Measure & Integration: A Concise Introduction to Real Analysis
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First Printing Errata & Addenda **Updated December 21, 2013.**

1. P. 13, Theorem 2.1.4. Add the word *nonempty*: A nonempty family $\mathfrak{A}$ of subsets of X is a Borel field if and only if ... .

2. P. 20. This is an addendum. It is common to abuse language by referring to a measure μ *on a set* X *if* there will be no confusion as to the choice of field $\mathfrak{A} \subseteq \mathfrak{P}(X)$ that is intended. See Exercise 2.16 for an example, in which part of the exercise is to pick a suitable field.

3. P. 24, Theorem 2.4.1. The uniqueness part of this theorem should say that the extension, μ^*, is unique when restricted to the Borel field generated by $\mathfrak{A}$. If two extensions are regular measures, however, then uniqueness on the Borel field generated by $\mathfrak{A}$ is sufficient to give uniqueness on any set that is measurable with respect to both measures, provided that the two measures assign the measure zero to the same sets. The extension provided by the proof of the Hopf Extension Theorem is regular.

4. P. 25, L. 12. The word *need* is repeated needlessly, and the redundant word should be deleted.

5. P. 28, Theorem 2.4.2. There is a missing statement. Rewrite the theorem as follows: Let μ be a Carathéodory outer measure on $\mathfrak{P}(X)$ that is regular with respect to the σ-field of μ-measurable sets and such that $\mu(X) < \infty$. Then a subset $A \subseteq X$ is μ-measurable if and only if $\mu(X) = \mu(A) + \mu(A^c)$. (Note that in this theorem $\mathbb{B}(\mathfrak{A}) = \mathfrak{A}$.)

6. P. 28, Exercise 2.21. We could add a third part to this exercise. Show that if both measures μ_1 and μ_2 are regular, and if they assign the measure zero to the same sets, then they agree on all sets on which they are defined, and each is defined on the same σ-field. The following also is not an error but an addendum. Exercise 6.20 can be extended in to show why uniqueness of the extension measure can fail without σ-finiteness. See Item #21 below.

7. P.37, Exercise 3.5. This exercise should say: Prove that $\mathfrak{B}(\mathbb{R})$ is the σ-field generated by the field $\mathbb{F}(\mathfrak{E}(\mathbb{R}))$. The $\mathbb{F}$ was missing in the text.

8. p. 48, Exercise 3.20. The hint contains an error. Please use the following version of this problem. The space $\mathcal{C}[a,b]$ of all continuous functions on the closed finite interval $[a,b] \subset \mathbb{R}$ is complete with respect to the metric
$$\rho(f,g) = \sup \left\{ |f(x) - g(x)| \,\middle|\, x \in [a,b] \right\}.$$
Use the Baire Category Theorem to prove that the set of *nowhere differentiable continuous* functions is of the second category in $\mathcal{C}[a,b]$ if $a < b$. Hints: Prove that for all $m, n \in \mathbb{N}$ the set $S_{m,n} = \left\{ f \in \mathcal{C}[a,b] \,\middle|\, \inf_{x \in [a,b]} \sup_{0<|h|<\frac{1}{m}} \left| \frac{f(x+h) - f(x)}{h} \right| > n \right\}$ is an open, dense subset of $\mathcal{C}[a,b]$. Consider *sawtooth* functions.

9. P. 52, Exercise 3.28. A clearer phrasing of the question would be as follows: Prove that the family $\mathfrak{J}$ of Jordan-measurable sets is *not* a σ-field.

10. P. 65, Theorem 4.3.4. Delete the word *finite* from the statement of the theorem. In fact, the proof does not use the hypothesis of finiteness of the measure $\mu(X)$, and this hypothesis is unnecessary.

11. P. 71, Theorem 5.1.2. The concludion of the theorem should say: Then $d(f,h) \leq d(f,g) + d(g,h)$.

12. P. 75. **Definition 5.2.4** should state: *On any measure space, an extended real-valued, nonnegative,* $\mathfrak{A}$*-measurable function* f *with* σ*-finite carrier is said to be* integrable *provided that* $\Phi(f) < \infty$. *The class of all such functions is denoted by* $\mathcal{L}^+$.
The omission of the phrase *with* σ*-finite carrier* in the first printing is my proofreading error. It is in fact presumed throughout this book that an integrable function on an arbitrary measure space has σ-finite carrier. The requirement in Definition 5.2.4 that f have σ-finite carrier ensures that there are enough functions in $\mathfrak{S}_0$ to approximate f adequately. This excludes such examples as the following: Let $X \neq \emptyset$, $\mathfrak{A} = \{\emptyset, X\}$, $\mu(\emptyset) = 0$ and $\mu(X) = \infty$. If $f(x) \equiv 1$ then $\Phi(f) = 0$ yet f lacks σ-finite carrier and is not zero almost everywhere.
Lemma 5.2.1 should state: *Let* $(X, \mathfrak{A}, \mu)$ *be a measure space with* μ *approximately finite and let* f *be an extended real-valued, nonnegative,* $\mathfrak{A}$*-measurable function defined on* X *for which* $\Phi(f) < \infty$. *Then* f *has* σ*-finite carrier.*

13. P.76, Theorem 5.2.1 (ii) Monotonicity. This should say that $0 \leq \Phi(g)$, not $<$.

14. P. 90, In the display for Theorem 5.4.2 and on the second line of the proof replace f by $|f|$.

15. P. 100, Exercise 5.46 parts (a) and (b) should read as follows: Let $(X, \mathfrak{A}, \mu)$ be a measure space and let $f : X \to \mathbb{C}$.
    (a) Prove that if $f \in L^1(X, \mathbb{C})$ then f is measurable and has σ-finite carrier.
    (b) Prove that $f \in L^1(X, \mathbb{C})$ if and only if $\|f\|_1 < \infty$, where f is measurable and has σ-finite carrier.

16. P. 101, Exercise 5.48. Replace the symbol $\|a_j\|_1$ by $\|z\|_1$.

17. P. 108, L 3. The word *improper* should be *proper.*

18. P. 109, Line 23. The integrand should be f_C as in $\int_X f_C \, d\mu = \ldots$. Note also, for clarification, the the application of Theorem 2.1.2 higher on this page means application to the field $\mathfrak{E}$ of elementary sets.

19. P. 115, Exercise 6.10. The Hint for part (a) should refer to Exercise 6.9, not 5.43.

20. P. 117, Exercise 6.20. Note that the order of iterated integration has a typo. It should read: Show that $\int_Y \int_X f(x, y) \, d\mu(x) \, d\lambda(y) \neq \int_X \int_Y f(x, y) \, d\lambda(y) \, d\mu(x)$ and explain why this does not violate Fubini's theorem.

21. P. 117, Exercise 6.20. This is not an error, but a new second part to extend the exercise further, as follows. Let $\mathfrak{A}$ be the field generated by the rectangular sets $A \times B$ where A is μ-measurable and B is λ-measurable. Show that $\mu \times \lambda$ has two *distinct* countably additive extensions to the σ-field $\mathbb{B}(\mathfrak{A})$. (See Item #6 above.)

22. P. 131. Insert the following exercise. Let $f \in L^1(\mathbb{R})$.
    (a) Suppose for each closed finite interval $[c, d] \subset \mathbb{R}$ we have $\int_{[c,d]} f \, dl = 0$. Prove that $f(x) = 0$ for almost all $x \in \mathbb{R}$.
    (b) Suppose now that $\int_{[c,d]} f \, dl \leq 0$. Prove that $f(x) \leq 0$ almost everywhere.

23. P. 147. This is an addendum to Exercise 7.17. It is an example to illustrate the importance of the hypotheses used in the exercise. Let $\phi : [0,1] \to [0,1]$ be the Cantor function. Let $D = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y \leq 1\}$. Let $f(x,y) = \phi(x+y)1_D(x,y)$ for all $(x,y) \in \mathbb{R}^2$. Show that $\frac{\partial}{\partial y} \int_{\mathbb{R}} f(x,y)\, dl(x) \neq \int_{\mathbb{R}} \frac{\partial}{\partial y} f(x,y)\, dl(x)$.

24. P. 176: The final equality in the displayed equation $T(f) = \lim_n T(\phi_n) = \lim_n \int_X \phi_n\, g\, d\mu = \int_X fg\, d\mu$ requires further explanation. Decompose $g = g^+ - g^-$ and let $P = \{x \mid g(x) \geq 0\}$ and $N = X \setminus P$. Then $f = f1_P + f1_N$ with both summands being in $L^p(X, \mathfrak{A}, \mu)$. Hence $T(f1_P) = \lim_n T(\phi_n 1_P) = \lim_n \int_X \phi_n 1_P\, g\, d\mu = \lim_n \int_X \phi_n g^+\, d\mu = \int_X fg^+\, d\mu$ by the Monotone Convergence Theorem. Similar reasoning applies to $f1_N$ and then the argument is completed by the linearity of T.

25. P. 180, Line 10B. This should read x' is congruent to *a multiple of* x modulo $ker(T)$.

26. P. 229, Reference #3. Delete the letter *a* from *sumas* to make the word *sums* in the title of Carleson's paper.