Print Your Name Here: ____________________

Show all work in the space provided. *Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if it is to be graded*. No books or notes are allowed. A scientific calculator is OK—but not needed. The maximum total score is 200.

Part I: Short Questions. Answer **12** of the 18 short questions: 8 points each. **Circle** the **numbers** of the 12 questions that you want counted—*no more than 12*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 96 points.

1. True or False: In a field $\mathbb{F}$, $1+1 \neq 1$.

2. A, B and C are the vertices of a triangle. The vector from A to the midpoint of BC can be written as $\alpha\overrightarrow{AB} + \beta\overrightarrow{AC}$. Find α and β.

3. True or False: $\{f \mid f \in C(\mathbb{R}), \int_0^1 f(x)\,dx = 0\}$ is a subspace of the vector space $C(\mathbb{R})$ of continuous functions on $\mathbb{R}$.

4. Find the dimension of the vector space of all twice differentiable functions f on $\mathbb{R}$ such that $\frac{d^2 f}{dx^2} = 0$.

5. Let S and T be subspaces of a vector space V such that $\dim(S+T) = 5$, $\dim(S \cap T) = 2$ and $\dim(S) = 3$. Find $\dim(T)$.

6. True or False: For a homogeneous system of 7 linear equations in 11 unknowns, the dimension of the solution space is at most 4.

7. If A is an $n \times n$ real matrix and the solution set of the equation $A\mathbf{x} = \mathbf{b}$ is a 2-dimensional linear manifold in $\mathbb{R}^n$, find the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$.

8. Let T be the linear transformation of $\mathcal{P}_n(\mathbb{R})$ to itself given by $Tp(x) = \frac{d}{dx}(xp(x))$. Find the rank and the nullity of T.

9. Let S be the linear transformation of $\mathcal{P}_n(\mathbb{R})$ to itself given by $Tp(x) = x\frac{d}{dx}p(x)$. Find the rank and the nullity of S.

10. A linear manifold $L = \mathbf{b} + W \subset \mathbb{R}^3$ where $W^\perp = \{t(2,1,2) \mid t \in \mathbb{R}\}$ and $\mathbf{b} = (1,-1,1)$. Write the linear equation in x_1, x_2, x_3 that is satisfied if and only if $\mathbf{x} = (x_1, x_2, x_3) \in L$.

11. Find the inverse of the matrix $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$, where x, y, z are real numbers.

12. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. If $AB = BA$ for all values of α and β, find the values of b and c.

13. Let D be the linear transformation of $\mathcal{P}_2(\mathbb{R})$ to itself given by $Tp(x) = \frac{d}{dx}p(x)$. Find the matrix of T with respect to the basis $\{1, x, x^2\}$.

14. Use the Gram-Schmidt Process to find an orthonormal basis for the span of the two vectors $\{1, x\}$ in $C[0,1]$ with respect to the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$.

15. If A and B are orthogonal transformations of an inner product space, find the value of $(AB)(AB)^t$.

16. Let $v \in V$, an inner product space, and let $\{u_1, u_2\}$ be an orthonormal set in V. Express the value of $\langle v - \langle v, u_1\rangle u_1 - \langle v, u_2\rangle u_2\,,\; v - \langle v, u_1\rangle u_1 - \langle v, u_2\rangle u_2\rangle$ in terms of $\|v\|, \langle v, u_1\rangle, \langle v, u_2\rangle$.

17. For the invertible matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 2 & 1 \\ 4 & 5 & 3 \end{pmatrix}$, what is the value of $D(A)D\left(A^{-1}\right)$?

18. Find the determinant of the matrix $A = \begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 1 & 6 & 14 \end{pmatrix}$.

Part II: Proofs. *Prove carefully* **4** of the following 6 theorems for 26 points each. **Circle** the *letters* of the 4 proofs to be counted in the list below—*no more than 4!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 104 points.

A. Let $f_1, f_2, f_3 \in \mathcal{F}(\mathbb{R})$, the vector space of all real-valued functions on $\mathbb{R}$, and let $x_1, x_2, x_3 \in \mathbb{R}$. If the *rows* of the matrix $[f_i(x_j)]_{3\times 3}$ comprise a linearly independent set in $\mathbb{R}^3$, prove that $\{f_1, f_2, f_3\}$ is a linearly independent set in $\mathcal{F}(\mathbb{R})$,

B. A system of m linear equations in n unknowns has coefficient matrix $C = [\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n]$ in which $\mathbf{c}_j$ is the jth column. Prove that the homogeneous system $x_1\mathbf{c}_1 + \cdots + x_n\mathbf{c}_n = \mathbf{0}$ has a nontrivial solution if and only if $\operatorname{rank}(C) < n$.

C. Show that the equation $A\mathbf{x} = \mathbf{b}$, where A is an $n \times n$ matrix, has a unique solution if and only if A is an invertible matrix. If A is invertible, show that the solution of the equation is $\mathbf{x} = A^{-1}\mathbf{b}$.

D. Let V be a finite dimensional vector space and let S and T be in $\mathcal{L}(V, V)$ such that $S \circ T = I$, the identity transformation of V. Prove that $T \circ S = I$ and $T = S^{-1}$. (Make it clear how you use the finite dimensionality of V.)

E. Let $\{\mathbf{u}, , \ldots, \mathbf{u}_n\}$ be an orthonormal basis for a finite dimensional inner product space V.

- **(i)** If $\mathbf{v} = \sum_{i=1}^{n} a_i\mathbf{u}_i$ and $\mathbf{w} = \sum_{j=1}^{n} b_j\mathbf{u}_j$ then $\langle \mathbf{v}, \mathbf{w}\rangle = \sum_{i=1}^{n} a_i b_i$.
- **(ii)** Show that every vector $\mathbf{v} \in V$ can be expressed uniquely as $\mathbf{v} = \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{u}_i\rangle \mathbf{u}_i$.

F. Let $T \in \mathcal{L}(V, V)$ be an orthogonal transformation of a finite dimensional inner product space V.

- **(i)** Prove that $D(T) = \pm 1$.
- **(ii)** Explain why both 1 and -1 actually occur as determinants of suitable orthogonal transformations.

Solutions and Class Statistics

1. True, since $1 \neq 0$.

2. $\alpha = \frac{1}{2} = \beta$.

3. True: It is closed under vector addition and scalar multiplication.

4. The dimension is 2 since $f(x) = a_1 x + a_2$.

5. $\dim(T) = 4$.

6. False: the dimension is at least 4, since the rank of the coefficient matrix is at most 7.

7. 2 is the dimension of the solution space since the rank of A must be $n-2$.

8. The rank is $n+1$ and the nullity is 0.

9. The nullity is 1 and the rank is n.

10. $2x_1 + x_2 + 2x_3 = 3$.

11. The inverse matrix is $\begin{pmatrix} 1 & -x & xy - z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}$

12. $b = c = 0$.

13. $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

14. Gram-Schmidt yields the orthonormal basis $\{1, \sqrt{3}(2x-1)\}$.

15. $(AB)(AB)^t = ABB^tA^t = AIA^t = AA^t = I$, the identity transformation.

16. $\|v\|^2 - \langle v, u_1 \rangle^2 - \langle v, u_2 \rangle^2$

17. $D(A)D\left(A^{-1}\right) = D(AA^{-1}) = D(I) = 1$.

18. 6

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	7	9	13	5	11
80-89 (B)	6	4	8	9	9
70-79 (C)	9	9	1	4	6
60-69 (D)	3	3	2	8	1
0-59 (F)	3	4	3	2	1
Test Avg	78.9%	77.9%	85.3%	77.2%	81.7%
HW Avg	6.3	5.7	5.4	5.4	5.4
HW/Test Correl	0.65	0.69	0.61	0.62	0.62