Chapter 4

Measurable Functions

If X is a set and $\mathfrak{A} \subseteq \mathfrak{P}(X)$ is a σ-field, then $(X, \mathfrak{A})$ is called a *measurable space*. If μ is a countably additive measure defined on $\mathfrak{A}$ then $(X, \mathfrak{A}, \mu)$ is called a *measure space*. In this chapter we will introduce the family of *measurable functions* for which we will seek to define the Lebesgue integral. We will prove the very important fact that pointwise limits of measurable functions must be measurable. This is encouraging because pointwise limits of Riemann integrable functions need not be Riemann integrable.[1]

4.1 Measurable Functions

Definition 4.1.1. Let $(X, \mathfrak{A}, \mu)$ be a measure space.

i. If $f : X \to \mathbb{R}$ we say that f is $\mathfrak{A}$-*measurable* provided that

$$f^{-1}(-\infty, a) = \{x \in X \mid f(x) < a\} \in \mathfrak{A}$$

for all $a \in \mathbb{R}$.[2]

ii. If $f : X \to \mathbb{C}$, the complex numbers, we write $f(x) = u(x) + iv(x)$ for real-valued functions $u = \Re f$ and $v = \Im f$. We say f is $\mathfrak{A}$-*measurable* provided that both u and v are $\mathfrak{A}$-measurable.

iii. If $f : X \to S$, where S is a *topological space*, we say that f is $\mathfrak{A}$-measurable provided that $f^{-1}(G) \in \mathfrak{A}$ for every *open* set $G \subseteq S$.

[1] See Example 1.2.3.

[2] This definition is motivated by Section 1.3.

It is necessary to prove that the three parts of the definition of measurability of a function are consistent.

Exercise 4.1.1. Show that if $f : X \to \mathbb{R}$ is $\mathfrak{A}$-measurable, then $f^{-1}(G) \in \mathfrak{A}$ for every open set $G \subseteq \mathbb{R}$. Show that the concepts of measurability in Definition 4.1.1 for both real and complex valued functions are consistent with the concept of measurability for a topological space valued function. For example, show that $f : X \to \mathbb{R}$ satisfies the definition of measurability if and only if $f^{-1}(G)$ is measurable for each open set $G \subseteq \mathbb{R}$.

The reader may note correctly that the concept of measurability has a formal similarity to the definition of continuity. A function between topological spaces is called continuous provided that the inverse image of each open set is open. Of course for measurability the inverse image of an open set must be measurable, and measurability is by no means synonymous with open. In $\mathbb{R}^n$, every open set is Lebesgue measurable, but the converse is clearly false. The following exercise gives a very useful equivalent form of measurability for functions.

Exercise 4.1.2. Let $f : X \to \mathbb{R}^n$, which can be regarded as an example of a topological space. Prove that f is measurable if and only if $f^{-1}(B) \in \mathcal{L}$ for each Borel set B. (Hint: Show that the family

$$\mathfrak{S} = \{A \in \mathfrak{P}(\mathbb{R}^n) \mid f^{-1}(A) \in \mathcal{L}\}$$

is a σ-field.)

It will be important to know that many combinations of measurable functions and many functions of measurable functions are again measurable. To investigate this we need the following definition.

Definition 4.1.2. Let $(X, \mathfrak{A}, \mu)$ be any measure space arising from the Hopf extension theorem, so that we have a concept of the field of Borel sets generated by a field of elementary sets. If $f : X \to S$, where S is a *topological space*, we say that f is a *Baire function* provided that $f^{-1}(G)$ is a Borel set for each *open* set $G \subseteq S$.

Clearly, every Baire function is measurable and every continuous function (from $\mathbb{R}^n$ to S) is a Baire function. The indicator function of a measurable set that is not a Borel set would be an example of a measurable function that is not a Baire function.

Theorem 4.1.1. *Suppose each of the functions $f_1, f_2, \ldots, f_n$ is an $\mathfrak{A}$-measurable real-valued function defined on X. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a Baire function. Then $F = \Phi(f_1, f_2, \ldots, f_n)$ is an $\mathfrak{A}$-measurable function defined on X.*

Proof. We need to show that for each open set $G \subseteq \mathbb{R}$ we have $F^{-1}(G) \in \mathfrak{A}$. Denote $f = (f_1, f_2, \ldots, f_n) : X \to \mathbb{R}^n$. We claim that f is measurable. In fact, each open set $G \subseteq \mathbb{R}^n$ can be written as a disjoint union of countably many open blocks of the form

$$B = \prod_{i=1}^{n} (a_i, b_i)$$

which is a Cartesian product of n open intervals. Thus

$$f^{-1}(B) = \bigcap_{i=1}^{n} f_i^{-1}(a_i, b_i) \in \mathfrak{A}.$$

Then

$$F^{-1}(G) = (\Phi \circ f)^{-1}(G) = f^{-1}\left(\Phi^{-1}(G)\right).$$

Since $\Phi^{-1}(G)$ is a Borel set, the theorem is true by the result of Exercise 4.1.2. □

We remark that in Theorem 4.1.1 it would have sufficed to have Φ defined on a set $D \subseteq \mathbb{R}^n$ provided $(f_1, \ldots, f_n) : X \to D$.

Remark 4.1.1. It follows from Theorem 4.1.1 that such combinations of measurable functions as the following must be measurable.

- $c_1 f_1 + c_2 f_2$ where c_1 and c_2 are constants
- $f_1 \cdot f_2$, the product of two measurable functions
- $\frac{f_1}{f_2}$ where f_2 is nowhere zero

In particular, $-f_1$ is also measurable. Thus f is measurable if and only if $\{x \mid f(x) > \alpha\} \in \mathfrak{A}$ for every $\alpha \in \mathbb{R}$.

Corollary 4.1.1. *If f_1 and f_2 are $\mathfrak{A}$-measurable real-valued functions, then each of the following functions is $\mathfrak{A}$-measurable.*

i. $\max(f_1, f_2) = f_1 \vee f_2$, *where* $f_1 \vee f_2(x) = \max(f_1(x), f_2(x))$ *for each* $x \in X$.

ii. $\min(f_1, f_2) = f_1 \wedge f_2$*, where* $f_1 \wedge f_2(x) = \min(f_1(x), f_2(x))$ *for each* $x \in X$*.*

iii. $f^+ = f \vee 0$*, known as the* positive part *of* f*.*

iv. $f^- = (-f) \vee 0$*, known as the* negative part *of* f*. (The reader should note that the negative part of a real-valued function is positive.)*

v. $|f| = f^+ + f^-$

Proof. It suffices to observe that $\max(x_1, x_2)$ and $\min(x_1, x_2)$ are both continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$, making each of these a Baire function. For the last part we use the fact that $\Phi(x_1, x_2) = x_1 + x_2$ is continuous and thus a Baire function. □

4.2 Limits of Measurable Functions

In the study of point-wise limits of measurable functions and integrable functions, we will consider sequences of functions for which $f_n(x)$ diverges to $\pm\infty$ for some values of x. Thus it is helpful to extend the concept of real numbers to the set $\mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$. Measurability for an extended real valued function means that for each $\alpha \in \mathbb{R}$ the set

$$f^{-1}[\alpha, \infty] = \{x \mid f(x) \geq \alpha\} \in \mathfrak{A}.$$

Theorem 4.2.1. *Let* $(X, \mathfrak{A}, \mu)$ *be a measure space and let* $\{f_n \mid n \in \mathbb{N}\}$ *be any sequence of measurable functions from* X *to* $\mathbb{R}^*$*. Then each of the five functions defined as follows is* $\mathfrak{A}$*-measurable.*

i. $f_*(x) = \inf\{f_n(x) \mid n \in \mathbb{N}\}$ *for all* $x \in X$

ii. $f^*(x) = \sup\{f_n(x) \mid n \in \mathbb{N}\}$ *for all* $x \in X$

iii. $\underline{f}(x) = \liminf f_n(x)$ *for all* $x \in X$

iv. $\bar{f}(x) = \limsup f_n(x)$ *for all* $x \in X$

v. $f(x) = \lim_{n\to\infty} f_n(x)$ *provided the limit exists for all* $x \in X$*.*

Proof. For the first part, we observe that each $a \in \mathbb{R}$,

$$f_*^{-1}[\alpha, \infty] = \{x \mid \inf_n f_n(x) \geq \alpha\} = \bigcap_{n \in \mathbb{N}} f_n^{-1}[\alpha, \infty] \in \mathfrak{A}.$$

Thus f_* is $\mathfrak{A}$-measurable. Since

$$f^*(x) \equiv -\inf_n(-f_n(x))$$

it follows that f^* is $\mathfrak{A}$-measurable as well. Note next that since

$$i_n = \inf\{f_k(x) \mid k \geq n\}$$

is an increasing sequence of extended real numbers i_n, we have

$$\liminf f_n(x) = \underline{f}(x) = \sup\{\inf\{f_k(x) \mid k \geq n\} \mid n \in \mathbb{N}\}$$

so that $\underline{f}$ is $\mathfrak{A}$-measurable being the supremum of a sequence of measurable functions given as infima. Also,

$$\limsup f_n(x) = \bar{f}(x) = \inf\{\sup\{f_k(x) \mid k \geq n\} \mid n \in \mathbb{N}\}$$

with the result that $\bar{f}$ is $\mathfrak{A}$-measurable. Finally, we note that

$$f(x) = \lim_{n \to \infty} f_n(x)$$

exists if and only if $\bar{f}(x) = \underline{f}(x)$. Thus f is $\mathfrak{A}$-measurable provided the point-wise limit exists on X. □

Exercise 4.2.1. Suppose $f_n : X \to \mathbb{R}$ is a measurable function for each $n \in \mathbb{N}$, where $(X, \mathfrak{A}, \mu)$ is a measure space. Prove that the set

$$S = \{x \mid \lim_{n \to \infty} f_n(x) \text{ exists}\}$$

is a measurable set.

The reader should be able to give examples of point-wise convergent sequences of continuous functions for which the limit is not continuous, and examples of point-wise convergent sequences of Riemann integrable functions for which the limit is not Riemann integrable. We see already that measurability must be a valuable concept for point-wise convergence since point-wise convergence does preserve measurability.

Definition 4.2.1. Let $(X, \mathfrak{A}, \mu)$ be a measure space and let $\mathfrak{N}$ be the set of all *null-sets*. We say that x has some property P $\mathfrak{A} - \mu$ *almost everywhere* if and only if there is a set $N \in \mathfrak{N}$ such that x has the property P for all $x \in X \setminus N$. This is commonly expressed as μ-*almost everywhere* (μ-*a.e.*) or as *almost everywhere* (*a.e.*) provided that there will be no confusion as to which measure μ or σ-algebra $\mathfrak{A}$ is in use.

Corollary 4.2.1. *Let f_n be a sequence of measurable functions on the* complete *measure space* $(X, \mathfrak{A}, \mu)$. *Suppose $f_n(x) \to f(x)$* almost everywhere *on X. Then the function defined by $f(x) = \lim_{n\to\infty} f_n(x)$ is defined almost everywhere and is* measurable.

Proof. This follows from Theorem 4.2.1. It is commonly understood in this context that although the function f is defined everywhere by the given limit except on a null-set N, we may assign f arbitrary values on N itself. Then the completeness of the measure space tells us that the resulting function f remains $\mathfrak{A} - \mu$ measurable regardless of how values are assigned to f within the null-set N. □

Exercise 4.2.2. Suppose $f : X \to \mathbb{R}^*$ is a measurable function that has finite values almost everywhere on the measure space $(X, \mathfrak{A}, \mu)$ where $\mu(X) > 0$. Prove that there is a set of positive measure on which f is bounded.

Exercise 4.2.3. Suppose the measurable function $f : \mathbb{R}^n \to \mathbb{R}$ has the special property that for each fixed vector $c \in \mathbb{R}^n$, the translation of f given by $f_c(x) = f(x + c)$ is equal almost everywhere to $f(x)$ itself. Prove that $f(x)$ is equal almost everywhere to a constant function. (Hints: Consider both the sum and the terms of $\sum_{n\in\mathbb{Z}} l(f^{-1}[n, n+1))$. Apply Exercise 3.5.3 to select the special value of n. *Divide and conquer!*)

4.3 Simple Functions & Egoroff's Theorem

Definition 4.3.1. A function $f : X \to \mathbb{R}$ is called $\mathfrak{A}$-*simple* (or *simple* if there will be no confusion regarding the σ-field $\mathfrak{A}$ that is under consideration) if and only if f is $\mathfrak{A}$-measurable *and*

$$\{f(x) \mid x \in X\}$$

is a finite set. The class of simple functions is denoted by $\mathfrak{S}$.