Print Your Name Here: ______________________________

Show all work in the space provided. *Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if it is to be graded.* No books or notes are allowed. A scientific calculator is OK—but not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. Let $\mathbb{F} = \{0, 1\}$ be the field of two elements, in which $1 + 1 = 0$. Either state the relationship of order (e.g. $<, =,$ or $>$) between 0 and 1 in $\mathbb{F}$, or state that no order relation exists in $\mathbb{F}$.

2. Express the vector (directed line segment) $\overrightarrow{AB}$ in terms of the vectors $\vec{A}$ and $\vec{B}$ in $\mathbb{R}^2$.

3. True or False: The set $S = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1\}$ is a vector space over the field $\mathbb{R}$.

4. Determine whether the set $\{(1, 1, 2), (3, 1, 2), (-1, 0, 0)\} \subset \mathbb{R}^3$ is linearly *independent* or linearly *dependent.*

5. Give an example of two subspaces $S_1, S_2 \subseteq \mathbb{R}^3$ such that $S_1 \cup S_2$ is not a subspace of $\mathbb{R}^3$.

6. Find the dimension of the subspace $S \subseteq C(\mathbb{R})$ consisting of all twice differentiable functions such that $f'' = 0 \in C(\mathbb{R})\}$.

7. If S and T are subspaces of a vector space for which $\dim(S + T) = 8$, $\dim(S) = 5$, $\dim(T) = 6$, find $\dim(S \cap T)$.

8. How many one-dimensional subspaces are there in $\mathbb{F}^3$ if $\mathbb{F} = \{0, 1\}$ is the two-element field

9. True or False: A system of 7 linear *homogeneous* equations in 8 unknowns must have a nontrivial solution.

10. True or False: A system of n homogeneous linear equations in n unknowns must have a nontrivial solution if the rank of the coefficient matrix is exactly n.

11. Find a *basis* for the solution space to the system

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 0 \\ x_1 + x_2 - x_3 + 2x_4 &= 0 \end{aligned}$$

12. For which value(s) of α do the 3 columns of the *coefficient matrix* of the system below span $\mathbb{R}^2$?

$$\begin{aligned} x_1 + 2x_2 + \alpha^2 x_3 &= b_1 \\ x_1 + 2x_2 + 4x_3 &= b_2 \end{aligned}$$

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Let A, B, C and D be points in $\mathbb{R}^2$ such that $\overrightarrow{AB} = \overrightarrow{CD}$. *Use vector algebra* to prove that $\overrightarrow{AC} = \overrightarrow{BD}$. (Suggestion: It may be helpful to express as $\vec{A}$ the vector pointing from the origin to the point A, and the same for the three other points, so that $\overrightarrow{AB}$ becomes the *difference* between two vectors, etc.)

B. Let $n \in \mathbb{N}$ and let $P_n(\mathbb{R}) = \{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \mid a_i \in \mathbb{R}\ \forall i = 0, 1, \ldots, n\}$. Prove:

(**i**) $P_n(\mathbb{R})$ is a *subspace* of the (known) vector space $P(\mathbb{R})$ of all polynomials on $\mathbb{R}$.

(**ii**) The set $\{1, x, x^2, \ldots, x^n\}$ is a *basis* for the subspace $P_n(x)$.

C. Suppose (α, β) and (γ, δ) are two *distinct* points in $\mathbb{R}^2$. Prove that the solution space S of the system of 2 linear homogeneous equations in 3 unknowns written in vector form as

$$x_1 \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} + x_2 \begin{pmatrix} \beta \\ \delta \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \vec{0} \in \mathbb{R}^2$$

has dimension *exactly* equal to 1. (Suggestion: Consider the possibilities for the rank of the coefficient matrix of this system.)

Solutions and Class Statistics

1. No order relation exists. In any ordered field, the multiplicative identity 1 must be positive, as shown in class and in the homework. But then the sum $1+1$ would need also to be positive, which is impossible since $1+1=0$ in $\mathbb{F}$.

2. $\overrightarrow{AB} = \vec{B} - \vec{A}$.

3. False: S is not closed under either scalar multiplication or vector addition, both of which are required to be a vector space.

4. The set is linearly dependent: row reduce the matrix of the three given row vectors to echelon form and the third row vector will be $\vec{0} \in \mathbb{R}^3$.

5. For example, let $S_1 = \{(a,0,0) \mid a \in \mathbb{R}\}$ and let $S_2 = \{(0,b,0) \mid b \in \mathbb{R}\}$.

6. The dimension is 2: f' must be a constant function c and $f(x) = cx + d$, so that S has the basis $\{1, x\}$.

7. $\dim(S+T) = \dim(S) + \dim(T) - \dim(S \cap T)$, so $\dim(S \cap T) = 3$.

8. F^3 has $8-1=7$ nonzero vectors and 7 one-dimensional subspaces.

9. True: the rank of the coefficient matrix cannot exceed 7, making the dimension of the solution space at least 1.

10. False: If the rank were n then the set of n column vectors must be linearly independent, so that the system has only the trivial solution.

11. Reducing the coefficient matrix to echelon form, we see that for a basis we may take $B = \{(1,0,1,0), (-3,1,0,1)\}$, for example.

12. Row reducing the augmented matrix to echelon form, we see that we need $\alpha \neq \pm 2$.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	7				
80-89 (B)	6				
70-79 (C)	9				
60-69 (D)	3				
0-59 (F)	3				
Test Avg	78.6%	%	%	%	%
HW Avg	6.3				
HW/Test Correl	0.65				