c) Prove that $\phi(t)f$ is continuous at the point $t = 0$: This means that for each fixed $f$ in $L^1(\mathbb{R})$, we have

$$\|\phi(t)f - \phi(0)f\|_1 \to 0,$$

as $t \to 0$. Prove also continuity at each value of $t \in \mathbb{R}$. In words, this exercise says that the mapping $t \to \phi(t)f$ is a continuous mapping from $\mathbb{R} \to L^1(\mathbb{R})$.

d) Fix any $t > 0$, no matter how small. Show that there exists a function $f$ in $L^1(\mathbb{R})$ such that $\|f\|_1 = 1$ yet

$$\|\phi(t)f - f\|_1 \to 2.$$

Thus $\|\phi(t) - \phi(0)\|$ fails to converge to 0 as $t \to 0$, using the concept of the norm of a linear transformation of a normed linear space.\(^{42}\)

**Definition 5.5.6** In a normed linear space $V$, a sequence of vectors $v_n$ is called a Cauchy sequence provided that for each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that, for all $m$ and $n$ greater than or equal to $N$, we have $\|v_n - v_m\| < \epsilon$. A normed linear space is called complete provided that, for each Cauchy sequence $v_n$ in $V$, there exists $v \in V$ such that $v_n \to v$, which means that $\|v_n - v\| \to 0$. A complete normed real linear space is called a real Banach space, and a complete normed complex linear space is called a Banach space.

**EXERCISE**

5.41 † Give an example of a Cauchy sequence of functions

$$f_n \in L^1[0, 1],$$

such that there does not exist any point $x \in [0, 1]$ for which $f_n(x)$ converges. Prove that your example has the properties that are claimed.

**Theorem 5.5.2** Let $(X, \mathcal{A}, \mu)$ be a measure space. Then $L^1(X, \mathcal{A}, \mu)$ is a complete normed linear space.

**Proof:** Let $f_n$ be a Cauchy sequence in $L^1(X, \mathcal{A}, \mu)$. We must show there exists $f \in L^1(X, \mathcal{A}, \mu)$ such that $\|f_n - f\|_1 \to 0$ as $n \to \infty$. The main difficulty in the proof is to find a suitable function $f$ in $L^1(X, \mathcal{A}, \mu)$. The reason this is challenging is that a Cauchy sequence in $L^1(X, \mathcal{A}, \mu)$ need not converge pointwise at any point $x \in X$, as is shown in Exercise 5.41. In order to remedy this difficulty, we will prove that if a sequence of $L^1$-functions is sufficiently rapidly Cauchy, then it must converge pointwise almost everywhere.

\(^{42}\) The point here is that continuity in analysis is a very delicate issue indeed. It is very much a matter of what is the mapping, what is the domain (and its norm or topology), and what is the range (and its norm or topology).
Note that since $f_n$ is Cauchy there exists an increasing sequence of natural numbers $n_k \in \mathbb{N}$ such that if $n$ and $m$ are greater than or equal to $n_k$ then

$$\|f_n - f_m\|_1 < \frac{1}{4^k}.$$  \hfill (5.5)

In particular,

$$\|f_{n_k} - f_{n_{k+1}}\|_1 < \frac{1}{4^k},$$  \hfill (5.6)

for each $k \in \mathbb{N}$. The following lemma will be very helpful.

**Lemma 5.5.1 (Pointwise Convergence Lemma)** Let $(X, \mathcal{A}, \mu)$ be a measure space, and suppose a sequence of functions $g_k \in L^1(X, \mathcal{A}, \mu)$ satisfies the equation

$$\|g_k - g_{k+1}\|_1 < \frac{1}{4^k},$$

for each $k \in \mathbb{N}$. Then there exists a function $g \in L^1(X, \mathcal{A}, \mu)$ such that $g_k(x) \to g(x)$ for almost all values of $x$. Moreover, $g_k \to g$ in the sense of $L^1$–norm convergence, meaning that $\|g_k - g\|_1 \to 0$.

**Proof:** We think of the sequence $g_k$ as being very rapidly Cauchy. Note that the set

$$A_k = \left\{ x \mid |g_k(x) - g_{k+1}(x)| \geq \frac{1}{2^k} \right\}$$

is measurable, since $|g_k - g_{k+1}|$ is $\mathcal{A}$–measurable. Furthermore,

$$\frac{1}{2^k} \mu(A_k) \leq \int_X |g_k - g_{k+1}| \, d\mu < \frac{1}{4^k},$$

so that $\mu(A_k) < \frac{1}{2^k}$, for each $k \in \mathbb{N}$. Next we define

$$N = \limsup A_k - \bigcap_{p=1}^\infty \bigcup_{k=p}^\infty A_k,$$

and we note that $N$ is the set of all points $x$ that lie in infinitely many of the sets $A_k$. Furthermore

$$\mu \left( \bigcup_{k=p}^\infty A_k \right) \leq \sum_{k=p}^\infty \frac{1}{2^k} = \frac{1}{2^{p-1}} \to 0,$$

as $p \to \infty$. It follows that $\mu(N) = 0$, so that $N$ is an $(\mathcal{A}, \mu)$–null set. We will show that the sequence $g_k(x)$ converges for each $x \in X \setminus N$. 

If \( x \notin N \) then there exists \( p \in \mathbb{N} \) such that \( k \geq p \) implies that
\[
|g_k(x) - g_{k+1}(x)| < \frac{1}{2^k}.
\]
Therefore, if \( k \) and \( l \) are greater than or equal to \( p \), repeated application of the triangle inequality tells us that
\[
|g_k(x) - g_l(x)| < \frac{1}{2^{p-1}},
\]
so that \( g_k(x) \) is a Cauchy sequence of real numbers. Thus the function given by
\[
g(x) = \lim_{k \to \infty} g_k(x)
\]
exists almost everywhere on \( X \) and is measurable.\(^{43}\) Moreover, we find that
\[
\|g_k - g\|_1 = \int_X |g_k - g| \, d\mu
\]
\[
\leq \int_X \liminf_l |g_k - g_l| \, d\mu
\]
\[
\leq \liminf_l \int_X |g_k - g_l| \, d\mu
\]
\[
\leq \frac{1}{4^{p-1}},
\]
by Fatou’s theorem.\(^{44}\) Thus \( g_k - g \in L^1(X) \), which implies that \( g \in L^1(X) \), and that \( g_k \to g \) in the \( L^1 \)-norm. This proves the Pointwise Convergence Lemma.\(^{\Box}\)

To finish the proof of Theorem 5.5.2, we substitute \( f_{n_k} \) from Equation (5.5) for \( g_k \) in the Lemma.

To show convergence in \( L^1 \)-norm of \( f_n \) to \( f - g \) from the Lemma, we note that if \( n \geq n_k \), then we have \( \|f_n - f_{n_k}\|_1 < \frac{1}{4^k} \). It follows that if \( n \geq n_k \), then we have
\[
\|f_n - f\|_1 \leq \|f_n - f_{n_k}\|_1 + \|f_{n_k} - f\|_1 < \frac{2}{4^{k-1}} \to 0,
\]
as \( k \to \infty \). Hence \( \|f_n - f\|_1 \to 0 \) as \( n \to \infty \), and \( L^1(X) \) is complete.\(^{\Box}\)

\(^{43}\)If \( (X, \mathcal{A}, \mu) \) happens to be a complete measure space, then it is clear that \( g \) is measurable. If the measure space is not complete, we can define \( g \) to be constant on the null-set \( N \) of non-convergence pointwise, and again \( g \) will be measurable. For the purposes of this proof, we need to find only one \( L^1 \)-function that can serve as the limit for the original Cauchy sequence.

\(^{44}\)Fatou’s theorem is useful here, since we do not have a dominating function that would be required for the Lebesgue Dominated Convergence theorem.