Example 0.1. In class we have calculated the coefficients of the Fourier-Bessel expansion of the function
\[ f(x) = \begin{cases} k & \text{if } 0 \leq x \leq a \\ 0 & \text{if } a < x \leq R \end{cases} \]
We determined that if \( \alpha_{mn} \) is the \( m \)th zero of the function \( J_n \) on the positive \( x \)-axis, then
\[
 f(x) = \sum_{m=1}^{\infty} c_m J_n\left( \frac{\alpha_{mn}x}{R} \right), \quad \text{where} \quad c_m = \frac{2}{R^2 J_{n+1}^2(\alpha_{mn})} \int_0^R f(x) J_n\left( \frac{\alpha_{mn}x}{R} \right) x \, dx.
\]
The convergence takes place in the square norm corresponding to the given scalar product with the weight function \( p(x) = x \).

In order to use Mathematica to compute the graphs of some partial sums of this Fourier-Bessel series, I specify in the following that \( k = 1 = a \) and \( R = 2 \). I chose to use \( n = 0 \). In case you use Mathematica, I will include the Mathematica input code that I employed for this example, together with the output.

Example 0.2. Here is the Mathematica code with the Output. First we calculate the first 50 zeros of \( J_0 \):
\[
\text{zeros} = \text{N}\left[\text{BesselJZero}[0, \text{Range}[50]]\right]
\]
\[
\]
Then we write the sum of the first ten terms of the Bessel series:
\[
\text{Sum}[\text{N}[\text{BesselJ}[1, \text{zeros}[k]/2]]/(\text{zeros}[k])*\text{N}[\text{BesselJ}[1, \text{zeros}[k]]]^2)\text{BesselJ}[0, \text{zeros}[k]]*x/2, \{k, 1, 10\}]
\]
\[
\text{output} = 0.769756 \text{BesselJ}[0, 1.20241 x] + 0.661472 \text{BesselJ}[0, 2.76004 x] - 0.282963 \text{BesselJ}[0, 4.32686 x] - 0.464336 \text{BesselJ}[0, 5.89577 x] + 0.198712 \text{BesselJ}[0, 7.46546 x] + 0.378402 \text{BesselJ}[0, 9.03553 x] - 0.160955 \text{BesselJ}[0, 10.6058 x] - 0.327418 \text{BesselJ}[0, 12.1762 x] + 0.138625 \text{BesselJ}[0, 13.7467 x] + 0.292704 \text{BesselJ}[0, 15.3173 x]
\]
Then we plot the sum of the first 10 terms of the Bessel series in Fig. 1, after which we perform the final two operations above, changing to the sum of the first 50 terms in Fig. 2:
\[
\text{Plot}[[\%, 1 - \text{HeavisideTheta}[x - 1]], \{x, 0, 2\}]
\]
Notice how little area there is between the graph of $y = f(x)$ and the graph of the 50th partial sum of the Fourier-Bessel series. In fact, we know that

$$\|f - S_n\|^2 \to 0, \text{ as } n \to \infty.$$ 

In Fig. 3 we have shown the square of $(S_{50} - f(x))^2$. Note that there is a narrow peak at $x = 1$ because the continuous partial sum $S_{50}(x)$ is about half-way between one and zero at $x = 1$. This is necessary since the jump must be bridged. In fact,

$$\|S_{50} - f\|^2 = 0.00400453$$

according to Mathematica.

**Example 0.3.** You may have noticed that it took a rather long partial summation of the Fourier-Bessel series to give a good approximation to a function that has a jump discontinuity at the
midpoint, $x = 1$. Here is a different example, without a discontinuity but with points of nondifferentiability. Define the tent function as follows:

$$f(x) = 1 - |x - 1|, \ 0 \leq x \leq 2.$$  

After tabulating the zeros of $J_0$ as in the first two examples, we use the following method of finding the tenth partial sum of the Fourier-Bessel series:

$$\text{Sum}\left[\frac{2}{(2^2 \times N[BesselJ[1,\text{zeros}[[k]]] ]^2)} \times \text{Integrate}\left[(1 - \text{Abs}[x - 1]) \times BesselJ[0, (\text{zeros}[[k]] \times x/2)] \times x, \{x, 0, 2\}\right] \times BesselJ[0, \text{zeros}[[k]] \times x/2], \{k, 1, 10\}\right]$$

Output: $1.01488 \times BesselJ[0, 1.20241 x] - 0.644837 \times BesselJ[0, 2.76004 x] - 0.302905 \times BesselJ[0, 4.32686 x] + 0.0365356 \times BesselJ[0, 5.89577 x] + 0.08648 \times BesselJ[0, 7.46546 x] - 0.0668066 \times BesselJ[0, 9.03553 x] - 0.0765271 \times BesselJ[0, 10.6058 x] + 0.0143768 \times BesselJ[0, 12.1762 x] + 0.0372709 \times BesselJ[0, 13.7467 x] - 0.0259684 \times BesselJ[0, 15.3173 x]$  

Notice that this time we have instructed Mathematica to perform the weighted integral to find the needed scalar products for the coefficients.

Plot\[
\{\% , 1 - \text{Abs}[x - 1]\}, \{x, 0, 2\}\]
See Fig. 4

In this example

$$\|f - S_{10}\|^2 = 0.000119022, \quad \|f - S_{10}\| = 0.0109$$

although $S_{10}$ is the sum of only the first ten terms of the Fourier Bessel series for the tent function shown. In Fig. 5 there is a graph of

$$(f - S_{10})^2.$$  

Note the exaggerated scale on the vertical axis.
Figure 4: 10th partial $S_{10}$, with $f(x) = 1 - |x - 1|$.

Figure 5: $(f(x) - S_{10}(x))^2$, with $f(x) = 1 - |x - 1|$.