CHAPTER 6

PRODUCT MEASURES AND FUBINI’S THEOREM

All students learn in elementary calculus to evaluate a double integral by iteration. The theorem justifying this process is called Fubini’s theorem. However, the cleanest and simplest form of Fubini’s theorem appears for the first time with the Lebesgue integral. We will see in this chapter that Fubini’s theorem is instrumental in proving many important theorems.

6.1 PRODUCT MEASURES

We will assume here that we are given two measure spaces, which may or may not be identical to one another. We will call the two measure spaces \((X, \mathcal{A}, \lambda)\) and \((Y, \mathcal{B}, \mu)\), respectively. We intend to construct the product measure on a suitable \(\sigma\)-field contained in the power set of the Cartesian product \(Z = X \times Y\). By a rectangular set \(R\) in \(Z\) we mean any set of the form \(R = A \times B\) where \(A \in \mathcal{A}\) and \(B \in \mathcal{B}\). We will take as the family of elementary sets for the product measure

\[
\mathcal{E} = \left\{ E : E = \bigcup_{i=1}^{n} R_i \mid R_i = A_i \times B_i, A_i \in \mathcal{A}, B_i \in \mathcal{B} \right\},
\]

where the rectangles \(R_i\) are mutually disjoint, and \(n\) is an arbitrary natural number.
EXERCISE

6.1 Show that the set $\mathcal{E}$ of Equation (6.1) is a field of subsets of $Z = X \times Y$. (Be sure to check closure under complementation.)

Definition 6.1.1 Define the product measure

$$\nu(E) = \sum_{i=1}^{n} \lambda(A_i) \mu(B_i),$$

for each elementary set $E \in \mathcal{E}$ as defined by Equation (6.1).

This definition requires justification, because the decomposition given in Equation (6.1) is not unique. Suppose

$$E = \bigcup_{i=1}^{m} A_i \times B_i = \bigcup_{j=1}^{n} C_j \times D_j.$$  

It follows from the finite additivity of each of the measures $\lambda$ and $\mu$ that no rectangular set of infinite measure can be expressed as a union of finitely many rectangular sets of finite measure. Thus

$$\sum_{i=1}^{m} \lambda(A_i) \mu(B_i) - \infty \Leftrightarrow \sum_{j=1}^{n} \lambda(C_j) \mu(D_j) = \infty.$$  

We can use the integral of a special simple function to rephrase Definition 6.1.1 in a way that expresses concisely why that definition is independent of the decomposition in the case in which $\sum_{i=1}^{m} \lambda(A_i) \mu(B_i) < \infty$.

Definition 6.1.2 If $S \subseteq X \times Y$, a Cartesian product space, we define the $x$–section of $S$ by

$$xS = \{ y \mid (x, y) \in S, y \in Y \},$$

and the $y$–section by

$$Sy = \{ x \mid (x, y) \in S, x \in X \}.$$  

If $E \in \mathcal{E}$, the field of elementary sets, then

$$E = \bigcup_{i=1}^{n} R_i - \bigcup_{i=1}^{n} A_i \times B_i.$$  

Define the $x$–section function by

$$f_E(x) = \mu(xE) = \sum_{\{i \in A_i\}} \mu(B_i)$$  

for

$$E = \bigcup_{i=1}^{n} A_i \quad \text{and} \quad B_i.$$  

(6.2)
If $\sum_{i=1}^{m} \lambda(A_i) \mu(B_i) < \infty$, then we see from Equations (6.2) that $f_E$ is a nonnegative special simple function on $X$. Moreover,

$$\int_X f_E \, d\lambda = \sum_{i=1}^{n} \lambda(A_i) \mu(B_i) - \nu(E). \quad (6.3)$$

From Equation (6.3), it is clear that Definition 6.1.1 is independent of the decomposition of the elementary set into a disjoint union of rectangular sets. In fact, if we consider two such decompositions, then we will have two different expansions of the same simple cross–section function $f_E$, so the result of the integral $\int_X f_E \, d\lambda$ must be the same either way.

Furthermore, if $E$ and $E'$ are any two mutually disjoint elementary sets in $X \times Y$, then the $x$–section function

$$f_{E \cup E'} = f_E + f_{E'}.$$ 

This establishes that

$$\nu(E \cup E') = \nu(E) + \nu(E'),$$

since integration over $X$ is a linear function of the integrand.

**EXERCISE**

6.2 Let $E \in \mathcal{E}$, the field of elementary sets in the product of two measure spaces, $(X, \mathcal{A}, \lambda)$ and $(Y, \mathcal{B}, \mu)$. Prove that the product measure $\nu = \lambda \times \mu$ is given on $\mathcal{E}$ by

$$\nu(E) = \int_Y \lambda(E_y) \, d\mu.$$ 

Our next goal is to show that, under suitable hypotheses, $\nu$ can be extended to a countably additive measure on $\mathcal{E} = B(\mathcal{E})$, the $\sigma$-field generated by $\mathcal{E}$.

**Theorem 6.1.1** Let $(X, \mathcal{A}, \lambda)$ and $(Y, \mathcal{B}, \mu)$ be any two measure spaces. There exists a countably additive measure $\nu$ defined on $\mathcal{E} = B(\mathcal{E})$ such that

$$\nu(A \times B) = \lambda(A) \mu(B),$$

for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Moreover, $\nu$ is unique, provided that $\lambda$ and $\mu$ are both $\sigma$-finite.

**Proof:** Uniqueness will follow at the end from Exercise 2.21, since $\nu$ will be $\sigma$–finite if both $\lambda$ and $\mu$ are $\sigma$–finite. Existence depends upon proving countable additivity within $\mathcal{E}$.

Each elementary set is a disjoint union of finitely many rectangular sets of the form

$$E = \bigcup_{i=1}^{n} A_i \times B_i,$$
with each \( A_i \) and each \( B_i \) measurable. To prove the countable additivity within \( \mathcal{E} \), it will suffice to prove for each rectangular set

\[ R = A \times B = \bigcup_{n \in \mathbb{N}} A_n \times B_n, \]

that

\[ \nu(A \times B) = \sum_{n \in \mathbb{N}} \lambda(A_n) \mu(B_n). \]

To this end, we define

\[ f_n(x) = \mu\left( x \left( \bigcup_{i=1}^{n} (A_i \times B_i) \right) \right), \]

so that \( f_n \) is a nonnegative simple function. Note that although the sets \( B_i \) need not be disjoint, we do have

\[ x(A \times B) = \bigcup_{n \in \mathbb{N}} x(A_n \times B_n), \]

which is a disjoint union. Since \( \mu \) is countably additive, it follows that the sequence \( f_n \) increases monotonically toward the limit \( f_R \), where

\[ f_R(x) = \mu(\cdot R) = \begin{cases} \mu(B) & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \]

By the Monotone Convergence Theorem (5.4.1) we see that

\[ \nu(R) = \int_A f \ d\lambda = \lim_{n \to \infty} \int_A f_n \ d\lambda 
= \lim_{n \to \infty} \sum_{i=1}^{n} \lambda(A_i) \mu(B_i) 
= \sum_{n \in \mathbb{N}} \lambda(A_n) \mu(B_n), \]

and this is true even if \( \nu(R) = \infty. \)

Note that in the preceding proof we have applied the Monotone Convergence only to nonnegative measurable functions defined on \( X \), on which there is already a countably additive measure \( \lambda \). (We are not applying Monotone Convergence to functions defined on the product space, for which we are in the process of establishing the existence of a countably additive measure.) We remark that we did not need to use the hypothesis of completeness that appeared in the Monotone Convergence theorem, because in the preceding proof, \( f_n \) converges everywhere to \( f \)—not merely almost everywhere.
Definition 6.1.3 Let \((X, \mathcal{A}, \lambda)\) and \((Y, \mathcal{B}, \mu)\) be any two measure spaces. We will denote the Cartesian product
\[
\mathcal{A} \times \mathcal{B} = \{ A \times B \mid A \in \mathcal{A}, B \in \mathcal{B} \},
\]
and the field that it generates is called \(\mathcal{E}\). But we will denote by
\[
\mathcal{A} \otimes \mathcal{B} = \mathcal{B}(\mathcal{A} \times \mathcal{B}),
\]
the Borel field generated by the field \(\mathcal{E}\). Finally, we denote by
\[
\mathcal{A} \otimes \mathcal{B}^\cdot
\]
the completion of \(\mathcal{A} \otimes \mathcal{B}\) with respect to the product measure \(\lambda \times \mu\).

It is reasonable to wonder whether it is necessary to form the completion \(\mathcal{A} \otimes \mathcal{B}\) if both \(\mathcal{A}\) and \(\mathcal{B}\) happen to be complete families of measurable sets for their respective measures. The answer—that the completion needs to be formed—is confirmed by the following special case.

Let \(X = Y = \mathbb{R}\) and \(Z = X \times Y = \mathbb{R}^2\), the Euclidean plane. Here we will assume that \(\lambda = \mu = l\), Lebesgue measure on the line. And \(\mathcal{A} = \mathcal{B} = \mathcal{L}(\mathbb{R})\) will be the \(\sigma\)-field of all Lebesgue measurable sets in the line. Recall that we have constructed earlier the family \(\mathcal{B}(\mathbb{R}^n)\) of Borel sets in \(\mathbb{R}^n\) and the family \(\mathcal{L}(\mathbb{R}^n)\) of Lebesgue measurable sets in \(\mathbb{R}^n\), for each \(n \in \mathbb{N}\).

Theorem 6.1.2 With the notations of Definition 6.1.3, we have
\[
\mathcal{B}(\mathbb{R}^2) \supseteq \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R}) \supseteq \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R}) \supseteq \mathcal{L}(\mathbb{R}^2).
\]

Proof: We will justify the claims numbered (i), (ii), and (iii) in order.

i. Each Cartesian product of two intervals lies in \(\mathcal{L}(\mathbb{R}) \times \mathcal{L}(\mathbb{R})\). Thus the \(\sigma\)-field \(\mathcal{B}(\mathbb{R}^2)\) that they generate is contained in \(\mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})\). Observe that the family
\[
\mathcal{S} = \left\{ S \subseteq \mathbb{R}^2 \bigg| S_y \in \mathcal{B}(\mathbb{R}) \forall y \in \mathbb{R} \right\}
\]
is a \(\sigma\)-field in the plane containing the elementary sets of the plane, as defined in Section 3.5. Thus it contains all the Borel sets of the plane. On the other hand, by Remark 3.3.1 there exists a set
\[
E \in \mathcal{L}(\mathbb{R}) \setminus \mathcal{B}(\mathbb{R}),
\]
so that
\[
E \times \mathbb{R} \in \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R}).
\]
Hence there is a set \(E \times \mathbb{R}\) that is in \(\mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})\) but not in \(\mathcal{B}(\mathbb{R}^2)\). This proves the first (improper) containment.\(^{47}\)

\(^{47}\)The strict containment (i) could be proven also by a cardinality argument, since the transfinite cardinal number of the right-hand side is greater than that of the left.
ii. For the second improper containment, let $M$ be any nonmeasurable subset of $\mathbb{R}$. Then the measure $(l \times l)([x] \times M) = 0$ for each singleton set $[x]$ in $\mathbb{R}$, being a subset of a null set. Hence

$$[x] \times M \in \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R}).$$

Thus it would suffice to show that $[x] \times M \notin \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$. Consider the class $\mathcal{S}$ of all sets $E \subset \mathbb{R}^2$ for which $x \in \mathcal{L}(\mathbb{R})$ for a fixed $x \in \mathbb{R}$. Then it is easy to check that $\mathcal{S}$ is a monotone class, which implies that $\mathcal{S}$ contains the $\sigma$-field $\mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$. But since the set $[x] \times M$ lacks this property, it follows that $[x] \times M \notin \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$.

iii. The space $\mathcal{L}(\mathbb{R}^2)$ contains both $\mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$ and $\mathcal{B}(\mathbb{R}^2)$. Since $\mathcal{L}(\mathbb{R}^2)$ is the (minimal) completion of $\mathcal{B}(\mathbb{R}^2)$, it must be also the unique, minimal completion of $\mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$, as is $\mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$.

6.2 FUBINI’S THEOREM

Theorem 6.2.1 (Fubini’s Theorem–First Form) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be complete $\sigma$–finite measure spaces. Let $\mathcal{E} = \mathcal{A} \otimes \mathcal{B}$. Then for each $\mu \times \nu$–measurable set $C \in \mathcal{E}$, the section $\chi_C$ is measurable for almost all $x$, the function $f_C(x) = \nu(\chi_C)$ is $\mathcal{A}$–measurable, and

$$(\mu \times \nu)(C) = \int_X f_C(x) \, d\mu(x). \quad (6.4)$$

Proof: Note that $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$ must be $\sigma$–finite as well, so that

$$X \times Y = \bigcup_{n \in \mathbb{N}} X_n \times Y_n,$$

where $\mu(X_n) \nu(Y_n) < \infty$ and $X_n \times Y_n$ is an ascending chain of sets that are rectangular and thus elementary and Borel as well. We remark that the first form of Fubini’s theorem expresses the product measure $\mu \times \nu$ of a set $C \in \mathcal{E}$ as the integral with respect to $\mu$ of the $\nu$–measures of the $x$–sections of $C$. This includes the possibility of both sides of Equation (6.4) being infinite.

Observe that the theorem follows easily from the definition of the product measure on the field $\mathcal{E}$ of elementary sets in the special case in which $C$ is an elementary set in the product space. For the latter sets, each section $\chi_C$ is measurable as well, being a union of finitely many measurable subsets of $Y$.

48It is not hard to see that because $\mathcal{L}(\mathbb{R}^2)$ contains the products of intervals, and because it is a $\sigma$–algebra, it must contain also all products of the form $B \times \mathbb{R} \text{ or } \mathbb{R} \times B$, where $B$ is a Borel set in the line. Thus it contains also the products of Borel sets. However, each Lebesgue measurable set is sandwiched between two Borel sets of the same measure. This, together with the completeness of $\mathcal{L}(\mathbb{R}^2)$, justifies the claim.