

Chapter 6

Product Measures and Fubini's Theorem

All students learn in elementary calculus to evaluate a double integral by iteration. The theorem justifying this process is called Fubini's theorem. However, the cleanest and simplest form of Fubini's theorem appears for the first time with the Lebesgue integral.

6.1 Product Measures

We will assume here that we are given two measure spaces, which may or may not be identical to one another. We let the two measure spaces be called $(X, \mathfrak{A}, \lambda)$ and $(Y, \mathfrak{B}, \mu)$. We intend to construct the product measure on a suitable σ -field contained in the power set of the Cartesian product $Z = X \times Y$. By a *rectangular set* R in Z we mean any set of the form $R = A \times B$ where $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. We will take as the family of *elementary sets* for the product measure

$$\mathfrak{E} = \left\{ E \in \mathfrak{P}(X \times Y) \mid E = \bigcup_{k=1}^n R_k = \bigcup_{i=1}^n A_i \times B_i, A_i \in \mathfrak{A}, B_i \in \mathfrak{B} \right\} \quad (6.1)$$

where R_i is a rectangular set for each i , and n is an arbitrary natural number.

Exercise 6.1.1. Show that the set $\mathfrak{E}$ of Equation 6.1 is a field of subsets of $Z = X \times Y$.

Definition 6.1.1. Define the product measure

$$\nu(E) = \sum_{i=1}^n \lambda(A_i) \mu(B_i)$$

for each elementary set $E \in \mathfrak{E}$ as defined by Equation 6.1.

This definition requires justification from the following theorem.

Theorem 6.1.1. *The product measure ν is well-defined on $\mathfrak{E}$ by Definition 6.1.1, despite the fact that the decomposition given in Equation 6.1 is not unique.*

Proof. Suppose that we have

$$E = \bigcup_{i=1}^n A_i \times B_i = \bigcup_{i=1}^m C_i \times D_i \in \mathfrak{E}.$$

The reader should verify that without loss of generality we can assume that the sets A_i are mutually disjoint, and similarly for the sets B_i , the sets C_i , and the sets D_i in turn. Furthermore, we can write

$$C_i = \bigcup_{j=1}^n C_i \cap A_j = \bigcup_{j=1}^n Q_{ij} \text{ and } A_j = \bigcup_{i=1}^m C_i \cap A_j$$

where the sets Q_{ij} are mutually disjoint. Define mutually disjoint sets R_{ij} similarly for the sets B_i and D_i . Then we have

$$\begin{aligned} E &= \sum_{ij} Q_{ij} \times R_{ij} \\ &= \sum_{i=1}^m \left(\sum_{j=1}^n Q_{ij} \right) \times \left(\sum_{j=1}^n R_{ij} \right) = \sum_{j=1}^n \left(\sum_{i=1}^m Q_{ij} \right) \times \left(\sum_{i=1}^m R_{ij} \right) \end{aligned}$$

and theorem is now established by the additivity of λ and μ . $\square$

The theorem just proven establishes also the *finite additivity* of $\nu = \lambda \times \mu$ on $\mathfrak{E}$. This finite additivity is built in to Definition 6.1.1. It was necessary only to know that the definition is valid, meaning that it is independent of how the elementary set is decomposed into a union of *disjoint* rectangular sets. Our next goal is to show that ν can be extended to a countably additive measure on $\mathfrak{C} = \mathbb{B}(\mathfrak{E})$, the σ -field generated by $\mathfrak{E}$. It will be helpful to make the following definition.

Definition 6.1.2. If $S \subseteq X \times Y$, a Cartesian product space, we define the x -section of S by

$${}_xS = \{y \mid (x, y) \in S, y \in Y\}$$

and the y -section by

$$S_y = \{x \mid (x, y) \in S, x \in X\}.$$

Theorem 6.1.2. *There exists a countably additive measure ν defined on $\mathfrak{C} = \mathbb{B}(\mathfrak{C})$ such that $\nu(A \times B) = \lambda(A)\mu(B)$ for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. Moreover, ν is unique provided that λ and μ are both σ -finite.*

Proof. Uniqueness will follow at the end from Exercise 2.4.1, since ν will be σ -finite if both λ and μ are σ -finite. Existence depends upon proving countable additivity *within* $\mathfrak{C}$.

Each elementary set is a *disjoint* union of finitely many rectangular sets of the form

$$E = \bigcup_{i=1}^n A_i \times B_i$$

with each A_i and each B_i measurable. To prove the countable additivity within $\mathfrak{C}$ it will suffice to prove for each of the sets of the form

$$R = A \times B = \dot{\bigcup}_{n \in \mathbb{N}} A_n \times B_n$$

that $\nu(A \times B) = \sum_{n \in \mathbb{N}} \lambda(A_n)\mu(B_n)$. Next define

$$f_n(x) = \mu({}_x(A_1 \times B_1 \cup \dots \cup A_n \times B_n))$$

so that $f_n \in \mathfrak{S}$ is a non-negative simple function. Note that although the sets B_n need not be disjoint, we do have

$${}_x(A \times B) = \dot{\bigcup}_{n \in \mathbb{N}} {}_x(A_n \times B_n)$$

which is a disjoint union. Since μ is countably additive, it follows that the sequence f_n increases monotonically toward the limit f , where

$$f(x) = \mu({}_xR) = \begin{cases} \mu(B) & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

By the Monotone Convergence Theorem (Theorem 5.4.1) we see that

$$\nu(R) = \int f d\lambda = \lim_{n \rightarrow \infty} \int f_n d\lambda = \sum_{n \in \mathbb{N}} \lambda(A_n) \mu(B_n)$$

and this is true even if $\nu(R) = \infty$. $\square$

Note that in the preceding proof we have applied the Monotone Convergence only to non-negative measurable functions defined on X , on which there is already a countably additive measure λ . (We are not applying it to functions defined on the product space, for which we are establishing the existence of a countably additive measure.) We remark that we did not need to use the hypothesis of completeness that appeared in the Monotone Convergence theorem because in the preceding proof f_n converges *everywhere* to f - not merely almost everywhere.

Definition 6.1.3. We will denote the Cartesian (or cross) product

$$\mathfrak{A} \times \mathfrak{B} = \{A \times B \mid A \in \mathfrak{A}, B \in \mathfrak{B}\}$$

and the field that it generates is called $\mathfrak{E}$. But we will denote by

$$\mathfrak{A} \otimes \mathfrak{B} = \mathbb{B}(\mathfrak{A} \times \mathfrak{B})$$

the Borel field generated by the field $\mathfrak{E}$. Finally, we denote by

$$\mathfrak{A} \otimes \mathfrak{B}$$

the *completion* of $\mathfrak{A} \otimes \mathfrak{B}$.

It is reasonable to wonder whether it is necessary to form the completion $\mathfrak{A} \otimes \mathfrak{B}$ if both $\mathfrak{A}$ and $\mathfrak{B}$ happen to be complete families of measurable sets for their respective measures. The answer—that the completion needs to be formed—is provided by the following special case.

Let $X = Y = \mathbb{R}$ and $Z = X \times Y = \mathbb{R}^2$, the Euclidean plane. Here we will assume that $\lambda = \mu = l$, Lebesgue measure on the line. And $\mathfrak{A} = \mathfrak{B} = \mathcal{L}(\mathbb{R})$ will be the σ -field of all Lebesgue measurable sets in the line.

Theorem 6.1.3. *With the notations of Theorem 6.1.3, we have*

$$\mathbb{B}(\mathbb{R}^2) \stackrel{(i)}{\subsetneq} \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R}) \stackrel{(ii)}{\subsetneq} \mathcal{L}(\mathbb{R}^2) \stackrel{(iii)}{=} \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R}).$$

Proof.

- i. Since the elementary sets of the plane are contained in $\mathcal{L}(\mathbb{R}) \times \mathcal{L}(\mathbb{R})$, it follows that $\mathbb{B}(\mathbb{R}^2) \subseteq \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$. Observe that the family

$$\mathcal{S} = \left\{ S \subseteq \mathbb{R}^2 \mid S_y \in \mathfrak{B}(\mathbb{R}) \forall y \in \mathbb{R} \right\}$$

is a σ -field in the plane containing the elementary sets of the plane, as defined in Section 3.5. Thus it contains all the Borel sets of the plane. On the other hand, by Remark 3.3.1 there exists a set

$$E \in \mathcal{L}(\mathbb{R}) \setminus \mathbb{B}(\mathbb{R})$$

so that

$$E \times \mathbb{R} \in (\mathcal{L}(\mathbb{R}) \setminus \mathbb{B}(\mathbb{R}^2)) \otimes \mathcal{L}(\mathbb{R}).$$

Hence there is a set $E \times \mathbb{R}$ that is in $\mathcal{L}(\mathbb{R}) \times \mathcal{L}(\mathbb{R})$ but not in $\mathfrak{B}(\mathbb{R}^2)$. This proves the first (improper) containment.¹

- ii. For the second improper containment, let M be any non-measurable subset of $\mathbb{R}$. Then the planar Lebesgue measure $l^{(2)}(\{x\} \times M) = 0$ for each singleton set $\{x\}$ in $\mathbb{R}$, being a subset of a plane Lebesgue null set. Hence

$$\{x\} \times M \in \mathcal{L}(\mathbb{R}^2).$$

Thus it would suffice to show that $\{x\} \times M \notin \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$. Consider the class $\mathcal{S}$ of all sets $E \subset \mathbb{R}^2$ for which $_xE \in \mathcal{L}(\mathbb{R})$ for a fixed $x \in \mathbb{R}$. Then it is easy to check that $\mathcal{S}$ is a monotone class, which implies that $\mathcal{S}$ contains the σ -field $\mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$. But since the set $\{x\} \times M$ lacks this property, it follows that $\{x\} \times M \notin \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$.

- iii. Since $\mathcal{L}(\mathbb{R}^2)$ is the (*minimal*) completion of $\mathbb{B}(\mathbb{R}^2)$, it must be also the completion of $\mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$, as is $\mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$.

□

¹The strict containment (i) could be proven also by a cardinality argument, since the transfinite cardinal number of the right-hand side is greater than that of the left.