Definition 0.1. *A* Sturm-Liouville Equation *is a second order differential equation that can be written in the form*

$$(r(x)y')' + (q(x) + \lambda p(x))y = 0. \tag{1}$$

Here the functions $y, r, q, p \in C^\infty[a, b]$, the vector space of all infinitely differentiable functions on the interval $[a, b]$. We require the function $p(x) > 0$ on the interval $[a, b]$. We use part of the left side of the Sturm-Liouville Equation to define a related concept.

Definition 0.2. *A* Sturm-Liouville Operator $T : C^\infty[a, b] \to C^\infty[a, b]$ *is a linear map given by*

$$T(y) = -(r(x)y')' - q(x)y.$$

In this terminology, we can rewrite the Sturm-Liouville Equation (1) as

$$T(y) = \lambda p(x)y.$$

Definition 0.3. *If y is a solution to Equation (1) that is not identically zero, we call the number λ an* eigenvalue *of the Sturm-Liouville operator T, and we call the corresponding solution function y an* eigenfunction *corresponding to the eigenvalue λ.*

We will define *two* different *scalar products* for vectors in the vector space $C^\infty[a, b]$ as follows.

Definition 0.4. *We define*

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx$$

as the standard scalar (or inner*) product, and we also define a* weighted scalar product

$$\langle f, g \rangle_p = \int_a^b f(x)g(x)p(x)\,dx.$$

Here p serves as a *weight function* in the integrand. The Sturm-Liouville operator T is called *self-adjoint* if and only if

$$\langle Tf, g \rangle = \langle f, Tg \rangle$$

for all $f, g \in C^\infty[a, b]$. The functions f and g are said to be *orthogonal* with respect to the weight function p if and only if $\langle f, g \rangle_p = 0$. Orthogonality is denoted by $f \perp_p g$. We have the following theorem.

Theorem 0.1. *If T is a* self-adjoint *Sturm-Liouville operator and if f and g are eigenfunctions corresponding to* two distinct *eigenvalues λ_1 and λ_2, then $f \perp_p g$.*

Proof. We are given that $Tf = \lambda_1 p(x) f$ and $Tg = \lambda_2 p(x) g$, where $\lambda_1 \neq \lambda_2$. Thus $\langle Tf, g \rangle = \langle \lambda_1 pf, g \rangle = \lambda_1 \langle f, g \rangle_p$. But $\langle Tf, g \rangle = \langle f, Tg \rangle = \lambda_2 \langle f, pg \rangle = \lambda_2 \langle f, g \rangle_p$. Hence $(\lambda_1 - \lambda_2)\langle f, g \rangle_p = 0$, which forces $f \perp_p g$. □

Next we develop necessary and sufficient *boundary conditions* for a Sturm-Liouville operator to be self-adjoint.

Theorem 0.2. *For a Sturm-Liouville operator T to be self-adjoint, it is necessary and sufficient that every pair of solution functions y_1, y_2 satisfy the following boundary condition:*

$$r(x) \left| \begin{array}{cc} y_1(x) & y_1'(x) \\ y_2(x) & y_2'(x) \end{array} \right| \Bigg|_{x=a}^{b} = 0 \tag{2}$$

Proof. We calculate readily that $\langle Ty_1, y_2 \rangle = \langle y_1, Ty_2 \rangle$ if and only if

$$\int_a^b y_2(x)(r(x)y_1'(x))'\,dx = \int_a^b y_1(x)(r(x)y_2'(x))'\,dx.$$

If we integrate each side by parts we see that it is necessary and sufficient that

$$r(x)y_1(x)y_2'(x)\Big|_a^b = r(x)y_1'(x)y_2(x)\Big|_a^b.$$

This is equivalent to Equation (2). ☐

Definition 0.5. *A* Sturm-Liouville Problem *is a Sturm-Liouville Equation together with boundary conditions satisfying Equation (2).*

There are many simple examples of boundary conditions that satisfy Equation (2). For example, it would suffice if we required that each solution function y to Equation (1) satisfy $y(a) = 0 = y(b)$, or that $y'(a) = 0 = y'(b)$, or that $r(a) = 0 = r(b)$. There are many other such examples. Here is a very simple Sturm-Liouville problem

Example 0.1. *The equation $y'' + \lambda y = 0$ with the boundary conditions $y(0) = 0 = y(\pi)$ is a Sturm-Liouville problem. The student should try to determine all the eigenfunctions and their corresponding eigenvalues.*