Print Your Name Here: ______________________________

- **INSTRUCTIONS:**
- **Show all work:** ***Answers without work are not sufficient.*** We can give credit *only* for what you write! ***Indicate clearly if you continue on the back side***, and write your **name at the top of the scratch sheet** if you will turn it in for grading.
- ***Books, notes (electronic or paper), cell phones, smart phones, and internet-connected devices are prohibited!*** A scientific calculator is allowed—but it is not needed. If you use a calculator, you must write out the operations performed on the calculator to show that you know how to solve the problem. *Please do not replace precise answers with decimal approximations.*
- There are **five (5)** problems plus an optional 10-point bonus question 1e: maximum total score = 200, or 210 with the optional bonus question.

1. (50) The differential equation $xy'' + 2y' - xy = 0$ has a regular singular point at $x = 0$. Thus there is *at least* one solution by the Method of Frobenius of the form $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ with $a_0 \neq 0$. Follow the instructions below.

a. (15) Rewrite the differential equation in terms of the series $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ and re-index as needed so that each series shows x to the same power. (Contiinued on next page.)

b. (15) Using the lowest value of the counter n, write the *indicial equation* and find the two values for r, which will turn out to differ by an integer.

c. (10) Using the *larger* value for r, write the recursion relation for a_n in terms of a_{n-2}.

d. (10) Choosing $a_0 = 1$ and $a_1 = 0$, write the series solution for y with the larger value of r.

e. (OPTIONAL 10 POINT BONUS) Recognize the solution for y in the preceding part as an elementary function.

2. (40) Solve the differential equation $x^2y'' + 2xy' + 4x^2y = 0$ by converting it into a Bessel equation of order ν (which has the form $x^2y'' + xy' + (x^2 - \nu^2)y = 0$) using the steps shown below.

a. (20) Let v be a new *dependent variable* given by $y = vx^{-\frac{1}{2}}$. *Express the derivatives* of y with respect to x in terms of derivatives of v with respect to x, and then rewrite the original equation in terms of v and its derivatives with respect to x. (Continued on next page.)

b. (10) Let $t = 2x$ be the new *independent variable* . Change all the derivatives with respect x in the preceding part into derivatives with respect to t and replace x by $t/2$. The result will be a Bessel equation of order ν.

c. (10) Solve for v in terms of t, and then rewrite this as a solution for y in terms of x.

3. (30) Let $f(x) = |x|$ on the interval $[-\pi, \pi]$. *State* whether f is even, odd, or neither, and then *expand* f as an appropriate Fourier series. Note: When computing the integrals for the Fourier coefficients, notice whether the *integrand* is even or odd, and make use of this to ease your work!

4. (20) Consider the Sturm-Liouville problem $y'' + y' + \lambda y = 0$ with the boundary conditions $y(0) = 0 = y(2)$.

a. (15) Use an integrating factor to place the differential equation into its self-adjoint form $(r(x)y')' + \lambda p(x)y = 0$.

b. (5) If $y_n(x)$ and $y_m(x)$ are eigenfunctions corresponding to distinct eigenvalues $\lambda_n \neq \lambda_m$, *rewrite* the following equation by putting in the correct values for a and b and the correct weight function $p(x)$:

$$\int_a^b y_n(x)y_m(x)\,p(x)dx = 0$$

(You are not asked to find $y_n(x)$ or $y_m(x)$.)

5. (60) Use the following steps to solve the *wave equation* $u_{tt} = u_{xx}$ on the interval $0 \le x \le \pi$ with the *boundary conditions* $u(0,t) = 0 = u(\pi, t)$ for all $t \ge 0$ and the *initial displacement* $u(x,0) = 0$ and the *initial velocity* $u_t(x,0) = x(\pi - x)$.

a. (25) We seek split solutions $u(x,t) = F(x)G(t)$. Separate variables using the separation constant $-\lambda$. Write the resulting ordinary differential equation for $F(x)$ with suitable boundary conditions on F. Find *all the positive eigenvalues* λ_n and the corresponding eigenfunctions $F_n(x)$. (Time-saving note: There are no eigenvalues $\lambda \le 0$.)

b. (15) For each eigenvalue λ_n found in part (a), write the ordinary differential equation for $G_n(t)$ with a suitable initial condition on $G_n(0)$. Find the corresponding eigenfunctions $G_n(t)$

c. (20) Express the final solution $u(x,t) = \sum_{n=1}^{\infty} b_n F_n(x) G_n(t)$ using the eigenfunctions you have found in parts (a) and (b). Then *find all the coefficients* b_n in order to satisfy the initial velocity condition on $u(x,t)$.

Solutions

Please remember to SHOW ALL WORK: ANSWERS ALONE ARE NOT SUFFICIENT. If your work does not fit into the spaces provided, continue on the scratch sheets and be sure to number the work on the scratch sheets with the problem number and problem part. And state "see work continued on scratch sheet" on the printed pages.

1. (50) This is part of WebAssign 5.2, question 13.

a. (15) $\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1} + 2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sum_{n=2}^{\infty} a_{n-2}x^{n+r-1} = 0$ or something equivalent to this. The most frequent mistakes were in the re-indexing of the series in the first part. Note that it was necessary to re-index only one of the three series, but some students re-indexed all three, increasing the risk for errors. When you re-index, you should check to make sure that your series starts with the same term as the original series. If not, there is an error.

b. (15) Using $n = 0$, the *indicial equation* is $r(r+1) = 0$ so that $r = 0$ or $r = -1$, which differ by an integer.

c. (10) Using the *larger* value $r = 0$, $a_n = \dfrac{a_{n-2}}{n(n+1)}$.

d. (10) Choosing $a_0 = 1$ and $a_1 = 0$, $y = \dfrac{1}{x}\sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{(2n+1)!}$ or an equivalent series showing the general nth term in closed form.

e. (OPTIONAL 10 POINT BONUS) $y = \dfrac{\sinh x}{x}$, which is an elementary function. We remark that with additional work one could show that the general solution is $y = \dfrac{C_1 \sinh x + C_2 \cosh x}{x}$

2. (40) This is WebAssign Special Functions question 3.

a. (20) $y = vx^{-\frac{1}{2}}$, $y' = v'x^{-\frac{1}{2}} - \frac{1}{2}vx^{-\frac{3}{2}}$, $y'' = v''x^{-\frac{1}{2}} - v'x^{-\frac{3}{2}} + \frac{3}{4}vx^{-\frac{5}{2}}$ so that the original equation becomes $x^2v'' + xv' + \left(4x^2 - \frac{1}{4}\right)v = 0$.

b. (10) Let $t = 2x$, $v' = 2\dfrac{dv}{dt}$, $v'' = 4\dfrac{d^2v}{dt^2}$. The resulting equation is $t^2\dfrac{d^2v}{dt^2} + t\dfrac{dv}{dt} + \left(t^2 - \frac{1}{4}\right)v = 0$. Note that I gave you a reminder as to the form of Bessel's equation. If you did not come out with a Bessel equation, that is a clue that a mistake has been made.

c. (10) $v = c_1 J_{\frac{1}{2}}(t) + c_2 J_{-\frac{1}{2}}(t)$ so that $y = \dfrac{c_1 J_{\frac{1}{2}}(2x) + c_2 J_{-\frac{1}{2}}(2x)}{\sqrt{x}}$. Since $\frac{1}{2}$ is not an integer, it is fine to use the two Bessel functions of the first kind. But one could also use a Bessel function of the second kind.

3. (30) This was WebAssign 12.3 question 4. $f(x) = |x|$ on the interval $[-\pi, \pi]$ is an even function. Thus $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$ where $a_0 = \frac{\pi}{2}$, which is the average value of f on the given interval. $a_n = \dfrac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(nx)\,dx = \dfrac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx$, where we have used the fact that the integrand is an even function. Thus $|x| = \dfrac{\pi}{2} + \sum_{n=1}^{\infty} \dfrac{2((-1)^n - 1)}{\pi n^2}\cos(nx)$. This could be written in different ways, since only the odd values of n yield non-zero coefficients. Note that since $|x|$ is an even function, the Fourier series is a pure cosine series automatically. So it was not necessary to calculate b_n.

4. (20) This is part of WebAssign 12.5 question 3.

a. (15) $(e^x y')' + \lambda e^x y = 0$.

b. (5) $\displaystyle\int_0^2 y_n(x) y_m(x)\, e^x\, dx = 0$ Notice that the endpoints of the interval are 0 and 2, according to the given boundary conditions.

5. (60) This is a slightly simplified version of WebAssign 13.4 question 2.

a. (25)$F''(x) + \lambda F(x) = 0$ with $F(0) = 0 = F(\pi)$. $\lambda_n = n^2$ for every strictly positive integer n and $F_n(x) = \sin(nx)$.

b. (15) For $\lambda_n = n^2$ we have $G_n''(t) + n^2 G_n(t) = 0$ with $G_n(0) = 0$. Thus $G_n(t) = \sin(nt)$.

c. (20) $u(x,t) = \displaystyle\sum_{n=1}^{\infty} b_n \sin(nx) \sin(nt)$ and $b_n = \dfrac{4}{\pi n^4}\left(1 - (-1)^n\right)$.

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	8	4	6	8	8
80-89 (B)	2	6	2	5	6
70-79 (C)	0	4	5	2	2
60-69 (D)	4	1	2	1	0
0-59 (F)	3	1	0	0	0
Test Avg	79.4%	81.6 %	83.6%	87.3%	88.82 %
HW Avg	95.76%	94.61%	91.9 %	94.11	94.11
HW/Test Correl.	0.22	0.67	0.68	0.38	0.38

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{16}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a weakly positive correlation with performance on the homework.