Print Your Name Here: ______________________________

Show all work in the space provided. *Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if it is to be graded.* No books or notes are allowed. A scientific calculator is OK—but not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.
On this test, $P(\mathbb{R})$ *is the vector space of all real polynomials, and* $P_n(\mathbb{R}) = \{p \in P(\mathbb{R}) \mid \deg(p) \leq n\}$.

1. True or False: $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T : (x_1, x_2) \to (x_1 + x_2,\, x_1 + 1 - x_2)$ is a linear transformation.

2. Define D and M mapping $P[\mathbb{R}] \to P[\mathbb{R}]$ by $(Dp)(x) = p'(x)$, the derivative of p, and $(Mp)(x) = xp(x)$. Find $(M \circ Dp)(x)$ and $(D \circ Mp)(x)$.

3. Let V and W be vector spaces over $\mathbb{F}$. True or False: If $T \in L(V, W)$ has the property that $T(x) = \mathbf{0} \in W$ implies $x = \mathbf{0} \in V$, then T is one-to-one.

4. Let $A = (a_{ij})_{n\times n}$ and define $T \in L(\mathbb{F}^n, \mathbb{F}^n)$ by $Tx = Ax$. True or False: If T maps $\mathbb{F}^n$ onto $\mathbb{F}^n$ then T must be an isomorphism.

5. True or False: There exists a linear transformation $T : P_3(\mathbb{R}) \to \mathbb{R}^3$ such that
$T(x^2 + x) = (1, 1, 0)$; $T(x^2 + 1) = (1, 0, 1)$; $T(x - 1) = (0, 1, 1)$.

6. Give an example of two maps $A, D \in L(P(\mathbb{R}), P(\mathbb{R}))$ such that $D \circ A = I$, the identity transformation, yet D is not invertible.

7. Give an example of 2×2 matrices A and B such that $AB \neq BA$.

8. Find a 2×2 elementary matrix E such that the result of the product EB is the result of adding $2 \times \text{row}_1(B)$ to $\text{row}_2(B)$, for each 2×2 matrix B.

9. Let $E = \{(1,0),(0,1)\}$ be the standard basis for $\mathbb{R}^2$ and let $F = \{(a,b),(c,d)\}$ be any other basis. Find the 2×2 matrix B such that for each $T \in L(\mathbb{R}^2,\mathbb{R}^2)$, the matrix $[T]_F = B^{-1}[T]_E B$.

10. Let $D \in L(P_2(\mathbb{R}), P_2(\mathbb{R}))$ defined by $(Dp)(x) = p'(x)$. Find the *rank* and the *nullity* of D.

11. Let $T \in L(P_3(\mathbb{R}),\mathbb{R}) \setminus \{\mathbf{0}\}$. Find the rank and the nullity of T.

12. Using the basis $\{1, x, x^2, x^3\}$ for $P_3(\mathbb{R})$, define an isomorphism $T : P_3(\mathbb{R}) \to \mathbb{R}^4$.

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Let A be an $m \times n$ matrix with coefficients in a field $\mathbb{F}$. Define $T \in L(\mathbb{F}^n, \mathbb{F}^m)$ by $Tx = Ax$.

- **(i)** Express Tx in terms of the column vectors $\mathrm{col}_j(A)$, $j = 1\ldots, n$.
- **(ii)** Prove that T maps $\mathbb{F}^n$ *onto* $\mathbb{F}^m$ if and only if the *rank* of the *matrix* A is m.

B. A matrix $D = (d_{ij})_{n\times n}$ is called a *diagonal matrix* over the field $\mathbb{F}$ if and only if $d_{ij} = 0 \in \mathbb{F}$ whenever $i \neq j$. Prove that a matrix $A = (a_{ij})_{n\times n}$ is a diagonal matrix if and only if $AD = DA$ for *every* $n \times n$ diagonal matrix D.

C. Let $E = \{e_1, e_2, \ldots, e_n\}$ be a basis for the vector space V over a field $\mathbb{F}$. Let $T \in L(V,V)$ and let $A = [T]_E$, the matrix of T with respect to E.

- **(i)** Consider the map $x \to [x]_E$, the *coordinate vector* of x with respect to E. Prove that a set $\{Te_{i_j} \mid j = 1,\ldots,K\} \subseteq \{Te_i \mid i = 1,\ldots,n\}$ is *linearly independent* in V *if and only* if $\{[Te_{i_j}]_E \mid j = 1,\ldots,K\}$ is *linearly independent* in $\mathbb{F}^n$.
- **(ii)** Use the first part to prove that the rank of T equals the rank of A.

Solutions and Class Statistics

1. False: $T(0) \neq 0$.

2. $(M \circ Dp)(x) = xp'(x)$ and $(D \circ Mp)(x) = p(x) + xp'(x)$.

3. True: If $Tx = Ty$ then $T(x-y) = 0$, so $x - y = 0$ and $x = y$.

4. True: the nullity of T must be 0 so that T is one-to-one and onto, making T an isomorphism.

5. False: $T(x-1) = T(x^2+x) - T(x^2+1) = (0, 1, -1)$.

6. For example, define $A : a_0 + a_1x + \cdots + a_nx^n \to a_0x + a_1\frac{x^2}{2} + \cdots + a_n\frac{x^{n+1}}{n+1}$ and $D : f \to f'$.

7. For example, let $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

8. $E = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

9. $B = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

10. The nullity of D is 1 since the 1-dimensional subspace of constant functions is the null space. Thus the rank of D is 2.

11. The rank of T is 1 and the nullity is 3.

12. Let T map $1 \to e_1$, $x \to e_2$, $x^2 \to e_3$, $x^3 \to e_4$, using the standard basis for $\mathbb{R}^4$.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	7	9			
80-89 (B)	6	4			
70-79 (C)	9	9			
60-69 (D)	3	3			
0-59 (F)	3	4			
Test Avg	78.9%	77.9%	%	%	%
HW Avg	6.3	5.7			
HW/Test Correl	0.65	0.69			