Definition 0.1. A Sturm-Liouville Equation is a second order differential equation that can be written in the form
\[(r(x)y')' + (q(x) + \lambda p(x))y = 0.\]  
(1)

Here the functions \(y, r, q, p \in C^\infty[a, b]\), the vector space of all infinitely differentiable functions on the interval \([a, b]\). We require the function \(p(x) > 0\) on the interval \([a, b]\). We use part of the left side of the Sturm-Liouville Equation to define a related concept.

Definition 0.2. A Sturm-Liouville Operator \(T : C^\infty[a, b] \to C^\infty[a, b]\) is a linear map given by
\[T(y) = -(r(x)y')' - q(x)y.\]

In this terminology, we can rewrite the Sturm-Liouville Equation (1) as
\[T(y) = \lambda p(x)y.\]

Definition 0.3. If \(y\) is a solution to Equation (1) that is not identically zero, we call the number \(\lambda\) an eigenvalue of the Sturm-Liouville operator \(T\), and we call the corresponding solution function \(y\) an eigenfunction corresponding to the eigenvalue \(\lambda\).

We will define two different scalar products for vectors in the vector space \(C^\infty[a, b]\) as follows.

Definition 0.4. We define
\[\langle f, g \rangle = \int_a^b f(x)g(x)\,dx\]

as the standard scalar (or inner) product, and we also define a weighted scalar product
\[\langle f, g \rangle_p = \int_a^b f(x)g(x)p(x)\,dx.\]

Here \(p\) serves as a weight function in the integrand. The Sturm-Liouville operator \(T\) is called self-adjoint if and only if
\[\langle Tf, g \rangle = \langle f, Tg \rangle\]
for all \(f, g \in C^\infty[a, b]\). The functions \(f\) and \(g\) are said to be orthogonal with respect to the weight function \(p\) if and only if \(\langle f, g \rangle_p = 0\). Orthogonality is denoted by \(f \perp_p g\). We have the following theorem.

Theorem 0.1. If \(T\) is a self-adjoint Sturm-Liouville operator and if \(f\) and \(g\) are eigenfunctions corresponding to two distinct eigenvalues \(\lambda_1\) and \(\lambda_2\), then \(f \perp_p g\).

Proof. We are given that \(Tf = \lambda_1 p(x)f\) and \(Tg = \lambda_2 p(x)g\), where \(\lambda_1 \neq \lambda_2\). Thus \(\langle Tf, g \rangle = \langle \lambda_1 pf, g \rangle = \lambda_1 \langle f, g \rangle_p\). But \(\langle Tf, g \rangle = \langle f, Tg \rangle = \lambda_2 \langle f, pg \rangle = \lambda_2 \langle f, g \rangle_p\). Hence \((\lambda_1 - \lambda_2)(\langle f, g \rangle_p = 0\), which forces \(f \perp_p g\). \(\square\)

Next we develop necessary and sufficient boundary conditions for a Sturm-Liouville operator to be self-adjoint.

Theorem 0.2. For a Sturm-Liouville operator \(T\) to be self-adjoint, it is necessary and sufficient that every pair of solution functions \(y_1, y_2\) satisfy the following boundary condition:

\[
\begin{array}{c|c|c|c}
  r(x) & y_1(x) & y_1'(x) & \bigg|_{x=a}^b \\
  y_2(x) & y_2'(x) & \bigg|_{x=a}^b \\
\end{array} = 0
\]  
(2)
Proof. We calculate readily that \( \langle Ty_1, y_2 \rangle = \langle y_1, Ty_2 \rangle \) if and only if
\[
\int_a^b y_2(x)(r(x)y_1'(x))' \, dx = \int_a^b y_1(x)(r(x)y_2'(x))' \, dx.
\]
If we integrate each side by parts we see that it is necessary and sufficient that
\[
r(x)y_1(x)y_2'(x) \bigg|_a^b = r(x)y_1'(x)y_2(x) \bigg|_a^b.
\]
This is equivalent to Equation (2).

Definition 0.5. A Sturm-Liouville Problem is a Sturm-Liouville Equation together with boundary conditions satisfying Equation (2).

There are many simple examples of boundary conditions that satisfy Equation (2). For example, it would suffice if we required that each solution function \( y \) to Equation (1) satisfy \( y(a) = 0 = y(b) \), or that \( y'(a) = 0 = y'(b) \), or that \( r(a) = 0 = r(b) \). There are many other such examples. Here is a very simple Sturm-Liouville problem

Example 0.1. The equation \( y'' + \lambda y = 0 \) with the boundary conditions \( y(0) = 0 = y(\pi) \) is a Sturm-Liouville problem. The student should try to determine all the eigenfunctions and their corresponding eigenvalues.