If $g(x,t) = \dfrac{\partial f}{\partial t}$, prove or disprove:

$$\int_{-\infty}^{\infty} \int_{s}^{\infty} g(x,t)\,dt\,dx \neq \int_{s}^{\infty} \int_{-\infty}^{\infty} g(x,t)\,dx\,dt.$$

What is the relevance of this example to the Fubini theorem?

6.20 Let $X = Y = [0,1]$. Let μ be Lebesgue measure on X and let λ be counting measure on Y. Let

$$f(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

Show that

$$\int_X \int_Y f(x,y)\,d\mu(x)\,d\lambda(y) \neq \int_Y \int_X f(x,y)\,d\lambda(y)\,d\mu(x)$$

and explain why this does not violate Fubini's theorem.

6.3 COMPARISON OF LEBESGUE AND RIEMANN INTEGRALS

Riemann integration corresponds to the concept of Jordan measure in a manner that is similar (but not identical) to the correspondence between the Lebesgue integral and Lebesgue measure. Although it is possible for an unbounded function to be Lebesgue integrable, this cannot occur with proper Riemann integration. Moreover, proper Riemann integrals are defined only for functions with a bounded domain D. Since a bounded domain D can always be contained in a rectangular block with edges parallel to the axes, and since we can let f be identically zero on the part of the block that is outside D, we will assume that f is defined on such a block. We will denote such a block by the suggestive notation $[\mathbf{a}, \mathbf{b}]$, with the understanding that $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$. Then the symbol we have chosen for a block has the form

$$[\mathbf{a}, \mathbf{b}] = \prod_{i=1}^{n} [a_i, b_i],$$

a Cartesian product of closed, finite intervals.

Let f be any bounded real–valued function on $[\mathbf{a}, \mathbf{b}]$. Since $f = f^+ - f^-$, a difference between two nonnegative functions, it will suffice to deal with the Riemann integration of nonnegative bounded functions f.[52]

Let Δ denote a partition of $[\mathbf{a}, \mathbf{b}]$ into the union of N rectangular blocks, the *interiors* of which are mutually disjoint:

$$\Delta = \{[\mathbf{x}_i, \mathbf{y}_i] \mid i = 1, \ldots, N\}.$$

[52] We assume the reader knows from advanced calculus that the positive and negative parts of a Riemann integrable function must be Riemann integrable. See [18].

Let Δx_i denote the *volume* of the box $[\mathbf{x}_i, \mathbf{y}_i]$. On each of the N blocks $[\mathbf{x}_i, \mathbf{y}_i]$ we let $m_i = \inf\{f(x) \mid x \in [\mathbf{x}_i, \mathbf{y}_i]\}$ and $M_i = \sup\{f(x) \mid x \in [\mathbf{x}_i, \mathbf{y}_i]\}$. We form the so–called lower and upper sums

$$s(\Delta) = \sum_1^N m_i \Delta x_i,$$

and

$$S(\Delta) = \sum_1^N M_i \Delta x_i.$$

Then we define the so–called lower and upper Riemann integrals by

$$\underline{\int_{\mathbf{a}}^{\mathbf{b}}} f(x)\,dx = \sup_{\Delta} s(\Delta),$$

which is a supremum over all possible finite partitions Δ of $[\mathbf{a}, \mathbf{b}]$, and

$$\overline{\int_{\mathbf{a}}^{\mathbf{b}}} f(x)\,dx = \inf_{\Delta} S(\Delta).$$

Definition 6.3.1 A bounded real–valued function f defined on $[\mathbf{a}, \mathbf{b}]$ is called Riemann integrable if and only if

$$\overline{\int_{\mathbf{a}}^{\mathbf{b}}} f(x)\,dx = \underline{\int_{\mathbf{a}}^{\mathbf{b}}} f(x)\,dx.$$

In the case of equality, this value is called $\displaystyle\int_{\mathbf{a}}^{\mathbf{b}} f(x)\,dx.$ [53]

Theorem 6.3.1 *(*Lebesgue's theorem*) A bounded real–valued function f on $[\mathbf{a}, \mathbf{b}]$ is Riemann integrable if and only if the set of points x at which f is* not continuous *is a Lebesgue null set.*

Proof: Without loss of generality, suppose that f is a nonnegative bounded function on $[\mathbf{a}, \mathbf{b}]$ and let

$$C(f) = \{(x, y) \mid 0 \leqslant y \leqslant f(x)\},$$

the region between the graph of f and the block $[\mathbf{a}, \mathbf{b}]$ in the $\mathbb{R}^n$. Observe that the Jordan inner and outer measure of $C(f)$ corresponds as follows to the upper and

[53]The reader who wishes to learn more about the Riemann integral in $\mathbb{R}^n$ can consult an advanced calculus text, such as the author's book, [18].

lower Riemann integrals of f:

$$\underline{v}(C(f)) = \underline{\int_{\mathbf{a}}^{\mathbf{b}}} f(x)\,dx = \sup_{0 \leqslant g \leqslant f} \int_{\mathbf{a}}^{\mathbf{b}} g(x)\,dx, \tag{6.6}$$

$$\overline{v}(C(f)) = \overline{\int_{\mathbf{a}}^{\mathbf{b}}} f(x)\,dx = \inf_{f \leqslant g} \int_{\mathbf{a}}^{\mathbf{b}} g(x)\,dx, \tag{6.7}$$

where g varies over the *step*-functions.[54] It follows that f is Riemann integrable if and only if $C(f)$ is Jordan-measurable, and the latter condition is equivalent to the *boundary* $\partial C(f)$ being a Jordan null-set, according to Theorem 3.6.3. Since the boundary is a closed set, this is equivalent to $\partial C(f)$ being a Lebesgue null set.

On the other hand, by Fubini's theorem, $\partial C(f)$ is a Lebesgue null set in the plane if and only if the x–section ${}_x\partial C(f)$ has linear Lebesgue measure equal to zero for almost all $x \in [\mathbf{a}, \mathbf{b}]$.

Suppose there were at least two cluster points, (x, y_1) and (x, y_2), of the graph of f with $y_1 < y_2$. It would follow that

$${}_x\partial C(f) \supseteq \{(x, y) \mid y_1 \leqslant y \leqslant y_2\},$$

so that ${}_x\partial C(f)$ has strictly positive linear Lebesgue measure because it contains the set

$$\{(x, y) \mid y_1 \leqslant y \leqslant y_2\}.$$

However, the reader can prove[55] as an exercise in advanced calculus that ${}_x\partial C(f)$ contains an interval of strictly positive length if and only if f fails to be continuous at x. See Exercise 6.23. ■

■ EXAMPLE 6.1

Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be defined by

$$f(x_1, x_2) = \begin{cases} \frac{1}{2} \sin \frac{\pi}{x_1 x_2} & \text{if } 0 < x_i \leqslant 1,\ i = 1, 2 \\ 0 & \text{if } x_1 x_2 = 0. \end{cases}$$

We claim that f is Riemann integrable on the closed rectangular box

$$[(0, 0), (1, 1)] = [0, 1]^2.$$

(See Figure 6.1[56].) In fact f is bounded and continuous except at the points on the two axes:

$$S = \{(x_1, x_2) \mid x_1 x_2 = 0\}.$$

Since S is a Lebesgue null set in the Lebesgue measure on $\mathbb{R}^2$, Lebesgue's theorem implies the integrability of f.

[54] In this context, by a step function we mean a finite linear combination of indicator functions of rectangular boxes.

[55] See, for example, [18].

[56] This illustration is from the author's book *Advanced Calculus: An Introduction to Linear Analysis* [18]

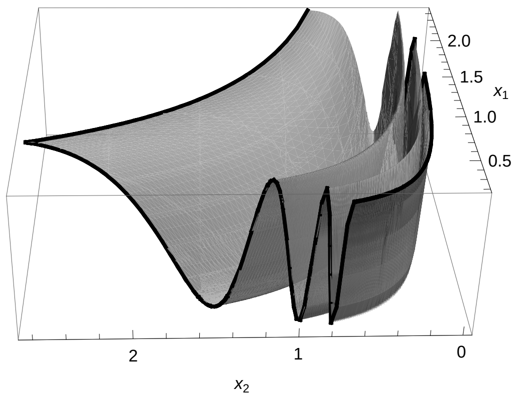

Figure 6.1 $f(\mathbf{x}) = \frac{1}{2}\sin\frac{\pi}{x_1 x_2}$

We remark that a Fubini theorem for the Riemann integral is much less general and more cumbersome in its statement than is the case for the Lebesgue integral. One reason for this is that one cannot be assured of the existence of the iterated integrals, and far fewer functions are Riemann integrable than Lebesgue integrable.

EXERCISES

6.21 Suppose $f : [0,1] \to \mathbb{R}$ is given by $f = 1_{\mathbb{Q}\cap[0,1]}$. Prove that f is not Riemann integral by applying Lebesgue's theorem.

6.22 Let $f : [0,1] \to \mathbb{R}$ by letting

$$f = 1_{\left\{\frac{1}{n}\,\middle|\, n\in\mathbb{N}\right\}}.$$

Prove by using Lebesgue's theorem that f is Riemann integrable.

6.23 Let $f : [\mathbf{a}, \mathbf{b}] \to \mathbb{R}$ be a nonnegative function defined on a block in Euclidean space. Define

$$C(f) = \{(\mathbf{x}, y) \mid 0 \leqslant y \leqslant f(\mathbf{x})\}$$

as in the proof of Theorem 6.3.1. Prove by the following steps that f is continuous $\mathbf{x}$ if and only if ${}_{\mathbf{x}}\partial C(f)$ contains no interval of positive length.

a) Show that if f is continuous at $\mathbf{p}$, then ${}_{\mathbf{p}}\partial C(f)$ consists of only two points.

b) Show that if it is false that $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})$ exists and equals $f(\mathbf{p})$, then ${}_{\mathbf{p}}\partial C(f)$ contains an interval of positive length.