).

a. (10) Choose the correct value of \( k \) and do the multiplication to write Eq. (2) in Sturm-Liouville (self-adjoint) format. Find the weight function \( p(x) \).

b. (15) Set \( y = e^{rx} \) to solve the original Eq. (2) to find a complete set of eigenvalues \( \lambda_n \) and corresponding (real-valued) eigenfunctions \( y_n(x) \) for the interval \([0, 1]\). (You will use Euler’s formula here to evaluate \( e^{(a+ib)x} = e^{ax}e^{ibx} \).)

c. (10) Write the orthogonality relation \( \langle y_n, y_m \rangle_p = 0 \) if \( m \neq n \) in the form of an integral vanishing if \( m \neq n \). (You do not need to evaluate the resulting integral.)

d. (5) If \( f \) is integrable on \([0, 1]\), it can be expanded in the form \( f(x) = \sum_{m=1}^{\infty} c_m y_m(x) \). Using what you know about orthogonal function expansions, express \( c_m \) in terms of \( \|y_m\| \) and an integral involving both \( f(x) \) and \( y_m(x) \). (Do not evaluate the integral or the value of \( \|y_m\| \) for this problem.)
5. (30) Consider the wave equation \( u_{tt} = u_{xx} \) on \([0, 1]\) with the boundary conditions
\[ u(0, t) = u(1, t) = 0, \quad t \geq 0 \]
and the initial conditions
\[ u(x, 0) = \sin 2\pi x, \quad u_t(x, 0) = \sin 3\pi x, \quad 0 \leq x \leq 1. \]

a. (15) Use the separation constant \( \lambda \) and find all eigenvalues \( \lambda_n \) with all corresponding split solutions
\[ u_n(x, t) = F_n(x) G_n(t) \]
satisfying the boundary conditions.

b. (15) Write the general solution satisfying the boundary conditions as a sum of all the split solutions
\[ u_n(x, t) \]
(using the principle of superposition) and determine all the necessary coefficients so as to find the unique solution satisfying the initial conditions as well as the boundary conditions.
6. (40) Consider the wave equation $u_{tt} = u_{xx} + u_{yy}$ on the square $0 \leq x, y \leq \pi$ with the boundary conditions $u(0, y, t) = u(\pi, y, t) = 0$ and $u(x, 0, t) = u(x, \pi, t) = 0$ for all $t \geq 0$, and with the initial conditions $u(x, y, 0) = f(x, y)$ and $u_t(x, y, 0) = 0$. We will begin by finding all the split solutions $u(x, y, t) = F(x)G(y)H(t)$ as follows.

a. (10) Use the constant $\lambda$ to separate the functions of $x$ from the functions of $y$ and $t$. (Time Saver: $\lambda$ will be negative.) Determine all the eigenvalues $\lambda_m$ and the corresponding eigenfunctions $F_m(x)$.

b. (10) Given $\lambda_m$ as in part (a), use the constant $\mu$ to separate the functions of $y$ from the functions of $t$. (Time Saver: $\mu$ will be negative.) Determine all the eigenvalues $\mu_n$ and the corresponding eigenfunctions $G_n(y)$.

c. (10) Given $\lambda_m$ and $\mu_n$ from parts (a) and (b), Find the corresponding $H_{m,n}(t)$ and write out the general split solutions $u_{m,n}(x, y, t)$ satisfying the boundary conditions.

d. (10) Write out the general solution $u(x, y, t)$ as the sum of an infinite series satisfying all the boundary conditions. Now express all the coefficients as double integrals involving $f(x, y)$ to find the unique solution satisfying both initial conditions as well as the boundary conditions.
Solutions

1.

a. \( a_{n+2} = -\frac{2(n-1)a_n}{(n+2)(n+1)} \)

b. \( y_1(x) = x \) or any nonzero constant multiple thereof. The recursion formula shows that \( a_3 = 0 \) and the same for all higher odd-indexed coefficients.

c. (5) \( y_2(x) = 1 + x^2 - \frac{1}{6}x^4 - \cdots \).

2.

a. \( 2r^2 - 3r = 0 \).

b. \( r_1 = \frac{3}{2} \) and \( r_2 = 0 \).

c. \( a_{n+1} = -\frac{2a_n}{(n+r+1)(2(n+r) - 1)} \).

3.

a. \( a_n = 0 \) for all \( n \) since \( f \) is an odd function. \( b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx = 0 \) if \( n \) is even, but \( b_n = \frac{4}{\pi n} \) for each odd value of \( n \). It helps to avoid errors if one remembers that the product of two odd functions is even, simplifying the domain of integration as shown above.

b. \( \int_{-\pi}^{\pi} f(x)^2 \, dx = \int_{-\pi}^{\pi} 1 \, dx = 2\pi \)

c. \( \int_{-\pi}^{\pi} f(x) \, dx = 2\pi = \left( \sum_{n \text{ odd}}^{} \frac{4}{\pi n} \sin nx \right) \frac{4}{\pi n} \sin nx \) and
\( \sum_{n \text{ odd}}^{} (b_n)^2 = \frac{\pi^2}{8} \).

4.

a. \( k = 4 \): The differential equation is rewritten as \((e^{4x}y)' + (e^{4x} + \lambda e^{4x}) y = 0 \). This has the Sturm-Liouville (self-adjoint) format, with the weight function \( p(x) = e^{4x} \).

b. \( \lambda_n = n^2\pi^2 \) and \( y_n(x) = e^{-2x} \sin nx \) for all \( n = 1, 2, 3, \ldots \).

c. \( \langle y_n, y_m \rangle_p = 0 = \int_{-\pi}^{\pi} e^{-2x} \sin nx e^{-2x} \sin m\pi x e^{4x} \, dx = \int_{-\pi}^{\pi} \sin nx \sin m\pi x \, dx = 0 \) if \( m \neq n \). Remember that the p-scale product includes the weight function \( p(x) \)!

d. \( c_m = \frac{1}{\|y_m\|^2} \int_{-\pi}^{\pi} f(x) e^{-2x} \sin nx e^{4x} \, dx = \frac{1}{\|y_m\|^2} \int_{-\pi}^{\pi} f(x) e^{2x} \sin m\pi x \, dx \). Remember that the p-scale product includes the weight function \( p(x) \)!

5. It is important to know that \( \sin nx \) is orthogonal to \( \sin nx \) on \([0, 1]\) when the positive integers \( m \) and \( n \) are different. Thus the Fourier coefficients are unique which makes \( a_m = 0 \) unless \( m = 2 \) and \( b_n = 0 \) unless \( n = 3 \). There was a WebAssign problem similar to this, so I did not anticipate that this question would cause so much difficulty! Several students erred with the sign of the eigenvalues thereby producing the wrong equation for \( G_n \) resulting in \( G_n \) being incorrectly identified as a hyperbolic function. This should be recognized as impossible because we are dealing with the wave equation for an oscillation!

a. \( \lambda_n = -n^2\pi^2 \) and \( u_n(x, t) = (a_n \cos n\pi t + b_n \sin n\pi t) \sin nx \), \( n \geq 1 \).

b. The general solution satisfying the boundary conditions is \( u(x, t) = \sum_{n=1}^{\infty} (a_n \cos n\pi t + b_n \sin n\pi t) \sin nx \) and the unique solution satisfying the initial conditions as well as the boundary conditions is \( u(x, t) = \cos 2\pi t \sin 2\pi x + \frac{1}{3\pi} \sin 3\pi t \sin 3\pi x \).
Several students erred in this problem as well with the sign of the eigenvalues thereby producing the wrong equation for $G_n$ resulting in $G_n$ being incorrectly identified as a hyperbolic function. This should be recognized as impossible because we are dealing with the wave equation for an oscillation!

a. $\lambda_m = -m^2$ and the corresponding eigenfunctions $F_m(x) = \sin mx$, $m \geq 1$.

b. $\mu_n = -n^2$ and the corresponding eigenfunctions $G_n(y) = \sin ny$, $n \geq 1$.

c. $H_{m,n}(t) = a_{m,n} \cos \sqrt{m^2 + n^2} t + b_{m,n} \sin \sqrt{m^2 + n^2} t$ and $u_{m,n}(x,y,t) = (a_{m,n} \cos \sqrt{m^2 + n^2} t + b_{m,n} \sin \sqrt{m^2 + n^2} t) \sin mx \sin ny$

d. $u(x,y,t) = \sum_{m,n \geq 1} (a_{m,n} \cos \sqrt{m^2 + n^2} t + b_{m,n} \sin \sqrt{m^2 + n^2} t) \sin mx \sin ny$ where all $b_{m,n} = 0$ and $a_{m,n} = \frac{4}{\pi \int_0^{\pi} \int_0^{\pi} f(x,y) \sin mx \sin ny \, dx \, dy}$.

### Class Statistics

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<th>% Grade</th>
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<th>Test #3</th>
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Test Avg 88.8% 86.8% 87.6% 85.7% 90.5%
HW Avg 93.6% 92.27% 96.27% 96.27% 96.27%
HW/Test Correl. - 0.71 0.65 0.45 0.45
Absences/Test Correl. - - - - -0.44

The Correlation Coefficient is the cosine of the angle between two data vectors in 18-dimensional space: one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a positive correlation with performance on the homework and a negative correlation with the number of absences from class.