Remark 4.3.1. We defined the concept of Baire function in Definition 4.1.2. There is an alternative, equivalent definition. The Baire class B_0 is the class of continuous functions. B_1 is the class of all pointwise limits of functions in B_0. For each $\alpha < \Omega$, the smallest uncountable ordinal number, we can define B_α to be the set of all pointwise limits of functions belonging to lower Baire classes. The family of all Baire functions as defined in Definition 4.1.2 can be shown to be

$$\bigcup_{\alpha<\Omega} B_\alpha.$$

The detailed explanation can be found in the book [3] by Casper Goffman. One part of the significance of the preceding theorem is that there are functions of Baire class B_2 that are not of class B_1. Thus almost-everywhere convergence is the best that can be expected in the theorem.

4.3.2 Convergence in Measure

There is a concept called *convergence in measure* for sequences of measurable functions $f_n \to f$ that is especially useful in the theory of probability. In that context, it is useful to to know that the probability of a random variable f_n differing from the random variable f by more than ϵ is very small.

Definition 4.3.2. A sequence of measurable functions f_n on a measure space $(X, \mathfrak{A}, \mu)$ is said to *converge in measure* to a measurable function f provided that for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies that

$$\mu\{x \mid |f_n(x) - f(x)| \geq \epsilon\} < \epsilon.$$

It follows readily from the definition that $f_n \to f$ in measure if and only if for each $\epsilon > 0$ and each $\eta > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\mu\{x \mid |f_n(x) - f(x)| \geq \eta\} < \epsilon.$$

Thus the definition is phrased as it is for simplicity. One gains nothing that is not already there if one uses two criteria, ϵ and η.

Exercise 4.3.2. Let $(X, \mathfrak{A}, \mu)$ be a measure space for which $\mu(X) < \infty$. Suppose f_n is a sequence of measurable functions such that $f_n \to f$ almost everywhere. Prove that $f_n \to f$ in measure. (You may assume that f is measurable, or you may assume the measure space is complete, explaining why this has the same effect.)

Exercise 4.3.3. Given an example of a sequence of Lebesgue measurable functions $f_n \to 0$ in measure on $[0,1]$, yet the sequence of numbers $f_n(x)$ fails to converge to zero for any $x \in [0,1]$.

Exercise 4.3.3 adds to the significance of the theorem below. The exercise explains why, in the following theorem, we will need need to pass to a subsequence that is sufficiently rapidly convergent in measure in order to guarantee pointwise convergence almost everywhere.

Theorem 4.3.4. *Suppose f and f_n are measurable on a finite measure space $(X, \mathfrak{A}, \mu)$ for all n, and that $f_n \to f$ in measure. Then there exists a subsequence $f_{n_\nu} \to f$ almost everywhere as $\nu \to \infty$.*

Proof. By hypothesis, for each $\nu \in \mathbb{N}$ there exists $n_\nu \in \mathbb{N}$ such that $n \geq n_\nu$ implies that

$$\mu\left\{x \middle| \, |f_n(x) - f(x)| \geq \frac{1}{2^\nu}\right\} < \frac{1}{2^\nu}.$$

The difficulty in establishing convergence pointwise almost everywhere is that these sets can slide around and cover a big region as we vary $n \geq n_\nu$. Thus we define the set

$$E_\nu = \left\{x \middle| \, |f_{n_\nu}(x) - f(x)| \geq \frac{1}{2^\nu}\right\}$$

for the single function f_{n_ν}. Define

$$S = \limsup E_\nu = \bigcap_{k=1}^{\infty} \bigcup_{\nu=k}^{\infty} E_\nu.$$

It is easy to check that S is a null set. Moreover, $x \notin S$ if and only if x lies in only finitely many of the sets E_ν. Thus if $x \notin S$ we know that for sufficiently big values of ν we have $x \notin E_\nu$. This implies that

$$\left|f_{n_\nu}(x) - f(x)\right| < \frac{1}{2^\nu}$$

It follows that for $x \notin S$ the sequence $f_{n_\nu}(x) \to f(x)$ as $\nu \to \infty$. $\square$

4.4 Lusin's Theorem

Lusin's theorem is paraphrased often as stating that a measurable function on $\mathbb{R}^n$ is *almost a continuous function.* Such phrasings can be useful as reminders of theorems that could help us in certain situations. But it is very important to remember to interpret the paraphrasing of Lusin's theorem as meaning exactly what the theorem states.

Theorem 4.4.1. (Lusin) *Let $f : X \to \mathbb{R}$ be a measurable function defined on a Lebesgue measurable set $X \subset \mathbb{R}^p$ for which the Lebesgue measure $l(X)$ is finite. Then for each $\eta > 0$ there exists a* compact *subset $K \subseteq X$ such that $l(X \setminus K) < \eta$ and f is* continuous *on K.*

Proof. We will undertake four restrictions of domain in order to reach a compact set on which f is continuous.

i. We wish to restrict f to a bounded subset of X so that the closed approximations of measurable sets from within will be compact. We can do this as follows. Since

$$l(X) = \lim_{k\to\infty} l\left(X \cap [-k,k]^{\times p}\right)$$

there exists a closed cube $Q = [-K,K]^{\times p} \subset \mathbb{R}^p$ large enough so that

$$l(X) \geq l(Q \cap X) > l(X) - \frac{\eta}{8}.$$

ii. We know from Exercise 4.3.1 that f is the pointwise limit of functions $f_n \in \mathfrak{S}$, the class of simple Lebesgue measurable functions. Write

$$f_n = \sum_{i=1}^{p_n} \alpha_i^n 1_{A_i^n}$$

a linear combination of indicator functions of *disjoint* measurable sets A_i^n. (The superscripts are labels only - not exponents.) By Exercise 3.2.2, for each i and n there exists a compact set $K_i^n \subseteq Q \cap A_i^n$ such that

$$l((Q \cap A_i^n) \setminus K_i^n) < \frac{\eta}{p_n 2^{n+1}}.$$

Each function f_n is continuous on K_i^n because it is constant there. Note also that the cluster points of K_i^n and those of K_j^n for $j \neq i$ must be distinct, since both sets are respectively closed subsets of disjoint measurable sets. Thus f_n is continuous also on

$$K^n = \bigcup_{i=1}^{p_n} K_i^n.$$

Moreover

$$l(X \setminus K^n) < \frac{\eta}{8} + \sum_{i=1}^{p_n} \frac{\eta}{p_n 2^{n+1}} = \frac{\eta}{8} + \frac{\eta}{2^{n+1}}.$$

iii. Define another compact set

$$K^* = \bigcap_{n=1}^{\infty} K^n$$

so that

$$l(X \setminus K^*) < \frac{\eta}{8} + \frac{\eta}{2} = \frac{5\eta}{8}.$$

The functions f_n are continuous on K^* and the sequence $f_n \to f$ pointwise on K^*.

iv. By Egoroff's theorem there exists a measurable set $B \subseteq K^*$ with $l(B) < \frac{\eta}{4}$ and $f_n \to f$ *uniformly* on $K^* \setminus B$. Since the set $K^* \setminus B$ is measurable, there exists a compact set $K \subseteq K^* \setminus B$ such that

$$l((K^* \setminus B) \setminus K) < \frac{\eta}{8}$$

which implies that $l(X \setminus K) < \eta$ and f is continuous on K.

□

Lusin's theorem tells us that the *restriction* $f\Big|_K$ is a continuous function. That is, $f \in \mathcal{C}(K, \mathbb{R})$. In other words, f is continuous as a function defined only on the restricted domain, K. But f need not be continuous at any $k \in K$ as a function defined on X. This is relevant to Exercise 4.4.1.

Exercise 4.4.1. Let f be the indicator function of the set of all irrational numbers in the interval $X = [0, 1]$.

a. Show that f is nowhere continuous on $[0, 1]$.

b. Let $\eta > 0$ and find a set B of measure less than η such that f is continuous on $K = X \setminus B$ and such that K is compact.

Exercise 4.4.2. Let $f : [a, b] \to \mathbb{R}$ be a measurable function. Let $\eta > 0$ and $\epsilon > 0$. Prove that there exists a measurable set B such that $l(B) < \eta$ and a polynomial p such that $\sup\{|f(x) - p(x)| \mid x \in [0, 1] \setminus B\} < \epsilon$. (Hint: Apply the Tietze Extension theorem and the Weierstrass Approximation theorem.)