d) Prove that the open sets and the closed sets are Borel sets in $\mathbb{R}^n$.

e) Prove Theorem 3.5.2: $\mathfrak{B}(\mathbb{R}^n)$ is the σ-field generated by the family of open sets, or, equivalently, by the family of closed sets, in $\mathbb{R}^n$.

3.24 † Suppose E is Lebesgue measurable in $\mathbb{R}^n$, and define

$$E + c = \{e + c \mid e \in E\},$$

the translate of E by c. (Addition refers to *vector* addition in this context.) We will prove the *translation-invariance* of Lebesgue measure.

a) Prove that E is a null set if and only if $E + c$ is a null set.

b) Prove that $E + c$ is Lebesgue measurable.

c) Suppose that E is an *elementary set* in $\mathbb{R}^n$, and prove that $l(E + c) = l(E)$.

d) Let E be any *measurable set*, and prove that $l(E + c) = l(E)$.

3.25 † Suppose E is a subset of $\mathbb{R}^n$ and let $-E = \{-e \mid e \in E\}$. We will prove the *invariance* of Lebesgue measure under *reflection* through the origin.

a) Suppose first that E is an elementary set in $\mathbb{R}^n$, and prove that $-E$ is a Borel set, and $l(-E) = l(E)$.

b) Finally, let E be a measurable set, and prove that $l(-E) = l(E)$.

3.26 † Suppose A and B are measurable subsets of $\mathbb{R}^n$, each one of strictly positive but finite measure. Prove that there exists a vector $c \in \mathbb{R}^n$ such that $l((A+c) \cap B) > 0$. (Hint: Consider the outer measure of A and B.)

3.27 Let $\epsilon > 0$. Construct an *open, dense* subset S of $\mathbb{R}^n$ for which the Lebesgue measure $l(S) < \epsilon$.

3.6 JORDAN MEASURE IN $\mathbb{R}^N$

Lebesgue measure will be the foundation for defining the Lebesgue integral, and for proving its properties. Jordan measure is the corresponding foundation for the Riemann integral, though Jordan measure (also called *Jordan content*) is often not taught explicitly in Advanced Calculus courses. Although not required for understanding the Lebesgue integral itself, the study of Jordan measure will help us to understand the relationship between the Lebesgue integral and the Riemann integral, which the Lebesgue integral supersedes. Moreover, Jordan measure will enable us to prove Lebesgue's theorem (6.3.1), classifying all Riemann integrable functions in terms of the Lebesgue measure of the set of points of discontinuity.

Definition 3.6.1 Let $\mathcal{E}$ be the collection of unions of finitely many closed blocks the closed cube $\bar{Q}_N$, where $N \in \mathbb{N}$ is fixed arbitrarily. Define the *outer Jordan measure* for each $A \in \mathfrak{P}(Q_N)$ by

$$v^*(A) = \inf\{l(E) \mid A \subseteq E,\ E \in \mathcal{E}\}$$

and let the *inner Jordan measure* be defined by

$$v_*(A) = \sup\{l(E) \mid A \supseteq E,\ E \in \mathcal{E}\}.$$

Here $l(E)$ denotes the volume, or Lebesgue measure, of the union of finitely many rectangular blocks that comprise E. We define the set $\mathfrak{J}$ to be the family of all Jordan measurable sets, where A is Jordan measurable if and only if $v^*(A) = v_*(A)$, in which case either number is called $v(A)$, the *Jordan measure* of A.

We will see that the weakness of Jordan measure stems from the need to cover a set using only unions of *finitely* many rectangular blocks. This has the unfortunate effect that Jordan measure is only finitely additive, and that is insufficient for the needs of analysis.

EXERCISES

3.28 Prove that Jordan measure is *not* countably additive.

3.29 Use Definition 3.2.1 and DeMorgan's Laws to prove the following properties of inner and outer Jordan measure, in relation to inner and outer Lebesgue measure, for all $A \in \mathfrak{P}(\bar{Q}_N)$. The sets E_i are in $\mathcal{E}$.

a) $l^*(A) = \inf \left\{ \sum_{i \in \mathbb{N}} l(E_i) \,\middle|\, A \subseteq \bigcup_{i \in \mathbb{N}} E_i \right\}$

b) $v_*(A) \leqslant l_*(A) \leqslant l^*(A) \leqslant v^*(A)$.

It follows that every Jordan measurable set is Lebesgue measurable and that its Lebesgue measure equals its Jordan measure. We will see that not every Lebesgue measurable set is Jordan measurable however. Thus Lebesgue measure is an extension of Jordan measure.

Theorem 3.6.1 *The family $\mathfrak{J}$ of all Jordan measurable sets in $X = Q_N$ is a field, and v is a finitely additive measure on $\mathfrak{J}$.*

Proof: Let A and B lie in $\mathfrak{J}$. Since

$$v(X) - v^*(X \backslash A) = v_*(A) \text{ and } v(X) - v_*(X \backslash A) = v^*(A),$$

it follows that $X \backslash A \in \mathfrak{J}$. Both $A \cup B$ and $A \cap B$ are in $\mathfrak{J}$ since the union and intersection of any two elementary sets is again an elementary set. And v is finitely additive on $\mathfrak{J}$ since on that field of sets v agrees with l. ■

EXERCISE

3.30 A subset A of $X = \bar{Q}_N$ is Jordan measurable if and only if for each $\epsilon > 0$ there exist in $\mathcal{E}$ sets $E_1 \subseteq A \subseteq E_2$ such that $v(E_2 \backslash E_1) < \epsilon$.

Definition 3.6.2 A set $N \subset X = \bar{Q}_N$ is called a *Jordan null set* if and only if $N \in \mathfrak{J}$ and $v(N) = 0$.

Theorem 3.6.2 *A Lebesgue* null *set $F \subset X = \bar{Q}_N$ that is* closed *is also a Jordan null set.*

Proof: It suffices to prove that $F \in \mathfrak{J}$. It is easy to see that if $\epsilon > 0$ there exists a set G that is open in $\mathbb{R}^n$, and such that $F \subset G$ and $l(G) < \epsilon$. But the set G is a countable union of *open* rectangular blocks. Since F is compact, the Heine-Borel theorem implies that F can be covered with finitely many of these open rectangular blocks with measure no greater than that of G. Thus there exists an $E \supset F$ such that $E \in \mathcal{E}$, and $v(E) = l(E) < \epsilon$. Thus F is a Jordan null set since $\varnothing \subseteq F \subset E$. ■

Theorem 3.6.3 *If $A \subseteq X = \bar{Q}_N$, then $A \in \mathfrak{J}$ if and only if the* boundary

$$\partial A = \bar{A} \backslash A^{\mathrm{o}},$$

the difference between the closure and the interior of A, is a Jordan null set.

Proof: Suppose first that $A \in \mathfrak{J}$. Since the boundary ∂A is a closed set, it will suffice to show that it is a Lebesgue null set, since that will imply it is also a Jordan null set. We know that for each $k \in \mathbb{N}$ there exist sets $E_k \subseteq A \subseteq F_k$, such that both E_k and F_k are in $\mathcal{E}$, and

$$l\left(F_k \backslash E_k\right) = l\left(\bar{F}_k \backslash E_k^{\mathrm{o}}\right) < \frac{1}{k}.$$

Let

$$F = \bigcap_{k \in \mathbb{N}} \bar{F}_k \text{ and } E = \bigcup_{k \in \mathbb{N}} E_k^{\mathrm{o}}.$$

It follows that $\partial A \subseteq F \backslash E$, and $l\left(F \backslash E\right) = 0$. Thus ∂A is a Lebesgue null set and also a Jordan null set.

Now we suppose that ∂A is a Jordan null set. For each $\epsilon > 0$, there exists in $\mathcal{E}$ a set $E^o \supseteq \partial A$, with $l(E) < \epsilon$.[33] We seek to prove that $A \in \mathfrak{J}$. Denote $X = Q^N$ and

$$E^c = X \backslash E^o = \bigcup_{i=1}^{K} B_k \in \mathcal{E}.$$

Suppose that one of the rectangular blocks B_k contained a point $p \in A$ and also a point $q \in A^c$. Then it would follow that the straight line segment from p to q must contain a boundary point of A, which contradicts the hypothesis that $\partial A \subseteq E$. Now let E_1 be the union of those blocks B_k that lie inside A and let E_2 be the union of the blocks that lie outside A. Then the elementary set $\overline{E_2^c} \backslash E_1^o = E$ and this means that $E_1 \subseteq A \subseteq \overline{E_2^c}$ and $l\left(\overline{E_2^c} \backslash E_1\right) < \epsilon$. Thus $A \in \mathfrak{J}$. ■

EXERCISE

3.31 Let $A = \mathbb{Q} \cap [0, 1]$, the set of all rational numbers in $[0, l)$. Prove that ∂A is neither a Lebesgue null set nor a Jordan null set, so $A \notin J$. On the other hand, show that $A \in \mathfrak{B}$, the family of Borel sets.

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[33] The set E^o denotes the *interior* of E.