Example 0.1. *In class we have calculated the coefficients of the Fourier-Bessel expansion of the function*
$f(x) = \begin{cases} k & \text{if } 0 \le x \le a \\ 0 & \text{if } a < x \le R \end{cases}$. *We determined that if* α_{mn} *is the* m*th zero of the function* J_n *on the positive* x*-axis, then*

$$f(x) = \sum_{m=1}^{\infty} c_m J_n\left(\frac{\alpha_{mn}x}{R}\right), \text{ where } c_m = \frac{2}{R^2 J_{n+1}^2(\alpha_{mn})} \int_0^R f(x) J_n\left(\frac{\alpha_{mn}x}{R}\right) x dx.$$

The convergence takes place in the square norm corresponding to the given scalar product with the weight function $p(x) = x$.

In order to use Mathematica to compute the graphs of some partial sums of this Fourier-Bessel series, I specify in the following that $k = 1 = a$ and $R = 2$. I chose to use $n = 0$. In case you use Mathematica, I will include the Mathematica input code that I employed for this example, together with the output.

Example 0.2. *Here is the Mathematica code with the Output. First we calculate the first 50 zeros of* J_0*:*

zeros = N[BesselJZero[0, Range[50]]]

output = 2.40483, 5.52008, 8.65373, 11.7915, 14.9309, 18.0711, 21.2116, 24.3525, 27.4935, 30.6346, 33.7758, 36.9171, 40.0584, 43.1998, 46.3412, 49.4826, 52.6241, 55.7655, 58.907, 62.0485, 65.19, 68.3315, 71.473, 74.6145, 77.756, 80.8976, 84.0391, 87.1806, 90.3222, 93.4637, 96.6053, 99.7468, 102.888, 106.03, 109.171, 112.313, 115.455, 118.596, 121.738, 124.879, 128.021, 131.162, 134.304, 137.446, 140.587, 143.729, 146.87, 150.012, 153.153, 156.295

Then we write the sum of the first ten terms of the Bessel series:

*Sum[N[BesselJ[1, zeros[[k]]/2]]/(zeros[[k]]*N[BesselJ[1, zeros[[k]]]]*^{2}*)BesselJ[0, zeros[[k]]*x/2], {k, 1, 10}]*

output = 0.769756 BesselJ[0, 1.20241 x] + 0.661472 BesselJ[0, 2.76004 x] - 0.282963 BesselJ[0, 4.32686 x] - 0.464336 BesselJ[0, 5.89577 x] + 0.198712 BesselJ[0, 7.46546 x] + 0.378402 BesselJ[0, 9.03553 x] - 0.160955 BesselJ[0, 10.6058 x] - 0.327418 BesselJ[0, 12.1762 x] + 0.138625 BesselJ[0, 13.7467 x] + 0.292704 BesselJ[0, 15.3173 x]

Then we plot the sum of the first 10 terms of the Bessel series in Fig. 1, after which we perform the final two operations above, changing to the sum of the first 50 terms in Fig. 2:

Plot[{%, 1 - HeavisideTheta[x - 1]}, {x, 0, 2}]

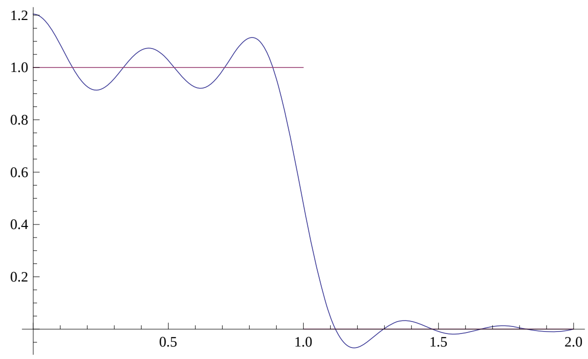

Figure 1: 10th partial sum, with $f(x)$.

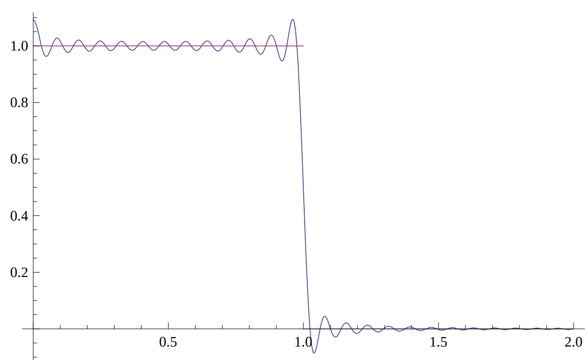

Figure 2: 50th partial sum S_{50}, with $f(x)$.

Notice how little area there is between the graph of $y = f(x)$ and the graph of the 50th partial sum of the Fourier-Bessel series. In fact, we know that

$$\|f - S_n\|^2 \to 0, \text{ as } n \to \infty.$$

In Fig. 3 we have shown the square of $(S_{50} - f(x))^2$. Note that there is a narrow peak at $x = 1$ because the continuous partial sum $S_{50}(x)$ is about half-way between one and zero at $x = 1$. This is necessary since the jump must be bridged. In fact,

$$\|S_{50} - f\|^2 = 0.00400453$$

according to Mathematica.

Example 0.3. *You may have noticed that it took a rather long partial summation of the Fourier-Bessel series to give a good approximation to a function that has a jump discontinuity at the*

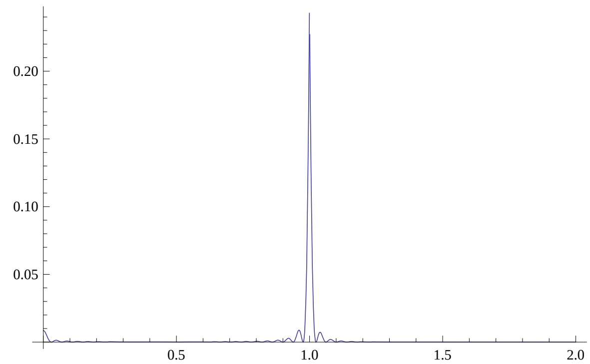

Figure 3: $(S_{50} - f(x))^2$, with step function $f(x)$.

midpoint, $x = 1$*. Here is a different example, without a discontinuity but with points of nondifferentiability. Define the* tent *function as follows:*

$$f(x) = 1 - |x - 1|, \ 0 \leq x \leq 2.$$

After tabulating the zeros of J_0 *as in the first two examples, we use the following method of finding the* tenth *partial sum of the Fourier-Bessel series:*
$Sum[2/(2^2 * N[BesselJ[1, zeros[[k]]]]^2) Integrate[(1 - Abs[x-1]) * BesselJ[0, (zeros[[k]] * x/2)] * x, \{x, 0, 2\}] * BesselJ[0, zeros[[k]] * x/2], \{k, 1, 10\}]$
Output= 1.01488 BesselJ[0, 1.20241 x] - 0.644837 BesselJ[0, 2.76004 x] - 0.302905 BesselJ[0, 4.32686 x] + 0.0365356 BesselJ[0, 5.89577 x] + 0.08648 BesselJ[0, 7.46546 x] - 0.0668066 BesselJ[0, 9.03553 x] - 0.0765271 BesselJ[0, 10.6058 x] + 0.0143768 BesselJ[0, 12.1762 x] + 0.0372709 BesselJ[0, 13.7467 x] - 0.0259684 BesselJ[0, 15.3173 x]

Notice that this time we have instructed Mathematica to perform the weighted integral to find the needed scalar products for the coefficients.
Plot[{%, 1 - Abs[x - 1]}, {x, 0, 2}] See Fig. 4
In this example

$$\|f - S_{10}\|^2 = 0.000119022, \ \|f - S_{10}\| = 0.0109$$

although S_{10} *is the sum of only the first ten terms of the Fourier Bessel series for the tent function shown. In Fig. 5 there is a graph of*

$$(f - S_{10})^2.$$

Note the exaggerated scale on the vertical axis.

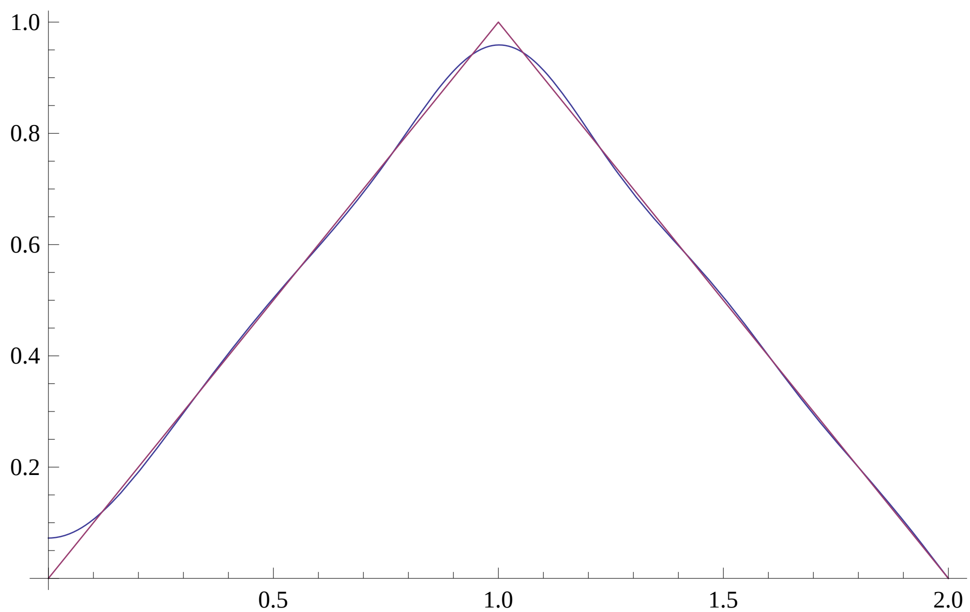

Figure 4: 10th partial S_{10}, with $f(x) = 1 - |x - 1|$.

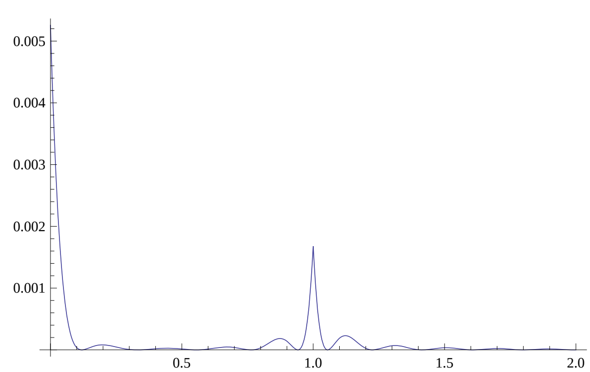

Figure 5: $(f(x) - S_{10}(x))^2$, with $f(x) = 1 - |x - 1|$.