7.4 Absolutely Continuous & Singular Functions

The concept of *absolute continuity* for a real-valued function of a real variable is particularly important when studying the various forms of the Fundamental Theorem of Calculus for the Lebesgue integral. We present the definition of this concept below, following a review of two more elementary concepts of continuity.

Definition 7.4.1. Let $f : [a, b] \to \mathbb{R}$. Then we have the following definitions regarding f.

i. f is *continuous* at $x_0 \in [a, b]$ if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that $x \in [a, b]$ and $|x - x_0| < \delta$ implies that $|f(x) - f(x_0)| < \epsilon$.

ii. f is *uniformly continuous on* $[a, b]$ if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that x and y in $[a, b]$ and $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$.

iii. f is *absolutely continuous on* $[a, b]$ if and only if for each $\epsilon > 0$ there exists a $\delta > 0$ such that for each $n \in \mathbb{N}$

$$a \le x_1 < y_1 \le x_2 < y_2 \le \ldots \le x_n < y_n \le b \text{ with } \sum_1^n (y_i - x_i) < \delta$$

implies that

$$\sum_1^n |f(y_i) - f(x_i)| < \epsilon.$$

The reader should take note that continuity at a point is a *local* concept, and the $\delta > 0$ that works in collaboration with a given $\epsilon > 0$ may depend upon where in $[a, b]$ the point x_0 is located. Uniform continuity requires that there exist a suitable δ corresponding to ϵ, regardless of where in $[a, b]$ the points x and y are located, provided they are within δ of one another. Absolute continuity demands even more, because $\delta > 0$ is required to be *independent* of both the location within $[a, b]$ of the $2n$ points $x_1, y_1, \ldots, x_n, y_n$ and also the number $n \in \mathbb{N}$ provided only that $\sum_1^n |y_i - x_i| < \delta$.

If we denote $E = \bigcup_1^n [x_i, y_i]$ in the definition of absolute continuity, then E is easily Lebesgue measurable and we are requiring that $l(E) < \delta$ imply $\sum_1^n |f(y_i) - f(x_i)| < \epsilon$. Thus the absolute continuity of f is commonly denoted as $f \prec l$ which is read as f *is absolutely continuous with respect to Lebesgue measure.*

Exercise 7.4.1. Let

$$f(x) = \begin{cases} x^n \sin \frac{2\pi}{x} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}$$

where $n \in \mathbb{N}$. Prove the following conclusions.

a. f is continuous at each point of $[0,1]$.

b. f is uniformly continuous on $[0,1]$.

c. f is *not* absolutely continuous on $[0,1]$ if $n = 1$ but f is absolutely continuous provided $n > 1$. (Hint: Compare with Exercise 7.1.4.)

Exercise 7.4.2. Show that the product of two absolutely continuous functions on a closed finite interval $[a,b]$ is absolutely continuous.

Definition 7.4.2. A continuous monotone function f is said to be *singular* with respect to Lebesgue measure (written $f \perp l$) provided that f is *non-constant* yet $f'(x) = 0$ almost everywhere.

Example 7.4.1. We will construct a singular function f, called the *Cantor function*, on $[0,1]$. Each number $x \in [0,1]$ can be expressed in a *ternary expansion*:

$$x = \sum_0^\infty \frac{a_n}{3^n} = a_0.a_1a_2\ldots a_n\ldots \tag{7.3}$$

where each coefficient $a_n \in \{0,1,2\}$. The coefficients a_n are not unique without some further restriction. For example, if we allow infinite tails of 2's and also allow 1's, this would render ternary expansions of x in a non-unique manner, since

$$\sum_p^\infty \frac{2}{3^n} = \frac{1}{3^{p-1}}.$$

For example, if $a_0 = 0$ and if $a_n = 2$ for all $n \geq 1$ then $x = 1$ and we could have used $a_0 = 1$ and $a_n = 0$ for all $n \geq 1$. The *Cantor set*, $\mathcal{C}$, defined in

Exercise 3.3.2, can be described arithmetically by prohibiting the use of the ternary digit 1, but allowing infinite tails of 2's. The effect is the removal of open middle thirds that results in the Cantor set. Thus

$$\mathcal{C} = \{x \in [0,1] \mid \forall n, a_n \neq 1\}.$$

The complement of the Cantor set can be pictured as follows. Delete from $[0,1]$ the open middle-third, which is $\left(\frac{1}{3}, \frac{2}{3}\right)$. This deletion eliminates all x with ternary expansions having $a_1 = 1$ and leaves two closed intervals of length $\frac{1}{3}$ each. Delete the open middle-third from each of the two remaining pieces, which eliminates all x for which the ternary expansion has $a_2 = 1$. Continue an infinite sequence of such deletions of open middle-thirds.

Let each $x \in [0,1]$ be expressed as in Equation 7.3. We will define the *Cantor function* f first *on* the Cantor set $\mathcal{C}$ by the following equation. If x is expanded in a ternary manner as in Equation 7.3, we define $f(x)$ by means of the *binary expansion*

$$f(x) = \frac{1}{2} \sum_{0}^{\infty} \frac{a_n}{2^n} \quad \forall x \in \mathcal{C}$$

One can see from this definition that f is monotone increasing on $\mathcal{C}$. On each of the missing open middle-thirds, we define f to be locally constant. In fact, if the missing open middle third (a_N, b_N) is defined by the requirement $a_N \neq 1$ on the N^{th} digit, one can show that $f(a_N) = f(b_N)$. For example, on the first deleted middle-third f will be constantly equal to $\frac{1}{2}$, and on the next two deleted thirds f will be $\frac{1}{4}$ and $\frac{3}{4}$ respectively.

A computer rendering of the Cantor function is shown in Figure 7.5. The computer was set to connect the plotted points. This is appropriate in the sense that the reader will prove in Exercise 7.4.3 that the Cantor function is continuous. However, the picture can be misleading as well, since it appears as though there were places on the graph with a slope different from zero. Actually, the derivative is zero wherever it is defined. If the picture were perfect, and if one could magnify it to an arbitrary degree, the seemingly upward-sloped parts of the graph would look just like the large-scale features, consisting of horizontal segments except on the null-set that is the Cantor set.

Exercise 7.4.3. † Show that the Cantor function defined in Example 7.4.1 maps the interval $[0,1]$ continuously onto itself and is a monotone increasing

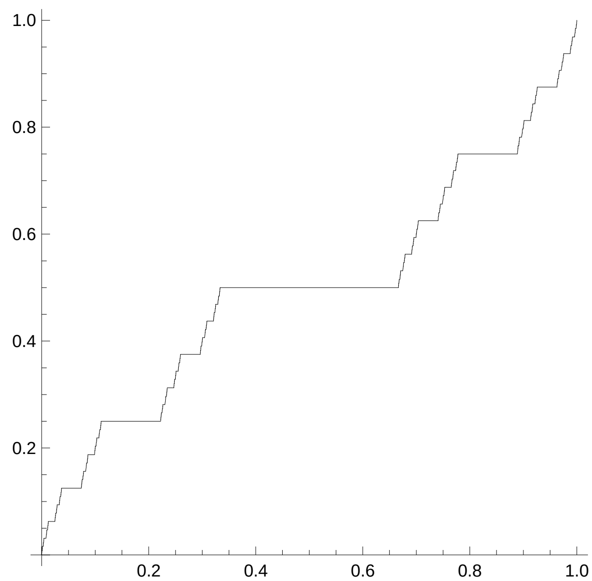

Figure 7.5: Cantor function.

function for which $f'(x)$ exists and equals zero almost everywhere, and such that $f(0) = 0$ and $f(1) = 1$.

Exercise 7.4.4. Let f be the Cantor function and define $\phi(x) = f(x) + x$ for all $x \in [0, 1]$. Let C denote the middle-thirds Cantor set.

a. Prove that $\phi : [0, 1] \to [0, 2]$ is a homeomorphism. That is, prove that ϕ is injective, surjective, and bi-continuous.

b. Prove that $l(\phi([0, 2] \setminus C)) = 1$ and that $l(\phi(C)) = 1$.

c. Let P be any non-measurable subset of $\phi(C)$. (See Exercise 3.4.4 for the existence of P.) Prove that $\phi^{-1}(P)$ is a Lebesgue measurable set but not a Borel set.

Exercise 7.4.5.

a. Provide an example of a function on $[0, 1]$ that is not absolutely continuous but is of bounded variation.

b. Provide examples of two different continuous functions on $[0, 1]$ that have the same derivative *a.e.* and that are both equal to zero at 0.

Theorem 7.4.1. *Let f be a monotone increasing real-valued function on $[a,b]$. Then f' exists almost everywhere on $[a,b]$, and*

i. $f(x) - f(a) \geq \int_a^x f'(t)\,dl(t)$ *for all* $x \in [a,b]$

ii. Equality holds in the inequality above if and only if f is absolutely continuous.

Proof. The existence of f' almost everywhere follows from Theorem 7.3.1. We need to prove the two parts concerning the inequality.

i. For almost all t,

$$f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}.$$

Thus we can pick a sequence $h_n \to 0+$, and for each t at which $f'(t)$ exists, we have

$$f'(t) = \lim_{n\to\infty} \frac{f(t+h_n) - f(t)}{h_n}.$$

It follows that f' is equal almost everywhere to the limit of a sequence of measurable functions, which implies that f' is measurable.

Note that the function

$$g_n(t) = \frac{f(t+h_n) - f(t)}{h_n}$$

is non-negative. Next we apply Fatou's theorem (5.4.3) as follows. For each $[c,d] \subset (a,b)$ we have

$$\int_c^d f'(t)\,dl(t) = \int_c^d \lim_{n\to\infty} \frac{f(t+h_n) - f(t)}{h_n}\,dl(t)$$

$$\leq \liminf_{n\to\infty} \int_c^d \frac{f(t+h_n) - f(t)}{h_n}\,dl(t)$$

$$= \lim_{n\to\infty} \frac{1}{h_n}\left(\int_d^{d+h_n} f(t)\,dl(t) - \int_c^{c+h_n} f(t)\,dl(t)\right) = f(d) - f(c)$$

for almost all c and d since f is differentiable (and hence continuous) almost everywhere. In the theorem, $\lim_{c\to a+} f(c) \geq f(a)$ because f is monotone increasing. If f is not continuous at a the inequality is true a fortiori.

ii. Suppose first that equality holds in the inequality of part (i). The reader will prove that f is absolutely continuous in Exercise 7.4.6.

Suppose for the opposite direction of implication that $f \prec l$, meaning that f is absolutely continuous with respect to Lebesgue measure. We need to prove that equality holds in Theorem 7.4.1. Define

$$g(x) = \int_a^x f'(t)\, dl(t)$$

so that $g \prec l$ also. Now let $h = f - g$, and we have

$$h'(x) = f'(x) - g'(x) = 0$$

almost everywhere. Also, it is easy to check that

$$h = f - g \prec l$$

as well. The key to the proof is to show now that an absolutely continuous function with derivative equal to zero almost everywhere must be constant, and in the present case, $f(a)$. Note that because $h' = 0$ almost everywhere, h would be singular if it were not constant. Thus we are about to show that an absolutely continuous function with zero derivative almost everywhere cannot be singular.

Since $h \prec l$, if $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum_1^n (b_k - a_k) < \delta \implies \sum_1^n |h(b_k) - h(a_k)| < \frac{\epsilon}{2}.$$

We *claim* that h must be constant. Let

$$\mathcal{I} = \left\{ I = [c,d] \subseteq [a,b] \,\middle|\, h(I) = h(d) - h(c) < \frac{\epsilon}{2(b-a)} |I| \right\}.$$

Then $\mathcal{I}$ covers $E = \{x \in [a,b] \mid h'(x) = 0\}$ in the sense of Vitali. Hence there exists a *disjoint* sequence $I_n \in \mathcal{I}$ such that

$$E \overset{\circ}{\subseteq} \bigcup_1^\infty I_n.$$

Thus

$$\sum_1^\infty h(I_n) < \frac{\epsilon}{2(b-a)} \sum_1^\infty |I_n| < \frac{\epsilon}{2}.$$

Since $[a,b] \setminus \bigcup_1^\infty I_n$ is a null-set, there exists a disjoint union of closed intervals J_n such that

$$[a,b] \setminus \bigcup_1^\infty I_n \subseteq \bigcup_1^\infty J_n$$

and such that $\sum_1^\infty |J_n| < \delta$. Because $h \prec l$ we know that

$$\sum_1^\infty h(J_n) \leq \frac{\epsilon}{2}.$$

Because h is monotone increasing,

$$h(b) - h(a) \leq \sum_1^\infty (h(I_n) + h(J_n)) < \epsilon$$

for all $\epsilon > 0$. Thus h must be a constant function. But then

$$f(x) - g(x) = h(x) \equiv h(a) = f(a) - g(a)$$

$$= f(a) - \int_a^a f'(t)\, dl(t) = f(a).$$

Hence

$$f(x) - f(a) = g(x) = \int_a^x f'(t)\, dl(t).$$

□

Exercise 7.4.6. **†** Let $f \in L^1(\mathbb{R})$.

a. If $\epsilon > 0$ prove that there exists $\delta > 0$ such that if E is Lebesgue measurable and if $l(E) < \delta$ then $\int_E f\, dl < \epsilon$. (Hint: Use Definition 5.2.3.)

b. Prove that equality holds in Theorem 7.4.1 then f is absolutely continuous.

Exercise 7.4.7. Prove that, if f is absolutely continuous on $[0,1]$, then the total variation of f on $[0,1]$ is equal to $\int_0^1 |f'|\, dl$.

Definition 7.4.3. A real-valued function f on a measure space $(X, \mathfrak{A}, \mu)$ is called *essentially bounded* if and only if there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for almost all x.

We denote the set of all essentially bounded functions as

$$L^\infty(X, \mathfrak{A}, \mu)$$

and we define the essential supremum of f by

$$\|f\|_\infty = \inf\{M \mid |f| \leq M\, ae\}.$$

Exercise 7.4.8. A real-valued function f on an interval I for which there exists a constant C such that

$$|f(x) - f(y)| \leq C|x - y|$$

for all x and y in I is called a *Lipschitz function.*

a. Show that a Lipschitz function is absolutely continuous.

b. Show that an absolutely continuous function f on an interval is Lipschitz if and only if f' is essentially bounded.

c. Give an example of a Lipschitz function that does not satisfy the Mean Value Theorem for derivatives.

Exercise 7.4.9.

a. Provide an example of a function of unbounded variation on $[0, 1]$ that has a derivative equal to zero at almost all $x \in [0, 1]$.

b. Provide an example of a function that is absolutely continuous on $[0, 1]$ but has an unbounded derivative.

We know already that if $f \in L^1[a, b]$ and if $F(x) = \int_a^x f(t)\, dl(t)$ then $F'(x)$ exists and $F'(x) = f(x)$ almost everywhere. Thus

$$\lim_{h \to 0} \int_x^{x+h} \frac{f(t) - f(x)}{h}\, dl(t) = 0$$

for almost all x. We have the following stronger theorem.

Theorem 7.4.2. (Lebesgue) *Let $f \in L^1[a,b]$. Then*

$$\lim_{h\to 0} \int_x^{x+h} \frac{|f(t)-f(x)|}{h}\, dl(t) = 0$$

for almost all x.

Proof. Suppose α is a given constant. We define the set N_α to be the set such that $x \in [a,b] \setminus N_\alpha$ implies that

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} |f(t)-\alpha|\, dl(t) = |f(x)-\alpha|.$$

We know that N_α is a Lebesgue null-set because of Theorem 7.2.1. We will show that we can choose the sets N_α independent of α. Write the set of all rational numbers as $\mathbb{Q} = \{\alpha_i \mid i \in \mathbb{N}\}$ and let $N = \bigcup_{i\in\mathbb{N}} N_{\alpha_i}$, which is a null-set.

Now let β be an arbitrary real number and pick $\alpha \in \mathbb{Q}$ such that

$$|\beta - \alpha| < \epsilon.$$

We apply the triangle inequality as follows.

$$\begin{aligned}
&\left|\frac{1}{h}\int_x^{x+h} |f(t)-\beta|\, dl(t) - |f(x)-\beta|\right| \\
&\leq \left|\frac{1}{h}\int_x^{x+h} |f(t)-\beta|\, dl(t) - \frac{1}{h}\int_x^{x+h} |f(t)-\alpha|\, dl(t)\right| \\
&+ \left|\frac{1}{h}\int_x^{x+h} |f(t)-\alpha|\, dl(t) - |f(x)-\alpha|\right| + \Big||f(x)-\alpha| - |f(x)-\beta|\Big| \\
&< 2\epsilon + \left|\frac{1}{h}\int_x^{x+h} |f(t)-\alpha|\, dl(t) - |f(x)-\alpha|\right| \to 2\epsilon
\end{aligned}$$

for all $x \in [a,b] \setminus N$. □

Next we consider an application.

Definition 7.4.4. Let A be a Lebesgue measurable subset of $\mathbb{R}$. A point $x \in \mathbb{R}$ is called a *density point* of A if and only if

$$\lim_{h\to 0+} \frac{l(A \cap [x-h, x+h])}{2h}$$

exists and equals 1.

A density point of A need not belong to A.

Exercise 7.4.10. Let A be a Lebesgue measurable subset of $\mathbb{R}$ of positive measure.

a. Apply Theorem 7.4.2 to the function $f = 1_A$, the indicator function of A in order to prove that almost every point $x \in A$ is a density point of A.

b. Prove that A and B are two sets of positive measure in $\mathbb{R}$. Apply the preceding part to prove that there exists a translation by some $a \in \mathbb{R}$ such that $l((A + h) \cap B) > 0$. (Hint: Consider two density points.)

The following surprising congruence theorem is a fairly simple consequence of Exercise 7.4.10.

Theorem 7.4.3. (Steinhaus) *Let A and B be any two subsets of $\mathbb{R}$ having identical, finite positive measure: $l(A) = l(B) = \alpha$ and $0 < \alpha < \infty$. Then there exist two sequences of* mutually disjoint *measurable sets A_n and B_n and null sets N and M such that*

$$
\begin{aligned}
A &= \dot{\bigcup}_1^{\infty} A_n \cup N \\
B &= \dot{\bigcup}_1^{\infty} B_n \cup M
\end{aligned}
$$

and there exist constants a_n such that $A_n + a_n = B_n$ for all $n \in \mathbb{N}$.

Proof. The function $f(x) = l((A+x) \cap B)$ is a continuous function of x which approaches zero as $|x| \to \infty$ and which achieves strictly positive values at least for some x. Thus there exists a number $x = a_1$ which maximizes the value of f. Let $B_1 = B \cap (A + a_1)$, and let $A_1 = B_1 - a_1$. Define $B^1 = B \setminus B_1$ and $A^1 = A \setminus A_1$. If B^1 and A^1 happen to be null sets, we are done.

If not, pick a_2 which maximizes $l((A^1 + x) \cap B^1)$ and define A_2, B_2, A^2 and B^2 in the same manner as in the first step. We proceed until the process terminates (in which case we are done) or else we generate in this way two infinite sequences of sets and translation numbers. In the latter case, observe that $l(A_n) = l(B_n) \to 0$ as $n \to \infty$. Let

$$N = A \setminus \dot{\bigcup}_1^{\infty} A_n$$

and let

$$M = B \setminus \dot{\bigcup}_1^\infty B_n.$$

It will suffice to prove that N and M, which must have the same measure, are null sets.

Suppose this conclusion were false. Then there exists $a \in \mathbb{R}$ such that

$$l((N + a) \cap M) > 0.$$

But then there exists n such that

$$l(A_n) = l(B_n) < l((N + a) \cap M).$$

This violates the maximality property in the choice of a_n. □