Print Your Name Here: ____________________

Show all work in the space provided. *Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if it is to be graded.* No books or notes are allowed. A scientific calculator is OK—but not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.
On this test, $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ *is the standard basis for* $\mathbb{F}^n$.

1. The Gram-Schmidt process is applied to a basis $\{\mathbf{a}_1, \mathbf{a}_2\}$ of a real inner product space $(V, \langle\cdot,\cdot\rangle)$ to produce an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$. Express $\mathbf{u}_2$ in terms of $\mathbf{a}_2$ and $\mathbf{u}_1$ using inner product space operations such as scalar products and norms.

2. The real vector space $C[0,1]$ has scalar product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$. Find the vector $\mathbf{u} = \frac{f}{\|f\|}$ and find $\|\mathbf{u}\|$ if $f(x) = x$.

3. If $\{\mathbf{u}_i \mid i = 1, \ldots, n\}$ is an orthonormal basis for the inner product space $(V, \langle\cdot,\cdot\rangle)$, each vector $\mathbf{v} \in V$ can be expressed as $\mathbf{v} = \sum_1^n a_i\mathbf{u}_i$. Express a_i, $i = 1, \ldots, n$, in terms of $\mathbf{v}$ and any needed basis vectors.

4. If $\{\mathbf{u}_i \mid i = 1, \ldots, n\}$ is an orthonormal basis for the inner product space $(V, \langle\cdot,\cdot\rangle)$, each vector $\mathbf{v} \in V$ can be expressed as $\mathbf{v} = \sum_1^n a_i\mathbf{u}_i$. Express $\|\mathbf{v}\|$ in terms of the coefficients a_i.

5. Let $\mathbf{u}$ be a unit vector in an inner product space V, and let $\mathbf{v} \in V$. Evaluate $\langle \mathbf{v} - \langle \mathbf{v}, \mathbf{u}\rangle\mathbf{u}, \mathbf{u}\rangle$.

6. If D is a determinant function of three vector variables from $\mathbb{F}^3$, find the value of $D(\lambda\mathbf{e}_1, \mathbf{e}_2 + \mu\mathbf{e}_3, \mathbf{e}_3)$ in terms of the scalars λ and μ.

7. Evaluate the determinant $\begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{vmatrix}$ by your favorite method. (Row reduction is suggested!)

8. Let $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{m\times m}$, where $\det A = \alpha$, $\det B = \beta$. Find the determinant of the $(n+m)\times(n+m)$ block-triangular matrix $C = \begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$.

9. Evaluate the determinant $\begin{vmatrix} 2 & 0 & 0 \\ 279 & 3 & 0 \\ 346 & 757 & 4 \end{vmatrix}$.

10. The determinant $\begin{vmatrix} 3 & 6 \\ -1 & 2 \end{vmatrix}$ can be expressed in the form $D(a\mathbf{e}_1 + b\mathbf{e}_2, c\mathbf{e}_1 + d\mathbf{e}_2)$. List in order the scalars a, b, c, d.

11. Two of the three defining properties of the determinant function D on n vectors in $\mathbb{F}^n$ are: (i) $D(\ldots, \lambda\mathbf{a}_i, \ldots) = \lambda D(\ldots, \mathbf{a}_i, \ldots)$ for all $\lambda \in \mathbb{F}$; (ii) $D(\ldots, \mathbf{a}_i + \mathbf{a}_j, \ldots, \mathbf{a}_j, \ldots) = D(\ldots, \mathbf{a}_i, \ldots, \mathbf{a}_j, \ldots)$ if $i \neq j$. State the third defining property of D.

12. Let $a_i \in \mathbb{F}^n$, $i = 1, \ldots, n$. Evaluate $D(a_1, \ldots, a_{n-1}, \lambda_1 a_1 + \cdots + \lambda_{n-1}a_{n-1})$.

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

- **A**. Let $(V, \langle\cdot,\cdot\rangle)$ be a vector space with an inner product, and $\dim V = n$, which is finite. Let W be a subspace of V and $W^\perp = \{v \in V \mid \langle v, w\rangle = 0\,\forall w \in W\}$. Prove: $\dim W + \dim W^\perp = n$.
- **B**. Let A be a triangular $n\times n$ matrix with diagonal coefficients $\alpha_1, \ldots, \alpha_n$, zeros below the diagonal and arbitrary coefficients above the diagonal. Using only the three defining properties of a determinant function and the property concerning adding a multiple of one row to another row, prove that $D(A) = \alpha_1\alpha_2\cdots\alpha_n$.
- **C**. Let $\mathbf{a}_1 = (\alpha_{1,1}, \alpha_{1,2}) \in \mathbb{F}^2$ and $\mathbf{a}_2 = (\alpha_{2,1}, \alpha_{2,2}) \in \mathbb{F}^2$. Prove, using only the three defining properties of the determinant together with the proven fact that the determinant is alternating (in sign) and linear in each of its variables, that $D(\mathbf{a}_1, \mathbf{a}_2) = \alpha_{1,1}\alpha_{2,2} - \alpha_{1,2}\alpha_{2,1}$.

Solutions and Class Statistics

1. $\mathbf{u}_2 = \dfrac{\mathbf{a}_2 - \langle \mathbf{a}_2, \mathbf{u}_1\rangle \mathbf{u}_1}{\|\mathbf{a}_2 - \langle \mathbf{a}_2, \mathbf{u}_1\rangle \mathbf{u}_1\|}$

2. $\mathbf{u} = x\sqrt{3}$ and $\|u\| = 1$.

3. $a_i = \langle \mathbf{v}, \mathbf{u}_i\rangle$.

4. $\|\mathbf{v}\| = \sqrt{\sum_1^n a_i^2}$.

5. 0

6. $D(\lambda \mathbf{e}_1, \mathbf{e}_2 + \mu \mathbf{e}_3, \mathbf{e}_3) = \lambda$.

7. 2

8. $\alpha\beta$.

9. Since $\det A = \det A^t$, the answer is 24.

10. $(a, b, c, d) = (3, 6, -1, 2)$.

11. $D(\mathbf{e}_1, \ldots, \mathbf{e}_n) = 1$.

12. $D(a_1, \ldots, a_{n-1}, \lambda_1 a_1 + \cdots + \lambda_{n-1} a_{n-1}) = 0$ since the last row is linearly dependent upon the preceding rows.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	7	9	13		
80-89 (B)	6	4	8		
70-79 (C)	9	9	1		
60-69 (D)	3	3	2		
0-59 (F)	3	4	3		
Test Avg	78.9%	77.9%	85.3%	%	%
HW Avg	6.3	5.7	5.4		
HW/Test Correl	0.65	0.69	0.61		