We will solve the wave equation for a membrane in the shape of a circular disc of radius $R > 0$ with *bounded* displacement given by $u(r,t)$, having no dependence on the polar angle θ (radial symmetry). Thus $u(R,t) \equiv 0$ for all $t \geq 0$, and we impose initial conditions $u(r,0) = f(r)$ (initial displacement), $u_t(r,0) = g(r)$ (initial velocity). This yields the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right). \tag{1}$$

We begin by separating variables, seeking solutions of the form $u(r,t) = F(r)G(t)$. It follows from Eq. (1) that

$$\frac{G''}{c^2 G} = K = \frac{F'' + \frac{1}{r}F'}{F}.$$

If $K \geq 0$ we see that there are no nontrivial solutions that are bounded in $t \geq 0$. Thus we let $K = -k^2 < 0$ and let $\lambda = ck$. This yields the two linear differential equations

$$\frac{d^2 G}{dt^2} + \lambda^2 G = 0$$
$$\frac{d^2 F}{dr^2} + \frac{1}{r}\frac{dF}{dr} + k^2 F = 0.$$

We make the change of variables $s = kr$ so that the second equation becomes a Bessel Equation of order zero:

$$s^2 \frac{d^2 F}{ds^2} + s\frac{dF}{ds} + (s^2 - 0^2)F = 0$$

which has the general solution $F = c_1 J_0(s) + c_2 Y_0(s)$. However, Y_0 is not bounded near the origin so we are left with solutions that are multiples of $F(r) = J_0(s) = J_0(kr)$. Because of the boundary condition, we will need to have $F(R) = 0$, so we set $k = \frac{\alpha_m}{R}$, producing a system of orthogonal eigenfunctions

$$F_m(r) = J_0\left(\frac{\alpha_m}{R} r\right)$$

where α_m is the mth zero of the Bessel function J_0. We let $\lambda_m = c\frac{\alpha_m}{R}$. Then we have

$$G_m(t) = a_m \cos \lambda_m t + b_m \sin \lambda_m t$$

and

$$u_m(r,t) = (a_m \cos \lambda_m t + b_m \sin \lambda_m t)\, J_0\left(\frac{\alpha_m}{R} r\right).$$

Finally, we seek to satisfy the initial conditions with the sum of an infinite Fourier-Bessel series

$$u(r,t) = \sum_{1}^{\infty} (a_m \cos \lambda_m t + b_m \sin \lambda_m t)\, J_0\left(\frac{\alpha_m}{R} r\right).$$

According to the two initial conditions, we will need to have

$$a_m = \frac{2}{R^2 J_1^2(\alpha_m)} \int_0^R f(r) J_0\left(\frac{\alpha_m}{R} r\right) r\,dr; \; b_m = \frac{2}{\lambda_m R^2 J_1^2(\alpha_m)} \int_0^R g(r) J_0\left(\frac{\alpha_m}{R} r\right) r\,dr.$$

Here, r is the weight function corresponding to the Sturm-Liouville problem satisfied by the functions $F_m(r) = J_0\left(\frac{\alpha_m}{R} r\right)$ on the interval $[0, R]$.