Part I: Short Questions. Answer 8 of the 12 short questions: 6 points each. Circle the numbers of the 8 questions that you want counted—no more than 8! Detailed explanations are not required, but they may help with partial credit and are risk-free! Maximum score: 48 points.

1. The Gram-Schmidt process is applied to a basis \{a_1, a_2\} of a real inner product space \((V, \langle \cdot, \cdot \rangle)\) to produce an orthonormal basis \{u_1, u_2\}. Express \(u_2\) in terms of \(a_2\) and \(u_1\) using inner product space operations such as scalar products and norms.

2. The real vector space \(C[0, 1]\) has scalar product \(\langle f, g \rangle = \int_0^1 f(x)g(x) \, dx\). Find the vector \(u = \frac{f}{\|f\|}\) and find \(\|u\|\) if \(f(x) = x\).

3. If \(\{u_i \mid i = 1, \ldots, n\}\) is an orthonormal basis for the inner product space \((V, \langle \cdot, \cdot \rangle)\), each vector \(v \in V\) can be expressed as \(v = \sum_{i=1}^{n} a_i u_i\). Express \(a_i, i = 1, \ldots, n\), in terms of \(v\) and any needed basis vectors.

4. If \(\{u_i \mid i = 1, \ldots, n\}\) is an orthonormal basis for the inner product space \((V, \langle \cdot, \cdot \rangle)\), each vector \(v \in V\) can be expressed as \(v = \sum_{i=1}^{n} a_i u_i\). Express \(\|v\|\) in terms of the coefficients \(a_i\).

5. Let \(u\) be a unit vector in an inner product space \(V\), and let \(v \in V\). Evaluate \(\langle v - \langle v, u \rangle u, u \rangle\).

6. If \(D\) is a determinant function of three vector variables from \(\mathbb{F}^3\), find the value of \(D(\lambda e_1, e_2 + \mu e_3, e_3)\) in terms of the scalars \(\lambda\) and \(\mu\).

7. Evaluate the determinant

\[
\begin{vmatrix}
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 1
\end{vmatrix}
\]

by your favorite method. (Row reduction is suggested!)
8. Let $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{m \times m}$, where $\det A = \alpha$, $\det B = \beta$. Find the determinant of the $(n + m) \times (n + m)$ block-triangular matrix $C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

9. Evaluate the determinant \[ \begin{vmatrix} 2 & 0 & 0 \\ 279 & 3 & 0 \\ 346 & 757 & 4 \end{vmatrix} \]

10. The determinant \[ \begin{vmatrix} 3 & 6 \\ -1 & 2 \end{vmatrix} \] can be expressed in the form $D(\alpha e_1 + \beta e_2, \gamma e_1 + \delta e_2)$. List in order the scalars $\alpha, \beta, \gamma, \delta$.

11. Two of the three defining properties of the determinant function $D$ on $n$ vectors in $\mathbb{F}^n$ are:
   (i) $D(\ldots, \lambda a_i, \ldots) = \lambda D(\ldots, a_i, \ldots)$ for all $\lambda \in \mathbb{F}$; (ii) $D(\ldots, a_i + a_j, \ldots, a_j, \ldots) = D(\ldots, a_i, \ldots, a_j, \ldots)$ if $i \neq j$. State the third defining property of $D$.

12. Let $a_i \in \mathbb{F}^n$, $i = 1, \ldots, n$. Evaluate $D(a_1, \ldots, a_{n-1}, \lambda_1 a_1 + \cdots + \lambda_{n-1} a_{n-1})$.

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**Part II: Proofs.** Prove carefully 2 of the following 3 theorems for 26 points each. Circle the letters of the 2 proofs to be counted in the list below—no more than 2! You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

**A.** Let $(V, \langle \cdot, \cdot \rangle)$ be a vector space with an inner product, and $\dim V = n$, which is finite. Let $W$ be a subspace of $V$ and $W^\perp = \{ v \in V \mid \langle v, w \rangle = 0 \forall w \in W \}$. Prove: $\dim W + \dim W^\perp = n$.

**B.** Let $A$ be a triangular $n \times n$ matrix with diagonal coefficients $\alpha_1, \ldots, \alpha_n$, zeros below the diagonal and arbitrary coefficients above the diagonal. Using only the three defining properties of a determinant function and the property concerning adding a multiple of one row to another row, prove that $D(A) = \alpha_1 \alpha_2 \cdots \alpha_n$.

**C.** Let $a_1 = (a_{1,1}, a_{1,2}) \in \mathbb{F}^2$ and $a_2 = (a_{2,1}, a_{2,2}) \in \mathbb{F}^2$. Prove, using only the three defining properties of the determinant together with the proven fact that the determinant is alternating (in sign) and linear in each of its variables, that $D(a_1, a_2) = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$. 

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2
Solutions and Class Statistics

1. \( u_2 = \frac{a_2 - (a_2, u_1)u_1}{\|a_2 - (a_2, u_1)u_1\|} \)

2. \( u = x\sqrt{3} \) and \( \|u\| = 1. \)

3. \( a_i = (v, u_i). \)

4. \( \|v\| = \sqrt{\sum a_i^2}. \)

5. 0

6. \( D(\lambda e_2, e_2 + \mu e_3, e_3) = \lambda. \)

7. 2

8. \( \alpha\beta. \)

9. Since \( \det A = \det A^t, \) the answer is 24.

10. \( (a, b, c, d) = (3, 6, -1, 2). \)

11. \( D(e_1, \ldots, e_n) = 1. \)

12. \( D(a_1, \ldots, a_{n-1}, \lambda a_1 + \cdots + \lambda_{n-1}a_{n-1}) = 0 \) since the last row is linearly dependent upon the preceding rows.

Class Statistics

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