Show all work in the space provided. Indicate clearly if you continue on the back. Write your name at the top of the scratch sheet if it is to be graded. No books or notes are allowed. A scientific calculator is OK—but not needed. The maximum total score is 200.

Part I: Short Questions. Answer 12 of the 18 short questions: 8 points each. Circle the numbers of the 12 questions that you want counted—no more than 12! Detailed explanations are not required, but they may help with partial credit and are risk-free! Maximum score: 96 points.

1. True or False: In a field \( F \), \( 1 + 1 \neq 1 \).

2. \( A, B \) and \( C \) are the vertices of a triangle. The vector from \( A \) to the midpoint of \( BC \) can be written as \( \alpha \overrightarrow{AB} + \beta \overrightarrow{AC} \). Find \( \alpha \) and \( \beta \).

3. True or False: \( \{ f \mid f \in C(\mathbb{R}), \int_0^1 f(x) \, dx = 0 \} \) is a subspace of the vector space \( C(\mathbb{R}) \) of continuous functions on \( \mathbb{R} \).

4. Find the dimension of the vector space of all twice differentiable functions \( f \) on \( \mathbb{R} \) such that \( \frac{d^2 f}{dx^2} = 0 \).

5. Let \( S \) and \( T \) be subspaces of a vector space \( V \) such that \( \dim(S + T) = 5 \), \( \dim(S \cap T) = 2 \) and \( \dim(S) = 3 \). Find \( \dim(T) \).

6. True or False: For a homogeneous system of 7 linear equations in 11 unknowns, the dimension of the solution space is at most 4.

7. If \( A \) is an \( n \times n \) real matrix and the solution set of the equation \( Ax = b \) is a 2-dimensional linear manifold in \( \mathbb{R}^n \), find the dimension of the solution space of \( Ax = 0 \).

8. Let \( T \) be the linear transformation of \( P_n(\mathbb{R}) \) to itself given by \( Tp(x) = \frac{d}{dx}(xp'(x)) \). Find the rank and the nullity of \( T \).
9. Let $S$ be the linear transformation of $P_n(\mathbb{R})$ to itself given by $T_p(x) = \frac{d}{dx}p(x)$. Find the rank and the nullity of $S$.

10. A linear manifold $L = b + W \subset \mathbb{R}^3$ where $W^\perp = \{t(2,1,2) \mid t \in \mathbb{R}\}$ and $b = (1,-1,1)$. Write the linear equation in $x_1, x_2, x_3$ that is satisfied if and only if $x = (x_1, x_2, x_3) \in L$.

11. Find the inverse of the matrix $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$, where $x, y, z$ are real numbers.

12. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. If $AB = BA$ for all values of $\alpha$ and $\beta$, find the values of $b$ and $c$.

13. Let $D$ be the linear transformation of $P_2(\mathbb{R})$ to itself given by $T_p(x) = \frac{d}{dx}p(x)$. Find the matrix of $T$ with respect to the basis $\{1, x, x^2\}$.

14. Use the Gram-Schmidt Process to find an orthonormal basis for the span of the two vectors $\{1, x\}$ in $C[0,1]$ with respect to the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x) \, dx$.

15. If $A$ and $B$ are orthogonal transformations of an inner product space, find the value of $(AB)(AB)^t$.

16. Let $v \in V$, an inner product space, and let $\{u_1, u_2\}$ be an orthonormal set in $V$. Express the value of $\langle v - \langle v, u_1 \rangle u_1 - \langle v, u_2 \rangle u_2, v - \langle v, u_1 \rangle u_1 - \langle v, u_2 \rangle u_2 \rangle$ in terms of $\|v\|, \langle v, u_1 \rangle, \langle v, u_2 \rangle$. 
17. For the invertible matrix 
\[ A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 2 & 1 \\ 4 & 5 & 3 \end{pmatrix}, \]
what is the value of \( D(A)D(A^{-1}) \)?

18. Find the determinant of the matrix 
\[ A = \begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 1 & 6 & 14 \end{pmatrix}. \]

Part II: Proofs. Prove carefully 4 of the following 6 theorems for 26 points each. Circle the letters of the 4 proofs to be counted in the list below—no more than 4! You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 104 points.

A. Let \( f_1, f_2, f_3 \in \mathcal{F}(\mathbb{R}) \), the vector space of all real-valued functions on \( \mathbb{R} \), and let \( x_1, x_2, x_3 \in \mathbb{R} \). If the rows of the matrix \( [f_i(x_j)]_{3 \times 3} \) comprise a linearly independent set in \( \mathbb{R}^3 \), prove that \( \{f_1, f_2, f_3\} \) is a linearly independent set in \( \mathcal{F}(\mathbb{R}) \).

B. A system of \( m \) linear equations in \( n \) unknowns has coefficient matrix 
\[ C = [c_{ij}] \] in which \( c_j \) is the \( j \)th column. Prove that the homogeneous system 
\[ x_1c_1 + \cdots + x_n c_n = 0 \]
has a nontrivial solution if and only if rank(\( C \)) < \( n \).

C. Show that the equation \( A \mathbf{x} = \mathbf{b} \), where \( A \) is an \( n \times n \) matrix, has a unique solution if and only if \( A \) is an invertible matrix. If \( A \) is invertible, show that the solution of the equation is \( \mathbf{x} = A^{-1}\mathbf{b} \).

D. Let \( V \) be a finite dimensional vector space and let \( S \) and \( T \) be in \( \mathcal{L}(V, V) \) such that \( S \circ T = I \), the identity transformation of \( V \). Prove that \( T \circ S = I \) and \( T = S^{-1} \). (Make it clear how you use the finite dimensionality of \( V \).)

E. Let \( \{u_1, \ldots, u_n\} \) be an orthonormal basis for a finite dimensional inner product space \( V \).

(i) If \( \mathbf{v} = \sum_{i=1}^{n} a_i u_i \) and \( \mathbf{w} = \sum_{j=1}^{n} b_j u_j \) then \( \langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^{n} a_i b_i \).

(ii) Show that every vector \( \mathbf{v} \in V \) can be expressed uniquely as \( \mathbf{v} = \sum_{i=1}^{n} \langle \mathbf{v}, u_i \rangle u_i \).

F. Let \( T \in \mathcal{L}(V, V) \) be an orthogonal transformation of a finite dimensional inner product space \( V \).

(i) Prove that \( D(T) = \pm 1 \).

(ii) Explain why both 1 and \( -1 \) actually occur as determinants of suitable orthogonal transformations.
Solutions and Class Statistics

1. True, since $1 \neq 0$.
2. $\alpha = \frac{1}{2} = \beta$.
3. True: It is closed under vector addition and scalar multiplication.
4. The dimension is 2 since $f(x) = a_1x + a_2$.
5. $\text{dim}(T) = 4$.
6. False: the dimension is at least 4, since the rank of the coefficient matrix is at most 7.
7. 2 is the dimension of the solution space since the rank of $A$ must be $n - 2$.
8. The rank is $n + 1$ and the nullity is 0.
9. The nullity is 1 and the rank is $n$.
10. $2x_1 + x_2 + 2x_3 = 3$.
11. The inverse matrix is
    $\begin{pmatrix}
    1 & -x & xy - z \\
    0 & 1 & -y \\
    0 & 0 & 1
    \end{pmatrix}$
12. $b = c = 0$.
13. $\begin{pmatrix}
    0 & 1 & 0 \\
    0 & 0 & 2 \\
    0 & 0 & 0
    \end{pmatrix}$
14. Gram-Schmidt yields the orthonormal basis $\{1, \sqrt{3}(2x - 1)\}$.
15. $(AB)(AB)^t = ABB^tA^t = AIA^t = AA^t = I$, the identity transformation.
16. $\|v\|^2 - \langle v, u_1 \rangle^2 - \langle v, u_2 \rangle^2$
17. $D(A)D(A^{-1}) = D(AA^{-1}) = D(I) = 1$.
18. 6

Class Statistics

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| Test Avg | 78.9% | 77.9% | 86.3% | 77.2% | 81.7% |
| HW Avg   | 6.3   | 5.7   | 5.4   | 5.4   | 5.4   |
| HW/Test Correl | 0.65 | 0.69 | 0.61 | 0.62 | 0.62 |