Chapter 9

Examples of Dual Spaces from Measure Theory

We have seen that $L^1(X, \mathfrak{A}, \mu)$ is a Banach space for any measure space $(X, \mathfrak{A}, \mu)$. We will extend that concept in the following section to identify an infinite family of Banach spaces, and we will see that some of these spaces succeed in representing all the continuous linear mappings of other such spaces into the field of scalars.

9.1 The Banach Space $L^p(X, \mathfrak{A}, \mu)$

Recall that for a measurable function f on a measure space $(X, \mathfrak{A}, \mu)$, we define the equivalence class $[f]$ to be the set of all measurable functions g such that $f = g$ almost everywhere. This equivalence is denoted also by $f \sim g$.

Definition 9.1.1. Let $(X, \mathfrak{A}, \mu)$ be any measure space. For each real number $p \in [1, \infty)$, we define the *vector space*

$$L^p(X, \mathfrak{A}, \mu) = \left\{ [f] \,\Big|\, \int_X |f|^p \, d\mu < \infty, \ f \ \ \mathfrak{A}\text{−measurable} \right\}.$$

It is common to say or write that some function f is an element of $L^p(X, \mathfrak{A}, \mu)$ although the elements of that vector space are actually equivalence classes.

Exercise 9.1.1. For each $1 \le p < \infty$, prove that $L^p(X, \mathfrak{A}, \mu)$ is a vector space. (Hint: Closure under scalar multiplication is easy. Show that if f and g are in L^p then $f + g \in L^p$ too. Express X as a disjoint union of two sets, depending on which of the two functions, $|f|$ or $|g|$, is larger.)

It will be useful to define a suitable norm on $L^p(X, \mathfrak{A}, \mu)$.

Definition 9.1.2. For each f in $L^p(X, \mathfrak{A}, \mu)$ we define

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}}.$$

Observe that $f \sim g$ in $L^p(X, \mathfrak{A}, \mu)$ if and only if $\|f - g\|_p = 0$. Also, $\|f\|_p$ is well-defined on equivalence classes in L^p.

One of our objectives is to prove that L^p is a normed linear space. The triangle inequality is the only property required to be a norm that is not very easy to check. To prove the triangle inequality we begin with an important inequality for the real numbers.

Lemma 9.1.1. (Jensen's Inequality) *Suppose*

$$\alpha \ge 0,\ \beta \ge 0,\ a > 0, \text{ and } b > 0$$

are real numbers such that $\alpha + \beta = 1$. *Then*

$$a^\alpha b^\beta \le \alpha a + \beta b.$$

Proof. Since the left and right sides of the desired inequality are strictly positive, we can give a proof by taking the natural logarithm on both sides[1] Thus the theorem is equivalent to the claim that

$$\alpha \ln a + \beta \ln b \le \ln(\alpha a + \beta b).$$

But the left side is the second coordinate of the convex combination

$$\alpha(a, \ln a) + \beta(b, \ln b)$$

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[1] The reason for requiring $a > 0$ and $b > 0$ is that we do wish to allow either α or β (but not both) to be zero. Thus we choose to avoid the difficulty of interpreting 0^0. If a^α means $e^{\alpha \ln a}$ then 0^0 is undefined. In some contexts 0^0 is defined as $\lim_{x \to 0+} x^x = 1$ or as $\lim_{x \to 0} x^0 = 1$. But it could be interpreted also as $\lim_{x \to 0+} 0^x = 0$. Thus we avoid this case by requiring that a and b to be strictly positive.

of the two vectors $(a, \ln a)$ and $(b, \ln b)$, and so it lies on the chord joining the points $(a, \ln a)$ and $(b, \ln b)$ on the graph of $y = \ln x$. And the right side of the displayed inequality is the value of the logarithm at the first coordinate of $\alpha(a, \ln a) + \beta(b, \ln b)$. The function $\ln x$ is said to be *concave* because the chords that join pairs of points on the graph must lie below the curve. (Such graphs are also called concave down in elementary calculus.) $\square$

Jensen's Inequality enables us to prove the following very important inequality.

Theorem 9.1.1. (Hölder's Inequality) *Let $p \geq 1$ and $q \geq 1$ be real numbers such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Let $f \in L^p(X, \mathfrak{A}, \mu)$ and $g \in L^q(X, \mathfrak{A}, \mu)$. Then the product $fg \in L^1(X, \mathfrak{A}, \mu)$ and

$$\int_X |fg| \, d\mu \leq \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}} \left(\int_X |f|^q \, d\mu \right)^{\frac{1}{q}}$$

which can be written more conveniently as

$$\|fg\|_1 \leq \|f\|_p \|g\|_q. \tag{9.1}$$

In the special case in which $p = 1$ we take $q = \infty$ and we denote by $\|f\|_\infty$ the essential supremum of $|f|$.

Proof. Note first that if $p = 1$ and $q = \infty$ then Hölder's inequality is very easy to prove. So we will suppose that $p > 1$ and $q > 1$.

To prove Hölder's inequality, let

$$E = \{x \mid |f(x)g(x)| > 0\}$$

and assume without loss of generality that $\mu(E) > 0$. We will need to consider the set E in order to have the strictly positive terms to which we can apply Jensen's inequality. For each x let

$$a(x) = \frac{|f(x)|^p}{\|f\|_p^p} \text{ and } b(x) = \frac{|g(x)|^q}{\|g\|_q^q}$$

where neither denominator can vanish because $\mu(E) > 0$. By Jensen's inequality we have for each $x \in E$

$$a(x)^{\frac{1}{p}} b(x)^{\frac{1}{q}} \leq \frac{1}{p} a(x) + \frac{1}{q} b(x)$$

or

$$\frac{|f(x)g(x)|}{\|f\|_p\|g\|_q} \leq \frac{|f(x)|^p}{p\|f\|_p} + \frac{|g(x)|^q}{q\|g\|_q}.$$

Next we integrate both sides over E to obtain

$$\frac{\|fg\|_{L^1(E)}}{\|f\|_p\|g\|_q} \leq \frac{1}{p} + \frac{1}{q} = 1$$

which implies that

$$\|fg\|_1 = \|fg\|_{L^1(E)} \leq \|f\|_p\|g\|_q.$$

□

We are ready now to prove the triangle inequality for the L^p norms.

Theorem 9.1.2. (Minkowski's Inequality) *If* $1 \leq p \leq \infty$ *and* f *and* g *are in* $L^p(X, \mathfrak{A}, \mu)$ *then*

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof. For $p = 1$ we have proven this already in Exercise 5.5.1. For $p = \infty$ the result is very easy. So we will suppose that $1 < p < \infty$. Because L^p is a vector space, we know that

$$0 \leq \|f+g\|_p < \infty.$$

If $\|f+g\|_p = 0$ the theorem is very easy. So we will suppose that $\|f+g\|_p > 0$.

We can write

$$\int |f+g|^p = \int |f+g||f+g|^{p-1} \leq \int |f||f+g|^{p-1} + \int |g||f+g|^{p-1}.$$

If we let $q = \frac{p}{p-1}$ then $\frac{1}{p} + \frac{1}{q} = 1$ and we have $|f+g|^{p-1} \in L^q$. We apply Hölder's inequality to each summand on the right side above, obtaining

$$\|f+g\|_p^p \leq (\|f\|_p + \|g\|_p) \left\| |f+g|^{p-1} \right\|_q \tag{9.2}$$

Since

$$\left\| |f+g|^{p-1} \right\|_q = \left(\int |f+g|^p \right)^{\frac{1}{q}} = \|f+g\|_p^{p-1} \neq 0$$

we can divide both sides of Equation 9.2 by $\|f+g\|_p^{p-1}$ to obtain Minkowski's inequality. □

If we introduce a metric $d(f,g) = \|f - g\|_p$, we see that L^p would be only a semi-metric space if we did not factor out the equivalence relation indicated in the definition. With this quotient space we have made L^p into a normed vector space. It remains to prove that this space is complete, and is therefore a Banach space.

Theorem 9.1.3. *The normed vector space $L^p(X, \mathfrak{A}, \mu)$ is a Banach space for each real number $p \geq 1$.*

Proof. We will model the proof on that for Theorem 5.5.2. Let f_n be a Cauchy sequence in the L^p-norm. For each $k \in \mathbb{N}$ there exists n_k such that for all n and m greater than or equal to n_k we have

$$\|f_n - f_m\|_p < \frac{1}{4^{\frac{k}{p}}}.$$

In particular

$$\|f_{n_k} - f_{n_{k+1}}\|_p < \frac{1}{4^{\frac{k}{p}}}.$$

Let

$$A_k = \left\{ x \,\middle|\, |f_{n_k}(x) - f_{n_{k+1}}(x)| \geq \frac{1}{2^{\frac{k}{p}}} \right\}.$$

Then

$$\frac{1}{2^k}\mu(A_k) \leq \int |f_{n_k} - f_{n_{k+1}}|^p \, d\mu < \frac{1}{4^k}$$

which implies that

$$\mu(A_k) < \frac{1}{2^k}.$$

Let

$$N = \limsup_k A_k = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k$$

which is the set of all those points x that appear in infinitely many sets A_k. It is easy to calculate that $\mu(N) = 0$ and for all $x \notin N$ the sequence $f_{n_k}(x) \to f(x)$, and f is measurable being the limit almost everywhere of a sequence of measurable functions. We need to prove that $f_n \to f$ in the L^p-norm.

By Fatou's Lemma we have

$$\int |f_{n_k} - f|^p = \int \lim_j |f_{n_k} - f_{n_j}|^p \leq \liminf_j \int |f_{n_k} - f_{n_j}|^p \leq \frac{1}{4^k}$$

so that $f_{n_k} - f \in L^p$ which implies that $f \in L^p$. Also

$$\|f_n - f\|_p \leq \|f_n - f_{n_k}\|_p + \|f_{n_k} - f\|_p$$

which can be made as small as we like by choosing k sufficiently big and $n \geq n_k$. Thus $\|f_n - f\|_p \to 0$ as $n \to \infty$. $\square$

9.2 The Dual of a Banach Space

Let B be a vector space over the field $\mathbb{F}$, which may be either $\mathbb{R}$ or $\mathbb{C}$. Suppose that B is equipped with a norm, as in Definition 5.5.1. We will assume that B is also a Banach space as in Definition 5.5.6. We have seen that for each real number $p \in [1, \infty)$ and for each measure space $(X, \mathfrak{A}, \mu)$ the space $L^p(X, \mathfrak{A}, \mu)$ is a Banach space.

Definition 9.2.1. If B is a Banach space equipped with a *norm* $\|\cdot\|$, we call $T : B \to \mathbb{F}$ a *linear functional* provided

$$T(\alpha\mathbf{x} + \mathbf{y}) = \alpha T(\mathbf{x}) + T(\mathbf{y})$$

for all x and y in B and for all $\alpha \in \mathbb{F}$. A linear functional T is called *continuous* at $\mathbf{x}$ if and only if for each sequence $\mathbf{x}_n$ in B such that $\mathbf{x}_n \to \mathbf{x}$ we have $T(\mathbf{x}_n) \to T(\mathbf{x})$. T is called *continuous* if and only if T is continuous at each $\mathbf{x} \in B$.

Example 9.2.1. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{E}^n$, the vector space $\mathbb{R}^n$ equipped with the Euclidean norm

$$\|\mathbf{x}\| = \sqrt{\sum_{j=1}^{n} x_j^2}.$$

Let $T_j : \mathbb{E}^n \to \mathbb{R}$ by the definition $T_j(\mathbf{x}) = x_j$. Then each T_j is a continuous linear functional on $\mathbb{E}^n$.

We see that the values of the continuous linear functionals T_j at $\mathbf{x}$, $1 \leq j \leq n$, determine $\mathbf{x}$ uniquely. In any n-dimensional vector space Banach space we could fix a basis arbitrarily and use $T_1(\mathbf{x}), \ldots, T_n(\mathbf{x})$ to determine the vector $\mathbf{x}$. Such a basis is not canonical however. A canonical version of this statement could be made as follows.

Let B^* denote the vector space of all continuous linear functionals on B. Then two vectors $\mathbf{x}$ and $\mathbf{y}$ in B are distinct if and only if there exists $T \in B'$ such that $T(\mathbf{x}) \neq T(\mathbf{y})$.

Lemma 9.2.1. *Let B be any Banach space over $\mathbb{F}$, which may be either $\mathbb{R}$ or $\mathbb{C}$, equipped with a norm $\|\cdot\|$. A linear functional $T : B \to \mathbb{F}$ is continuous if and only if T is continuous* at $\mathbf{0}$.

Proof. If T is continuous (at all $\mathbf{x} \in B$) then it must be continuous at $\mathbf{0}$. So we prove the opposite implication. Note that since

$$T(\mathbf{0}) = T(\mathbf{0}+\mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$$

we must have $T(\mathbf{0}) = 0$. Suppose T is continuous at $\mathbf{0}$: That is

$$\|\mathbf{x}_n - \mathbf{0}\| = \|\mathbf{x}_n\| \to 0$$

implies $T(\mathbf{x}_n) \to 0 = T(\mathbf{0})$. Let $\mathbf{x} \in B$ be arbitrary and suppose $\mathbf{x}_n \to \mathbf{x}$: ie, $\|\mathbf{x}_n - \mathbf{x}\| \to 0$. By hypothesis

$$T(\mathbf{x}_n - \mathbf{x}) = T(\mathbf{x}_n) - T(\mathbf{x}) \to 0$$

so $T(\mathbf{x}_n) \to T(\mathbf{x})$. □

Definition 9.2.2. A linear functional T on a Banach space B is called *bounded* if and only if there exists a positive number $K \in \mathbb{R}$ such that $|T(\mathbf{x})| \leq K\|\mathbf{x}\|$, for all $x \in B$.

Theorem 9.2.1. *If T is a linear functional on a Banach space B, then T is continuous if and only if T is* bounded.

Proof. In the direction from right to left, suppose T is bounded. It will suffice to prove T is continuous at $\mathbf{0}$. So suppose $\|\mathbf{x}_n\| \to \mathbf{0}$. Then

$$|T(\mathbf{x}_n)| \leq K\|\mathbf{x}_n\| \to 0.$$

Now suppose T is continuous. We will prove T is bounded by contradiction. So suppose the claim were false. Then for all $n \in \mathbb{N}$ there exists $\mathbf{x}_n \in B$ such that

$$|T(\mathbf{x}_n)\| > n\|\mathbf{x}_n\|.$$

Let

$$\mathbf{y}_n = \frac{\mathbf{x}_n}{\|\mathbf{x}_n\|\sqrt{n}}$$

(since $\mathbf{x}_n \neq \mathbf{0}$) so that $|T(\mathbf{y}_n)| > \sqrt{n}$ for all $n \in \mathbb{N}$. Note also that

$$\|\mathbf{y}_n\| = \frac{1}{\sqrt{n}} \to 0$$

yet $T(\mathbf{y}_n)$ fails to converge to 0: in fact

$$|T(\mathbf{y}_n)| > \sqrt{n}$$

which is unbounded. This is a contradiction. $\square$

Because of this theorem, continuous linear functionals on normed linear spaces are often called *bounded linear functionals*.

Exercise 9.2.1. Let $f \in L^1(X, \mathfrak{A}, \mu)$ and define $T(f) = \int_X f\, d\mu$. Prove that T is a bounded linear functional. Show that there is a smallest bound K and find it.

Exercise 9.2.2. Let V be a *finite*-dimensional real or complex normed vector space. Prove that every linear functional $T : B \to \mathbb{F}$ must be bounded.

Exercise 9.2.3. Let $\mathcal{P}$ be the vector space of all *polynomials* on the interval $[0, 1]$. Define a linear functional $T : \mathcal{P} \to \mathbb{R}$ such that T is not bounded with respect to the sup-norm.

Definition 9.2.3. Let B denote any (real or complex) Banach space. Let B' be the set of all $T : B \to \mathbb{F}$ such that T is linear and bounded. We call B' the *dual space* of B. If $T \in B'$, it has a norm defined by

$$\|T\| = \inf\{K \mid |T(\mathbf{v})| \leq K\|\mathbf{v}\| \ \forall \mathbf{v} \in B\}.$$

Exercise 9.2.4. Show that $|T(\mathbf{v})| \leq \|T\|\,\|\mathbf{v}\|$ for all $\mathbf{v} \in B$.

Theorem 9.2.2. *If B be any Banach space, then B' is a Banach space.*

Proof. The reader will recall from a course in linear algebra that the sum of any two linear maps is linear and that any constant times a linear map is linear, so we will see that B' is a vector space if we can show that the function $\|\cdot\|$ defined on B' is a norm, which will prove also that the sum of two bounded linear functionals is again bounded, and the same for scalar multiples. Observe first that $|T(\mathbf{v})| \leq \|T\| \cdot \|\mathbf{v}\|$ for all $\mathbf{v} \in B$, so that

$$|cT(\mathbf{v})| = |c||T(\mathbf{v})| \leq |c| \cdot \|T\| \cdot \|\mathbf{v}\|.$$

This implies that

$$\|cT\| \leq |c| \cdot \|T\| < \infty.$$

But $T = \frac{1}{c}(cT)$, so that

$$\|T\| \leq \frac{1}{|c|}\|cT\|.$$

Thus $\|cT\| = |c|\|T\|$. Observe next that

$$|(T_1 + T_2)(\mathbf{v})| \leq |T_1(\mathbf{v})| + |T_2(\mathbf{v})| \leq (\|T_1\| + \|T_2\|)\|\mathbf{v}\|.$$

Thus we see that

$$\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$$

To complete the proof that $\|\cdot\|$ is a norm on V', the reader should show that $\|T\| \geq 0$ for all T, and that $\|T\| = 0$ if and only if $T = 0$.

It remains to be shown that B' is complete in the given norm. Let T_n be any Cauchy sequence in B'. Let $\epsilon > 0$. Then there exists N such that m and $n \geq N$ implies $\|T_m - T_n\| < \frac{\epsilon}{2}$. Thus, for all $v \in B$,

$$|T_m(\mathbf{v}) - T_n(\mathbf{v})| < \frac{\epsilon}{2}\|\mathbf{v}\|.$$

Hence $\{T_n(\mathbf{v})\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{F}$ and we can define

$$T(\mathbf{v}) = \lim_{n\to\infty} T_n(\mathbf{v})$$

for all $\mathbf{v} \in B$. The proof that T is linear is an informal exercise for the student. Finally, we must show that T is bounded and that $\|T_n - T\| \to 0$. But if m and $n \geq N$, we know that

$$|T_m(\mathbf{v}) - T_n(\mathbf{v})| \leq \frac{\epsilon}{2}\|\mathbf{v}\|.$$

Letting $n \to \infty$, we see that $|T_m(\mathbf{v}) - T(\mathbf{v})| \leq \frac{\epsilon}{2}\|\mathbf{v}\|$ for all $\mathbf{v} \in B$. Thus $\|T_m - T\| \leq \frac{\epsilon}{2} < \epsilon$, so $T_m \to T$ in norm. Moreover,

$$\|T\| = \|T_m - (T_m - T)\| \leq \|T_m\| + \|T_m - T\| < \infty$$

so T is bounded as claimed. Thus B' is a Banach space. ☐

Exercise 9.2.5. Let $(X, \mathfrak{A}, \mu)$ be a measure space and suppose $1 < p < \infty$ and

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Let $g \in L^q(X, \mathfrak{A}, \mu)$ and define $T_g(f) = \int_X fg\, d\mu$.

a. Prove that T_g is a bounded linear functional on the Banach space $L^p(X, \mathfrak{A}, \mu)$.

b. Prove that $\|T_g\| \leq \|g\|_q$.

Definition 9.2.4. If f is a measurable function on the measure space $(X, \mathfrak{A}, \mu)$, define the *essential supremum* of f and denote it by

$$\|f\|_\infty = \inf\{K \in \mathbb{R} \cup \{\infty\} \mid |f(x)| \leq K \text{ almost everywhere}\}.$$

Define $L^\infty(X, \mathfrak{A}, \mu)$ to be the set of all measurable functions with finite essential supremum.

Exercise 9.2.6. Let (X, A, μ) be a measure space.

a. Prove that $L^\infty(X, \mathfrak{A}, \mu)$ is a Banach space.

b. Prove that for each $g \in L^\infty(X, \mathfrak{A}, \mu)$ the function

$$T_g(f) = \int_X f g \, d\mu$$

is a bounded linear functional on the Banach space $L^1(X, \mathfrak{A}, \mu)$.

c. Prove that $\|T_g\| \leq \|g\|_\infty$.

9.3 The Dual Space of $L^p(X, \mathfrak{A}, \mu)$

In Exercises 9.2.5 and 9.2.6 we saw that for each $g \in L^q(X, \mathfrak{A}, \mu)$ there is a corresponding bounded linear functional

$$T_g : f \to \int_X f g \, d\mu$$

acting on $L^p(X, \mathfrak{A}, \mu)$, provided[2] that $1 \leq p < \infty$ and $1 < q \leq \infty$ and

$$\frac{1}{p} + \frac{1}{q} = 1.$$

In this section we will prove that all bounded linear functionals on L^p arise in this way, thereby characterizing the dual space of L^p.

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[2] The equation that follows is interpreted informally when $p = 1$ and $q = \infty$.

Theorem 9.3.1. *Let $(X, \mathfrak{A}, \mu)$ be any σ-finite measure space. Let $1 \leq p < \infty$ and suppose that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Let T be any bounded linear functional on the Banach space $L^p(X, \mathfrak{A}, \mu)$. Then there exists a unique *function g in $L^q(X, \mathfrak{A}, \mu)$ such that*

$$T(f) = \int_X fg \, d\mu = T_g(f)$$

for all $f \in L^p(X, \mathfrak{A}, \mu)$. Moreover

$$\|T_g\| = \|g\|_q$$

and the map $g \to T_g$ is a Banach space isomorphism *of L^q onto the dual space of L^p.*

Remark 9.3.1. This theorem can be described as a *representation theorem*, because it represents the dual space $L^p(X)'$ as being $L^q(X)$. It is clear that the mapping $g \to T_g$ is linear, so that this mapping will be a Banach space isomorphism if it is onto and norm-preserving.

Proof. We will break the proof into three cases.

i. Suppose $\mu(X) < \infty$ and suppose also that $p > 1$, so that $q < \infty$.

   For each set $E \in \mathfrak{A}$ define $\sigma(E) = T(1_E)$. (Note that $1_E \in L^p$ because $\mu(X) < \infty$, so that 1_E does lie in the domain of definition of T.) We will show that σ is a countably additive set function on $\mathfrak{A}$. Let

   $$E = \dot{\bigcup}_{k \in \mathbb{N}} E_k$$

   be a disjoint union of sets in $\mathfrak{A}$. Let

   $$A_n = \bigcup_1^n E_k.$$

   Since $\sum_k \mu(E_k) = \mu(E) < \infty$ we know that $\mu(E \setminus A_n) \to 0$ as $n \to \infty$. It follows that

   $$\|1_{A_n} - 1_E\|_{L^p(X, \mathfrak{A}, \mu)} \to 0$$

as $n \to \infty$. Since T is bounded and thus continuous, we have

$$\sigma(E) = T(1_E) = \lim_n T(1_{A_n}) = \lim_n \sigma(A_n) = \sum_1^\infty \sigma(E_k)$$

proving that σ is countably additive on $\mathfrak{A}$.

Moreover, if $\mu(E) = 0$ then $\|1_E\|_p = 0$ forcing $T(1_E) = \sigma(E) = 0$. Thus σ is absolutely continuous with respect to μ. By the Radon-Nikodym theorem (Theorem 8.2.1) there exists a function $g \in L^1(X, \mathfrak{A}, \mu)$ such that

$$\sigma(E) = T(1_E) = \int_X 1_E \, g \, d\mu$$

for all $E \in \mathfrak{A}$. It follows easily that for each $\phi \in \mathfrak{S}_0$ we have

$$T(\phi) = \int_X \phi \, g \, d\mu.$$

Now let $f \in L^p$, which we can suppose without loss of generality to be non-negative as well. Then there exists a sequence $\phi_n \in \mathfrak{S}_0$ which increases pointwise towards f as a pointwise limit almost everywhere. It follows that $\|f - \phi_n\|_p \to 0$ as $n \to \infty$. Hence $T(\phi_n) \to T(f)$. Thus

$$T(f) = \lim_n T(\phi_n) = \lim_n \int_X \phi_n \, g \, d\mu = \int_X f g \, d\mu.$$

Thus $T = T_g$ as claimed.

We need to prove that $g \in L^q$ and that $\|T_g\| = \|g\|_q$. We define a *truncated* function f_n in such a way that $f_n g$ approximates $|g|^q$ from below as follows:

$$f_n(x) = \begin{cases} |g(x)|^{q-1} \operatorname{sgn} g(x) & \text{if } |g(x)|^{q-1} \leq n \\ n \operatorname{sgn} g(x) & \text{if } |g(x)|^{q-1} > n \end{cases}$$

for each $n \in \mathbb{N}$. Being bounded on a space of finite measure, f_n lies in L^p for each $n \in \mathbb{N}$. Thus

$$|T(f_n)| = \left| \int_X f_n g \, d\mu \right| \leq \|T\| \|f_n\|_p.$$

But

$$f_n g = |f_n||g| \geq |f_n||f_n|^{\frac{1}{q-1}} = |f_n|^p.$$

Thus

$$\int_X |f_n|^p \, d\mu \leq \|T\| \left(\int_X |f_n|^p \, d\mu \right)^{\frac{1}{p}}$$

Hence

$$\|T\| \geq \left(\int_X |f_n|^p \, d\mu \right)^{1-\frac{1}{p}} = \left(\int_X |f_n|^p \, d\mu \right)^{\frac{1}{q}} \to \|g\|_q.$$

The opposite inequality is contained in Exercise 9.2.5, and this completes the proof of the first case.

ii. Suppose again that $\mu(X) < \infty$, but that $p = 1$, so that $q = \infty$.

We obtain $Tf = \int_X fg \, d\mu$ as before, with $g \in L^1$. We need to show that $g \in L^\infty$, which will be contained in L^1 since $\mu(X) < \infty$. Suppose this were false. Then for each $K > 0$ the set

$$A_k = \{x \mid |g(x)| > K\}$$

has strictly positive measure. Define

$$f_K = \frac{1}{\mu(A_k)} 1_{A_k} \operatorname{sgn} g$$

so that $\|f_K\|_1 = 1$. Then we would have

$$|T(f_K)| = \int_X f_K g \, d\mu > K$$

for all $K > 0$. This contradicts the boundedness of T. Hence $g \in L^\infty$ as claimed. The reader will complete the proof in Exercise 9.3.1.

iii. Suppose $\mu(X) = \infty$ but $X = \bigcup_1^\infty A_k$ where $A_k \subseteq A_{k+1}$ for all k and $\mu(A_k) < \infty$.

In this case

$$T : f|_{A_k} \to \int_X f|_{A_k} \, g_k \, d\mu$$

determines $g_k \in L^q(A_k)$ and $\|g_k\|_q \leq \|T\|$ for each $n \in \mathbb{N}$. It is clear that g_k *extends* g_{k-1} for each k, so that $g = \lim_k g_k$ exists pointwise. By Exercise 9.3.2 we know that $T(f) = \int_X f\, g\, d\mu$.

We need to show that $g \in L^q$. The functions g_k^+ increase towards the limit g^+ and also $g_k^- \nearrow g^-$. Thus

$$\int_X |g|^q\, d\mu = \infty \Leftrightarrow \lim_k \int_X |g_k|^q\, d\mu = \infty$$

because of the Monotone Convergence theorem (Theorem 5.4.1). Thus $\int_X |g|^q\, d\mu < \infty$ if $q < \infty$.

If $q = \infty$, we have $\|g_k\|_\infty < \|T\|$ for all n. Thus $\|g\|_\infty < \infty$ also.

□

Exercise 9.3.1. † Complete the proof of case 2 of Theorem 9.3.1 by proving that $\|T\| = \|g\|_\infty$.

Exercise 9.3.2. † Complete the proof of case 3 of Theorem 9.3.1 by proving that $T(f) = \int_X fg\, d\mu$ for all $f \in L^p(X, \mathfrak{A}, \mu)$.

9.4 Riesz-Markov-Saks-Kakutani Theorem

Early in the twentieth century, Frigyes Riesz discovered a full classification (or *representation*) of the dual space for the vector space $\mathcal{C}[a,b]$, consisting of the continuous functions on $[a,b]$ and equipped with the L^∞ norm, or sup-norm. This space with the given norm it is a complete normed linear space and thus a Banach space in modern terminology. Riesz proved that each bounded linear functional on that space can be described as $T(f) = \int_a^b f\, d\mu$ using a suitable finite Borel measure μ. Later, Andrei A. Markov extended this theorem to the compactly supported functions on the infinite real line. A version was proven by Stanislaw Saks for $\mathcal{C}(X)$ in [13] with the hypothesis that X is a compact metric space. And Shizuo Kakutani generalized the theorem to cover the vector space of all continuous functions on any compact Hausdorff space. Kakutani produced this theorem as part of a paper [7] that provides a classification of all objects known as Banach lattices. *We will prove the Kakutani version of the theorem.* The proof we give is adapted slightly from the one presented by Kakutani in [7]. (Given its august lineage,