Chapter 8

General Countably Additive Set Functions

In Theorem 5.2.2 the reader saw that if $f : X \to \mathbb{R}$ is integrable on the measure space $(X, \mathfrak{A}, \mu)$ then we can define a countably additive set function ν on $\mathfrak{A}$ by the formula

$$\nu(A) = \int_A f \, d\mu \tag{8.1}$$

and we see that it can take both positive and negative values. In this chapter we will study general countably additive set functions that can take both positive and negative values. Such set functions are known also as *signed measures*. In the *Radon-Nikodym* theorem we will characterize all those countably additive set functions that arise from integrals as in Equation 8.1 as being *absolutely continuous* with respect to the measure μ. And in the *Lebesgue Decomposition* theorem we will learn how to decompose any signed measure into its *absolutely continuous* and *singular* parts. These concepts for signed measures will be defined as part of the work of this chapter.

8.1 Hahn Decomposition Theorem

Definition 8.1.1. Given a σ-algebra $\mathfrak{A}$ of subsets of X, a function $\mu : \mathfrak{A} \to \mathbb{R}$ is called a *countably additive set function* (or a *signed measure*[1]) provided

[1] Some authors allow a signed measure to be *extended* real-valued. In that case it is necessary to require that μ take only one of the two values $\pm\infty$ in order to ensure that μ is well-defined on $\mathfrak{A}$.

that for every sequence of mutually disjoint sets $A_n \in \mathfrak{A}$ we have

$$\mu\left(\dot{\bigcup}_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n).$$

We prove first the following theorem.

Theorem 8.1.1. *If μ is a countably additive set function on a σ-field $\mathfrak{A}$, then μ is bounded on $\mathfrak{A}$. That is, there exists a real number M such that $|\mu(A)| \leq M$ for all $A \in \mathfrak{A}$.*

Proof. We begin by restating the theorem as follows, bearing in mind that μ can have *both* positive and negative values on $\mathfrak{A}$, and that consequently μ *need not be monotone.* Let

$$\mu_*(A) = \sup\{|\mu(B)| \mid B \subset A, B \in \mathfrak{A}\}$$

for each $A \in \mathfrak{A}$. Then the theorem claims that $\mu_*(X) < \infty$.

i. We claim that both $|\mu(A)|$ and $\mu_*(A)$ are sub-additive as functions of $A \in \mathfrak{A}$[2]. The first inequality follows immediately from the triangle inequality for real numbers, combined with the additivity of μ: If A and B are $\mathfrak{A}$-measurable and disjoint, then

$$|\mu(A \dot{\cup} B)| = |\mu(A) + \mu(B)| \leq |\mu(A)| + |\mu(B)|.$$

For the second inequality, we note that if $A = A_1 \dot{\cup} A_2$, a disjoint union, then $\mu_*(A) \leq \mu_*(A_1) + \mu_*(A_2)$ because if an $\mathfrak{A}$-measurable set $B \subset A$ then

$$|\mu(B)| \leq |\mu(B \cap A_1)| + |\mu(B \cap A_2)| \leq \mu_*(A_1) + \mu_*(A_2)$$

where we have used in the first inequality the sub-additivity of $|\mu|$.

ii. We will suppose that $\mu_*(X) = \infty$ and deduce a contradiction. By hypothesis, $\mu(X) \in \mathbb{R}$. So there exists a set $B \in \mathfrak{A}$ such that

$$|\mu(B)| > |\mu(X)| + 1 \geq 1.$$

[2] We do not denote $|\mu(A)|$ in the form $|\mu|(A)$ because the latter symbol will be given a special meaning in Definition 8.1.2.

By the additivity of μ,

$$|\mu(X \setminus B)| = |\mu(X) - \mu(B)| \geq |\mu(B)| - |\mu(X)| > 1.$$

Because B and $X \setminus B$ are disjoint, it follows from sub-additivity that either

$$\mu_*(B) = \infty \text{ or } \mu_*(X \setminus B) = \infty.$$

Thus there exists $B_1 \in \mathfrak{A}$ such that $|\mu(B_1)| > 1$ *and* $\mu_*(X \setminus B_1) = \infty$. Hence there exists $B_2 \in \mathfrak{A}$, disjoint from B_1, such that $|\mu(B_2)| > 1$ and $\mu_*(X \setminus (B_1 \cup B_2)) = \infty$. This process generates an infinite sequence of mutually disjoint sets $B_n \in \mathfrak{A}$ such that $|\mu(B_n)| > 1$ for each $n \in \mathbb{N}$. Let

$$\mathcal{B} = \dot{\bigcup}_{n \in \mathbb{N}} B_n$$

so that

$$\mu(\mathcal{B}) = \sum_{n \in \mathbb{N}} \mu(B_n) \tag{8.2}$$

and the latter series is conditionally convergent, meaning that it is convergent, but *not* absolutely convergent. Therefore, by a familiar exercise or theorem from Advanced Calculus[3], both the sum of the positive terms and the sum of the negative terms in Equation 8.2 must diverge. Hence there exists a subsequence B_{n_j} such that

$$\mu\left(\bigcup_{j \in \mathbb{N}} B_{n_j}\right) \notin \mathbb{R}$$

which is a contradiction.

□

We are ready now to state and prove the Hahn Decomposition theorem.

Theorem 8.1.2. (Hahn Decomposition) *Let μ be a countably additive set function on a σ-algebra $\mathfrak{A}$. Then there exists a partition $X = P \dot{\cup} N$ into disjoint sets with the following properties. If $A \in \mathfrak{A}$ then if $A \subseteq P$ we must have $\mu(A) \geq 0$, whereas if $A \subseteq N$ then $\mu(A) \leq 0$. This partition is* essentially unique, *in the sense that if $X = P' \cup N'$ is another such decomposition, then $|\mu|(P \triangle P') = 0$ and $|\mu|(N \triangle N') = 0$, where $|\mu|$ will be defined in Definition 8.1.2.*

[3] See for example [10]. There it is shown that if a series is conditionally convergent, then the sum of the positive terms diverges, and the sum of the negative terms diverges.

Proof. Let $\alpha = \sup\{\mu(A) \mid A \in \mathfrak{A}\}$. Then $0 \le \alpha < \infty$ by Theorem 8.1.1 and because $\mu(\emptyset) = 0$. For each $n \in \mathbb{N}$ there exists $A_n \in \mathfrak{A}$ such that $\mu(A_n) > \alpha - \frac{1}{2^n}$. Let

$$P = \liminf A_n = \bigcup_{p=1}^{\infty} \bigcap_{n=p}^{\infty} A_n$$

so that $P \in \mathfrak{A}$ and P is the set of all those $x \in X$ such that x is present in all but a finite number of the sets A_n. We will show that $\mu(P) = \alpha$. First we need the following lemma.

Lemma 8.1.1. *Under the hypotheses of Theorem 8.1.2, if* $\mu(B_1) > \alpha - \epsilon_1$ *and if* $\mu(B_2) > \alpha - \epsilon_2$ *then*

$$\mu(B_1 \cap B_2) > \alpha - (\epsilon_1 + \epsilon_2).$$

Proof. Because μ is (countably) additive,

$$\mu(B_1 \cap B_2) = \mu(B_1) + \mu(B_2) - \mu(B_1 \cup B_2) > (\alpha - \epsilon_1) + (\alpha - \epsilon_2) - \alpha$$

$$= \alpha - (\epsilon_1 + \epsilon_2)$$

since $\alpha = \sup\{\mu(A) \mid A \in \mathfrak{A}\}$. □

It follows for each $q \ge p$ that

$$\mu\left(\bigcap_{n=p}^{q} A_n\right) > \alpha - \sum_{n=p}^{q} \frac{1}{2^n} \ge \alpha - \frac{1}{2^{p-1}}$$

for all $q \ge p$, from which we deduce[4] that

$$\mu\left(\bigcap_{p}^{\infty} A_n\right) \ge \alpha - \frac{1}{2^{p-1}}.$$

[4] The reader should take note that μ need not be monotone, being a signed measure. We use here the boundedness of μ together with its countable additivity to show that

$$\mu\left(\bigcap_{n=p}^{q} A_n\right) \to \mu\left(\bigcap_{n=p}^{\infty} A_n\right)$$

as $q \to \infty$.

In a similar manner one can show that

$$\mu\left(\bigcup_{p=1}^{\infty}\left[\bigcap_{n=p}^{\infty} A_n\right]\right) \geq \lim_{p\to\infty}\left(\alpha - \frac{1}{2^{p-1}}\right) = \alpha.$$

It follows that $\alpha \leq \mu(P) \leq \alpha$, which implies that $\mu(P) = \alpha$ as claimed.

Moreover, if there were a set $A \in \mathfrak{A}$ such that $A \subseteq P$ and $\mu(A) < 0$, then we would have

$$\mu(P \setminus A) = \mu(P) - \mu(A) > \mu(P)$$

which is a contradiction. It follows that if $A \subseteq P$ then $\mu(A) \geq 0$. Now let $N = X \setminus P$. Suppose there were a measurable set $A \subseteq N$ such that $\mu(A) > 0$. Then it would follow that $\mu(P \cup A) > \mu(P)$, which is impossible. Hence if $A \in \mathfrak{A}$ and $A \subseteq N$ it follows that $\mu(A) \leq 0$. We leave the proof of essential uniqueness to Exercise 8.1.1. □

Definition 8.1.2. Let μ be a countably additive set function on a σ-algebra $\mathfrak{A}$, and let P and N be (for μ) as in Theorem 8.1.2. Define the *positive part*, the *negative part*, and the *variation* of μ as follows:

$$\begin{aligned}\mu^+(A) &= \mu(A \cap P)\\ \mu^-(A) &= |\mu(A \cap N)|\\ |\mu|(A) &= \mu^+(A) + \mu^-(A)\end{aligned}$$

The number $|\mu|(X) = \|\mu\|$ is called the *total variation norm* of μ.

Exercise 8.1.1. Suppose that we have two Hahn decompositions as in Theorem 8.1.2: $X = P \cup N = P' \cup N'$. Prove that $|\mu|(P \vartriangle P') = 0 = |\mu|(N \vartriangle N')$.

Exercise 8.1.2. *(Jordan Decomposition Theorem.)* Prove that μ^+, μ^-, and $|\mu|$ are countably additive non-negative measures. Prove also the decomposition

$$\mu = \mu^+ - \mu^-$$

and that this *decomposition is minimal* in the following sense. If μ_1 and μ_2 are measures such that $\mu = \mu_1 - \mu_2$, then $\mu^+ \leq \mu_1$ and $\mu^- \leq \mu_2$.

Exercise 8.1.3. † Prove that the total variation norm satisfies all the requirements to be a norm on the vector space $\mathcal{M}$ of all countably additive set functions on $(X, \mathfrak{A}, \mu)$.

Exercise 8.1.4. Prove that $\mathcal{M}$ is complete in the total variation norm.

8.2 Radon-Nikodym Theorem

Definition 8.2.1. If λ and μ are *measures* on a σ-algebra $\mathfrak{A}$ of subsets of X, we call λ *absolutely continuous* with respect to μ, written as $\lambda \prec \mu$, if and only if $A \in \mathfrak{A}$ and $\mu(A) = 0$ implies $\lambda(A) = 0$.

We have a similar definition for countably additive set functions (signed measures).

If a non-negative function f is in $L^1(X, \mathfrak{A}, \mu)$, and if we define

$$\lambda(E) = \int_E f \, d\mu$$

for each $E \in \mathfrak{A}$, then λ will be absolutely continuous with respect to μ, written $\lambda \prec \mu$, as in the foregoing definition.

Definition 8.2.2. If λ and μ are *countably additive set functions* on $\mathfrak{A}$, we call λ absolutely continuous with respect to μ, written as $\lambda \prec \mu$, if and only if $\lambda(E) = 0$ for each $E \in \mathfrak{A}$ such that $|\mu|(E) = 0$.

Exercise 8.2.1. † Let λ and μ be countably additive set functions. Prove that the following three statements are equivalent.

i. $\lambda \prec \mu$

ii. $\lambda^+ \prec \mu$ and $\lambda^- \prec \mu$

iii. $|\lambda| \prec |\mu|$

Theorem 8.2.1. (Radon-Nikodym) *Suppose λ and μ are finite (non-negative)* measures *on a σ-algebra $\mathfrak{A}$ of subsets of X. Then we have the following conclusions.*

i. There exists a non-negative function f in $L^1(X, \mathfrak{A}, \mu)$ such that for each $A \in \mathfrak{A}$ we have

$$\lambda(A) = \int_A f \, d\mu$$

if and only if $\lambda \prec \mu$.

ii. Moreover, the $L^1(X, \mathfrak{A}, \mu)$-equivalence class of a Radon-Nikodym derivative is uniquely determined.

Remark 8.2.1. For f as in the Radon-Nikodym theorem, it is common to call f the Radon-Nikodym derivative, and to denote this as

$$f = \frac{d\lambda}{d\mu}.$$

This notation yields the formula

$$\lambda(A) = \int_A 1\, d\lambda = \int_A \frac{d\lambda}{d\mu}\, d\mu$$

which suggests a change of variables formula, and a chain rule. See Exercises 8.2.6 and 8.2.3.

Proof of Theorem. The implication from left to right is inherent in the fourth conclusion of Theorem 5.2.2. So we will suppose here that $\lambda \prec \mu$ and give a proof from right to left. We begin with a lemma.

Lemma 8.2.1. *Under the hypotheses of the Radon-Nikodym theorem, if $\lambda \prec \mu$, and if λ is not the identically zero measure, then there exists $\epsilon > 0$ and there exists $P \in \mathfrak{A}$ with $\mu(P) > 0$ such that if $A \in \mathfrak{A}$ and $A \subset P$ we have $\lambda(A) \geq \epsilon\mu(A)$.*

In the context of this proof, we will write the conclusion of the lemma as an inequality as follows:

$$\lambda \overset{P}{>} \epsilon\mu.$$

Proof of Lemma. Since $\lambda \prec \mu$ and λ is not identically zero, neither is μ identically zero. We claim that there exists sufficiently small $\epsilon > 0$ such that

$$(\lambda - \epsilon\mu)^+ \neq 0.$$

Suppose this were false. Then we would have $\lambda(A) \leq \epsilon\mu(A)$ for all $A \in \mathfrak{A}$ and for all $\epsilon > 0$. But this would force $\lambda = 0$ which is a contradiction. Thus there exists ϵ such that $(\lambda - \epsilon\mu)^+ > 0$. Hence there is a Hahn decomposition $X = P\dot{\cup}N$ for the signed measure $(\lambda - \epsilon\mu)$ such that if a measurable set $A \subset P$ implies that $(\lambda - \epsilon\mu)(A) > 0$, showing that

$$\lambda \overset{P}{>} \epsilon\mu.$$

□

If $f \in L^+(\mu)$ we define $\lambda_f(A) = \int_A f\, d\mu$ and we let

$$L^+(\mu, \lambda) = \left\{ f \in L^+(\mu) \mid \lambda_f(A) \leq \lambda(A)\ \forall A \in \mathfrak{A} \right\}$$

noting that the inequalities that define $L^+(\mu, \lambda)$ apply to all $A \in \mathfrak{A}$ and are not limited to the subsets of P. By Lemma 8.2.1 we know that there exist $\epsilon > 0$ and $P \in \mathfrak{A}$ with $\mu(P) > 0$ and such that $\epsilon 1_P \in L^+(\mu, \lambda)$, which therefore has a non-trivial element if λ is not identically zero. If λ were zero, still the set $L^+(\mu, \lambda)$ would be non-empty since it would contain the zero function.

Let

$$\alpha = \sup\{\lambda_f(X) \mid f \in L^+(\mu, \lambda)\}$$

so that $\alpha \leq \lambda(X) < \infty$. Thus for each $n \in \mathbb{N}$ there exists $f_n \in L^+(\mu, \lambda)$ such that $\lambda_{f_n}(X) > \alpha - \frac{1}{n}$. Let

$$g_n = \max(f_1, \ldots, f_n) = f_1 \vee \ldots \vee f_n.$$

Then g_n is a monotone increasing sequence of measurable functions, and $g_n \in L^+(\mu, \lambda)$ because this is true for each function f_j. Moreover, $\lambda_{g_n} \leq \lambda$ and $\lambda_{g_n}(X) > \alpha - \frac{1}{n}$.

We can define $g = \lim_n g_n$, which is defined almost everywhere, and $\lambda_g \leq \lambda$. We know also that $\lambda_g(X) = \alpha$. It will suffice to prove that $\lambda_g = \lambda$. We will suppose the latter equation is false and deduce a contradiction.

Suppose that $\lambda - \lambda_g > 0$, which means that $\lambda - \lambda_g$ is non-negative and not the identically zero measure. Let $\lambda^* = \lambda - \lambda_g$. Then $\lambda^* > 0$ and $\lambda^* \prec \mu$. By Lemma 8.2.1 we conclude that there exists $\epsilon > 0$ and $P' \in \mathfrak{A}$ such that $\mu(P') > 0$ and such that $A \in \mathfrak{A}$ and $A \subset P'$ implies that

$$\lambda(A) - \lambda_g(A) = \lambda^*(A) \geq \epsilon\mu(A).$$

Let $h = g + \epsilon 1_{P'}$. Then $\int_A h\, d\mu = \int_A g\, d\mu + \epsilon\mu(A)$ for each $A \in \mathfrak{A}$ such that $A \subset P'$. That is

$$\lambda \geq \lambda_g + \lambda_{\epsilon 1_{P'}}.$$

Hence $\int_X h\, d\mu > \alpha$, which is impossible since $h \in L^+(\mu, \lambda)$.

The uniqueness of the Radon-Nikodym derivative up to L^1-equivalence is shown in Exercise 8.2.2. □

Remark 8.2.2. We remark that if $f \in L^1(X, \mathfrak{A}, \mu)$ for some measure μ, then the carrier of f must be σ-finite. Thus in order to characterize those measures expressible in the form $\lambda(A) = \int_A f\,d\mu$ it would be appropriate to limit our attention to σ-finite measure spaces $(X, \mathfrak{A}, \mu)$. It is easy to extend the Radon-Nikodym theorem to the case in which μ is a σ-finite measure and λ is a finite measure.

It is simple also to give an extension of the Radon-Nikodym theorem to signed measures because each signed measure is the difference between two positive measures and so the Radon-Nikodym derivative in this more general context is the difference between two Radon-Nikodym derivatives for positive measures.

Exercise 8.2.2. † Show that $\frac{d\lambda}{d\mu}$, the Radon-Nikodym derivative of λ with respect to μ in Theorem 8.2.1, is uniquely determined as an element of $L^1(X, \mathfrak{A}, \mu)$.

Exercise 8.2.3. † Suppose that the measures λ, μ, ν on a σ-field $\mathfrak{A} \subset \mathfrak{P}(X)$ have the relationship

$$\lambda \prec \mu \prec \nu$$

where λ and μ are finite and ν is σ-finite. Prove that $\lambda \prec \nu$ and that

$$\frac{d\lambda}{d\nu} = \frac{d\lambda}{d\mu}\frac{d\mu}{d\nu}$$

by means of the following steps.

a. Let
$$f = \frac{d\lambda}{d\mu},\ g = \frac{d\mu}{d\nu},\ h = \frac{d\lambda}{d\nu}$$
and show that there is a sequence $f_n \in \mathfrak{S}_0$ such that $f_n \nearrow f$ pointwise almost everywhere.

b. Show that
$$\left|\lambda(A) - \int_A f_n\,d\mu\right| \to 0$$
for all $A \in \mathfrak{A}$ as $n \to \infty$.

c. Show that
$$\left|\int_A f_n\,d\mu - \int_A f_n\,g\,d\nu\right| \to 0$$
as $n \to \infty$ for all $A \in \mathfrak{A}$.

d. Use Exercise 8.2.2 to complete the proof that $f_n g_n \to h$.

Exercise 8.2.4. Let l denote Lebesgue measure on the unit interval, and let ϕ be the Cantor function from Example 7.4.1. Define $\lambda = l \circ \phi$ by $\lambda(A) = l(\phi(A))$ for each measurable set $A \subset [0, 1]$. Is it true that $\lambda \prec l$? Prove your conclusion.

Exercise 8.2.5. Let ϕ be a continuously differentiable monotone increasing function defined on $[a, b] \subset \mathbb{R}$. Define a measure λ on the Lebesgue measurable sets of $[a, b]$ by $\lambda(A) = l(\phi(A))$. Prove that $\lambda \prec l$ and find $\frac{d\lambda}{dl}$.

Exercise 8.2.6. Suppose $(X, \mathfrak{A}, \mu)$ is a complete measure space and

$$f \in L^1(X, \mathfrak{A}, \mu).$$

Suppose $\phi : X \to X$ is a bijection for which $\phi(E) \in \mathfrak{A}$ if and only if $E \in \mathfrak{A}$, and suppose ϕ maps Lebesgue null sets to Lebesgue null sets. Define the measure

$$\mu \circ \phi(E) = \mu(\phi(E)).$$

Prove that $\mu \circ \phi \prec \mu$ and the change of variables formula

$$\int_E (f \circ \phi) \frac{d(\mu \circ \phi)}{d\mu} \, d\mu = \int_{\phi(E)} f \, d\mu.$$

Exercise 8.2.7. Suppose μ is a σ-finite *measure* on a σ-algebra $\mathfrak{A}$ of subsets of X. Suppose λ is another measure on $\mathfrak{A}$ such that $\lambda \prec \mu$. Prove that there exists a non-negative μ-measurable function f on X such that $\lambda(A) = \int_A f \, d\mu$ for all $A \in \mathfrak{A}$. Prove that λ is a finite measure if and only if $f \in L^1(X, \mathfrak{A}, \mu)$.

Exercise 8.2.8. Suppose μ is a σ-finite *measure* on a σ-algebra $\mathfrak{A}$ of subsets of X. Suppose λ is a *signed real*-valued measure on $\mathfrak{A}$ such that $\lambda \prec \mu$. Prove that there exists a (signed) $f \in L^1(X, \mathfrak{A}, \mu)$ such that $\lambda(A) = \int_A f \, d\mu$ for all $A \in \mathfrak{A}$.

Exercise 8.2.9. It is interesting to consider the relationship between the concept of absolute continuity of functions given in Definition 7.4.1 and that of absolute continuity of measures.

a. If λ and μ are any two finite measures on a σ-field $\mathfrak{A} \subset \mathfrak{P}(X)$, prove that $\lambda \prec \mu$ if and only if they satisfy the following condition: for each $\epsilon > 0$ there exists a $\delta > 0$ such that $\mu(A) < \delta$ implies that $\lambda(A) < \epsilon$. (Hint: For one direction, use the Radon-Nikodym theorem.)

b. Suppose now that the finite measure λ is defined on the Lebesgue measurable sets of $([a,b), \mathfrak{L})$. Define $f(x) = \lambda[a,x)$ for all $x \in [a,b]$. Prove that f is an absolutely continuous function on $[a,b]$ if and only if $\lambda \prec l$. (Hint: From right to left is easy by part (a). For the other direction, prove that $\lambda(A) = \int_A f' \, dl$ for all $A \in \mathfrak{L}$.)

8.3 Lebesgue Decomposition Theorem

The Radon-Nikodym Theorem addressed the classification of measures absolutely continuous with respect to a given measure. Here we study a quite different (symmetrical) relationship of singularity between two measures or countably additive set functions.

Definition 8.3.1. If λ and μ are *countably additive set functions* on $\mathfrak{A}$, we call λ *singular* with respect to μ if and only if $X = E \dot{\cup} F$, a disjoint union of $\mathfrak{A}$-measurable sets, such that $|\lambda|(E) = 0 = |\mu|(F)$. This is denoted as

$$\lambda \perp \mu.$$

Theorem 8.3.1. *Let $\mathfrak{A}$ be a σ-algebra of subsets of X. Let μ and ν be two signed measures. Then there exist two* unique *signed measures ν_0 and ν_1 such that*

$$\nu = \nu_0 + \nu_1$$

with the properties that $\nu_0 \perp \mu$ and $\nu_1 \prec \mu$.

Proof. It follows from Definitions 8.2.2 and 8.3.1 that singularity or absolute continuity with respect to a signed measure μ means singularity or absolute continuity with respect to $|\mu|$. Thus we can assume without loss of generality that μ is a measure. Because of Exercise 8.2.1 and Definition 8.3.1 we can assume without loss of generality that ν is a measure as well[5].

The proof of the theorem is based upon the observation that $\nu \prec (\mu + \nu)$. Thus there exists a non-negative measurable function f such that

$$\nu(E) = \int_E f \, d\mu + \int_E f \, d\nu$$

[5]By Definition 8.1.1 our assumption implies that μ and ν are finite measures. For measures that are not signed, the present theorem can be generalized readily to the σ-finite case. See Exercise 8.3.4.

for all $E \in \mathfrak{A}$. Since

$$0 \leq \nu(E) \leq \mu(E) + \nu(E)$$

we have $0 \leq f \leq 1$ ν-almost everywhere and also $(\mu+\nu)$-almost everywhere. Let $A = f^{-1}(1)$ and $B = f^{-1}[0,1)$. Thus

$$\nu(A) = \mu(A) + \nu(A).$$

It follows that $\mu(A) = 0$. Define $\nu_0(E) = \nu(A \cap E)$ and $\nu_1(E) = \nu(E \cap B)$. Thus $\nu_0 \perp \mu$ because ν_0 vanishes on subsets of A^c and $\mu(A) = 0$.

Suppose next that $\mu(E) = 0$. Then

$$\nu(E \cap B) = \int_{E \cap B} 1 \, d\nu = \int_{E \cap B} f \, d(\mu + \nu) = \int_{E \cap B} f \, d\nu$$

since $\mu(E) = 0$ by hypothesis. This implies that

$$\int_{E \cap B} (1 - f) \, d\nu = 0.$$

Since $1 - f \geq 0$ ν-almost everywhere, with strict inequality on B, it follows that $\nu_1(E) = \nu(E \cap B) = 0$, so that $\nu_1 \prec \mu$.

Finally, we prove uniqueness. Let

$$\nu = \nu_0 + \nu_1 = \bar{\nu}_0 + \bar{\nu}_1 \tag{8.3}$$

be two Lebesgue decompositions. Thus $\nu_0 - \bar{\nu}_0 = \bar{\nu}_1 - \nu_1$ with one side singular and the other side absolutely continuous with respect to μ. (See Exercise 8.3.1.) This forces both sides to be zero, which completes the proof. (See Exercise 8.3.2.) □

Exercise 8.3.1. Suppose that the sum of two measures that are absolutely continuous with respect to μ on $(X, \mathfrak{A}, \mu)$ must be absolutely continuous. Prove also that the sum of two measures that are singular with respect to μ on $(X, \mathfrak{A}, \mu)$ must be singular.

Exercise 8.3.2. Let μ and ν be non-negative finite measures on $(X, \mathfrak{A})$. If $\nu \perp \mu$ and $\nu \prec \mu$, prove that $\nu = 0$, the identically zero measure on $\mathfrak{A}$.

Exercise 8.3.3. This exercise continues the work begun in Exercise 8.2.9. Let f be a monotone increasing function on $[a, b]$ and define a measure μ by letting it assign to an interval $[a, x)$ the measure $\mu[a, x) = f(x) - f(a)$.

a. Let μ_1 be the absolutely continuous part of μ with respect to Lebesgue measure, and find the Radon-Nikodym derivative

$$\frac{d\mu_1}{dl}.$$

b. Show that the singular part μ_0 and the absolutely continuous part μ_1 of μ_f can be used to define absolutely continuous and singular parts of the function f.

Exercise 8.3.4. Let ν be any σ-finite measure on the measure space $(X, \mathfrak{A}, \mu)$, where μ is σ-finite. Prove that there exist two *unique* measures ν_0 and ν_1 such that

$$\nu = \nu_0 + \nu_1$$

with the properties that $\nu_0 \perp \mu$ and $\nu_1 \prec \mu$.