Because of regularity, there exists a measurable set $V \supseteq W$ such that $\mu(V) = \mu(W)$. Observe that, since $V$ is measurable, we have

\[
\begin{align*}
\mu(A) &= \mu(A \cap V) + \mu(A \cap V^c), \\
\mu(A^c) &= \mu(A^c \cap V) + \mu(A^c \cap V^c)
\end{align*}
\]

(2.5)

By hypothesis and by Equations (2.5),

\[
\begin{align*}
\mu(X) - \mu(A) &= \mu(A^c) - \mu(V) + \mu(V^c) \\
&= \left[\mu(A \cap V) + \mu(A^c \cap V)\right] + \left[\mu(A \cap V^c) + \mu(A^c \cap V^c)\right].
\end{align*}
\]

On the other hand

\[
\begin{align*}
\mu(A \cap V) + \mu(A^c \cap V) &\geq \mu(V), \quad \text{and} \\
\mu(A \cap V^c) + \mu(A^c \cap V^c) &\geq \mu(V^c),
\end{align*}
\]

by subadditivity. Thus

\[
\begin{align*}
\mu(A \cap V) + \mu(A^c \cap V) &= \mu(V), \quad \text{and} \\
\mu(A \cap V^c) + \mu(A^c \cap V^c) &= \mu(V^c).
\end{align*}
\]

By the choice of $V$, and by the preceding equations,

\[
\begin{align*}
\mu(W) &= \mu(V) - \mu(V \cap A) + \mu(V \cap A^c) \\
&\geq \mu(W \cap A) + \mu(W \cap A^c) \\
&\geq \mu(W).
\end{align*}
\]

Hence

\[
\mu(W \cap A) + \mu(W \cap A^c) = \mu(W),
\]

and $A$ is measurable.

\section*{2.4.1 Fields, $\sigma$-Fields, and Measures Inherited by a Subset}

In Definition 3.3.1, we will see that a triplet, $(X, \mathcal{A}, \mu)$, is called a measure space, provided that $X$ is a set, $\mathcal{A} \subseteq \mathcal{P}(X)$ is a $\sigma$-field, and $\mu$ is a countably additive measure defined on $\mathcal{A}$.

\textbf{Definition 2.4.2} A triplet $(X, \mathcal{A}, \mu)$, is a pre-measure space, provided that $X$ is a set, $\mathcal{A} \subseteq \mathcal{P}(X)$ is a field, and $\mu$ is a finitely additive measure defined on $\mathcal{A}$.

Thus the Hopf Extension Theorem provides necessary and sufficient conditions for a pre-measure space to be extended to a full-fledged measure space. Note that there exist pre-measure spaces that cannot be extended to measure spaces.\footnote{For example, see Exercise 2.16.}
It is often useful to consider the restriction of a measure \( \mu \), given to us in either a pre-measure space or a measure space, to a subfield of the power set \( \mathcal{P}(S) \) for some set \( S \in \mathfrak{A} \). An especially important instance is the situation in which \((X, \mathfrak{A}, \mu)\) is \( \sigma \)-finite, so that \( X = \bigcup_{i \in \mathbb{N}} X_i \), with \( \mu(X_i) < \infty \), for each \( i \in \mathbb{N} \).

**Definition 2.4.3** If \((X, \mathfrak{A}, \mu)\) is any pre-measure space, define the pre-measure space inherited by \( S \in \mathfrak{A} \) to be the triplet
\[
(S, \mathfrak{A}_S, \mu),
\]
where
\[
\mathfrak{A}_S = \{ A \cap S \mid A \in \mathfrak{A} \} \subseteq \mathcal{P}(S),
\]
and we retain the symbol \( \mu \) for the restriction to \( \mathfrak{A}_S \) of the given measure on \( \mathfrak{A} \).

Since \( \mathfrak{A} \) is a field, it is clear that \( \mathfrak{A}_S \subseteq \mathfrak{A} \), so that \( \mu \) is defined on \( \mathfrak{A}_S \). Moreover, it is easily checked that \( \mathfrak{A}_S \) is itself a subfield of the field \( \mathcal{P}(S) \), with the understanding that complementation in \( \mathfrak{A}_S \) will be with respect to \( S \), not with respect to \( X \). That is, for \( A \in \mathfrak{A}_S \), we define \( A^c = S \setminus A \). Again because \( \mathfrak{A} \) is a field, the set \( S^c \) also inherits a pre-measure space from \((X, \mathfrak{A}, \mu)\).

**Theorem 2.4.3** If \((X, \mathfrak{A}, \mu)\) is any pre-measure space and if \( S \in \mathfrak{A} \), then an arbitrary set \( B \) belongs to \( \mathcal{B}(\mathfrak{A}) \) if and only if \( B = B_1 \cup B_2 \), where \( B_1 \in \mathcal{B}(\mathfrak{A}_S) \) and \( B_2 \in \mathcal{B}(\mathfrak{A}_{S'}) \).

We are to understand in this theorem that \( \mathcal{B}(\mathfrak{A}_S) \subseteq \mathcal{P}(S) \). That is, we treat \( S \) as the universal set in the definition of the Borel field generated by \( \mathfrak{A}_S \).

**Proof:** We observe that if \( \mathcal{B} \) is a \( \sigma \)-field in \( \mathcal{P}(X) \) containing \( \mathfrak{A} \), then both of the following two conditions are met: \( \mathcal{B}_S \) is a \( \sigma \)-field in \( \mathcal{P}(S) \) containing \( \mathfrak{A}_S \), and \( \mathcal{B}_{S'} \) is a \( \sigma \)-field in \( \mathcal{P}(S^c) \) containing \( \mathfrak{A}_{S'} \).

Conversely, if \( \mathcal{B}_1 \) is a \( \sigma \)-field in \( \mathcal{P}(S) \) containing \( \mathfrak{A}_S \), and if \( \mathcal{B}_2 \) is a \( \sigma \)-field in \( \mathcal{P}(S^c) \) containing \( \mathfrak{A}_{S'} \), then we define
\[
\mathcal{B} = \{ B_1 \cup B_2 \mid B_1 \in \mathcal{B}_1, \ B_2 \in \mathcal{B}_2 \}.
\]
Then it is clear that \( \mathcal{B}_S = \mathcal{B}_1 \) and \( \mathcal{B}_{S'} = \mathcal{B}_2 \), and that \( \mathcal{B} \) is a \( \sigma \)-field in \( \mathcal{P}(X) \) containing \( \mathfrak{A} \).

The conclusion follows from Definition 2.1.5, in which the Borel field generated by a given field is the intersection of all \( \sigma \)-fields containing the given field. \( \blacksquare \)

We note that the preceding theorem does not involve \( \mu \), and relates only to the pair \((X, \mathfrak{A})\), which is called a **pre-measurable space**.

**Corollary 2.4.1** Suppose
\[
X = \bigcup_{i \in \mathbb{N}} X_i,
\]
with each \( X_i \in \mathfrak{A} \), a field contained in \( \mathcal{P}(X) \). Then \( B \in \mathcal{B}(\mathfrak{A}) \) if and only if
\[
B = \bigcup_{i \in \mathbb{N}} B_i, \ B_i \in \mathcal{B}(\mathfrak{A}_{X_i}) \subseteq \mathcal{P}(X_i) \forall i \in \mathbb{N}.
\]
Proof: This is a countable adaptation of the proof of Theorem 2.4.3. A set $B$ is a $\sigma$-field containing $\mathcal{A}$ if and only if the following condition is met for each $i \in \mathbb{N}$: $\mathcal{B}_{X_i} \supseteq \mathcal{A}_{X_i}$, with $\mathcal{B}_{X_i}$ being a $\sigma$-field in $\mathcal{P}(X_i)$.

EXERCISE

2.21† Suppose both $\mu_1$ and $\mu_2$ are countably additive extensions of the measure $\mu$ from the field $\mathcal{A}$ to $\mathcal{B}(\mathcal{A})$. Suppose that $\mu$ is countably additive on $\mathcal{A}$.

a) Suppose that $\mu(X) < \infty$. Prove that $\mu_1(B) = \mu_2(B)$ for all $B \in \mathcal{B}(\mathcal{A})$.

(Hint: Show that the set

$$\mathcal{B}(\mathcal{A}) - \{ B \in \mathcal{B}(\mathcal{A}) \mid \mu_1(B) - \mu_2(B) \}$$

is closed under complementation, and under taking unions of increasing sequences, making $\mathcal{B}(\mathcal{A})$ a monotone class that contains $\mathcal{A}$. The finiteness of $\mu(X)$ will be helpful for complementation.)

b) Now replace the hypothesis that $\mu(X) < \infty$ in part (a) with the hypothesis that $\mu$ is $\sigma$-finite on $X$. Prove that $\mu_1(B) = \mu_2(B)$ for all $B \in \mathcal{B}(\mathcal{A})$.

(Hint: Use Corollary 2.4.1.)