1. Let $F = \{0, 1\}$ be the field of two elements, in which $1 + 1 = 0$. Either state the relationship of order (e.g. $<$, $=$, or $>$) between 0 and 1 in $F$, or state that no order relation exists in $F$.

2. Express the vector (directed line segment) $\overrightarrow{AB}$ in terms of the vectors $\vec{A}$ and $\vec{B}$ in $\mathbb{R}^2$.

3. True or False: The set $S = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1\}$ is a vector space over the field $\mathbb{R}$.

4. Determine whether the set $\{(1, 1, 2), (3, 1, 2), (-1, 0, 0)\} \subseteq \mathbb{R}^3$ is linearly independent or linearly dependent.

5. Give an example of two subspaces $S_1, S_2 \subseteq \mathbb{R}^3$ such that $S_1 \cup S_2$ is not a subspace of $\mathbb{R}^3$.

6. Find the dimension of the subspace $S \subseteq C(\mathbb{R})$ consisting of all twice differentiable functions such that $f'' = 0 \in C(\mathbb{R})$.

7. If $S$ and $T$ are subspaces of a vector space for which $\dim(S + T) = 8$, $\dim(S) = 5$, $\dim(T) = 6$, find $\dim(S \cap T)$.

8. How many one-dimensional subspaces are there in $F^3$ if $F = \{0, 1\}$ is the two-element field
9. True or False: A system of 7 linear homogeneous equations in 8 unknowns must have a nontrivial solution.

10. True or False: A system of \( n \) homogeneous linear equations in \( n \) unknowns must have a nontrivial solution if the rank of the coefficient matrix is exactly \( n \).

11. Find a basis for the solution space to the system

\[
\begin{align*}
    x_1 + 2x_2 - x_3 + x_4 &= 0 \\
    x_1 + x_2 - x_3 + 2x_4 &= 0
\end{align*}
\]

12. For which value(s) of \( \alpha \) do the 3 columns of the coefficient matrix of the system below span \( \mathbb{R}^2 \)?

\[
\begin{align*}
    x_1 + 2x_2 + \alpha^2 x_3 &= b_1 \\
    x_1 + 2x_2 + 4x_3 &= b_2
\end{align*}
\]

Part II: Proofs. Prove carefully 2 of the following 3 theorems for 26 points each. Circle the letters of the 2 proofs to be counted in the list below—no more than 2! You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Let \( A, B, C \) and \( D \) be points in \( \mathbb{R}^2 \) such that \( \overrightarrow{AB} = \overrightarrow{CD} \). Use vector algebra to prove that \( \overrightarrow{AC} = \overrightarrow{BD} \).

(Suggestion: It may be helpful to express as \( \vec{A} \) the vector pointing from the origin to the point \( A \), and the same for the three other points, so that \( \overrightarrow{AB} \) becomes the difference between two vectors, etc.)

B. Let \( n \in \mathbb{N} \) and let \( P_n(\mathbb{R}) = \{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \mid a_i \in \mathbb{R} \ \forall i = 0, 1, \ldots, n\} \). Prove:

(i) \( P_n(\mathbb{R}) \) is a subspace of the (known) vector space \( P(\mathbb{R}) \) of all polynomials on \( \mathbb{R} \).

(ii) The set \( \{1, x, x^2, \ldots, x^n\} \) is a basis for the subspace \( P_n(x) \).

C. Suppose \((\alpha, \beta)\) and \((\gamma, \delta)\) are two distinct points in \( \mathbb{R}^2 \). Prove that the solution space \( S \) of the system of 2 linear homogeneous equations in 3 unknowns written in vector form as

\[
\begin{align*}
    x_1 \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} + x_2 \begin{pmatrix} \beta \\ \delta \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \vec{0} \in \mathbb{R}^2
\end{align*}
\]

has dimension exactly equal to 1. (Suggestion: Consider the possibilities for the rank of the coefficient matrix of this system.)
Solutions and Class Statistics

1. No order relation exists. In any ordered field, the multiplicative identity 1 must be positive, as shown in class and in the homework. But then the sum 1 + 1 would need also to be positive, which is impossible since 1 + 1 = 0 in $\mathbb{F}$.

2. $\overline{AB} = \overline{B} - \overline{A}$.

3. False: $S$ is not closed under either scalar multiplication or vector addition, both of which are required to be a vector space.

4. The set is linearly dependent: row reduce the matrix of the three given row vectors to echelon form and the third row vector will be $\overrightarrow{0} \in \mathbb{R}^3$.

5. For example, let $S_1 = \{(a,0,0) \mid a \in \mathbb{R}\}$ and let $S_2 = \{(0,b,0) \mid b \in \mathbb{R}\}$.

6. The dimension is 2: $f'$ must be a constant function $c$ and $f(x) = cx + d$, so that $S$ has the basis $\{1, x\}$.

7. $\dim(S + T) = \dim(S) + \dim(T) - \dim(S \cap T)$, so $\dim(S \cap T) = 3$.

8. $F^3$ has $8 - 1 = 7$ nonzero vectors and 7 one-dimensional subspaces.

9. True: the rank of the coefficient matrix cannot exceed 7, making the dimension of the solution space at least 1.

10. False: If the rank were $n$ then the set of $n$ column vectors must be linearly independent, so that the system has only the trivial solution.

11. Reducing the coefficient matrix to echelon form, we see that for a basis we may take $B = \{(1,0,1,0), (-3,1,0,1)\}$, for example.

12. Row reducing the augmented matrix to echelon form, we see that we need $\alpha \neq \pm 2$.

### Class Statistics

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