Non-commutative, Non-cocommutative Semisimple Hopf Algebras arise from Finite Abelian Groups

Siu-Hung Ng
Mathematics Department, Towson University, Towson, MD 21252.

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Abstract

Given any nontrivial alternating tri-character f on a finite abelian group G, one can construct a finite dimensional non-commutative and non-cocommutative semisimple Hopf algebra H. The group of group-like elements of H is an abelian central extension of B by $\hat{G}$ where B is the radical of f.

1 Introduction

In this exposition, we will discuss the Hopf algebra structure of the twisted quantum double $D^\omega(G)_0$ constructed from a finite abelian group G and a normalized 3-cocycle ω of G [DPR92] (Section 2). These Hopf algebras are semisimple and self-dual [MN01]. Moreover, $D^\omega(G)_0$ is non-commutative and non-cocommutative if there exist $x, y, x \in G$ such that

$$\frac{\omega(x,y,z)\omega(y,z,x)\omega(z,x,y)}{\omega(y,x,z)\omega(z,y,x)\omega(x,z,y)} \neq 1\,. \tag{1}$$

The formula on the left hand side of equation (1) actually defines an alternating tri-character of G. If we write $\psi^*([\omega])(x,y,z)$ for the left hand side of equation (1), then $[\omega] \mapsto \psi^*(\omega)$ defines a split epimorphism from $H^3(G, \mathbb{C}^*)$ onto $\mathrm{Hom}(\bigwedge^3 G, \mathbb{C}^*)$, where $[\omega]$ denotes the cohomology class of ω. Moreover, $D^\omega(G)$ is commutative if, and only if, $[\omega] \in \ker\psi^*$ (Section 4).

Let f be non-trivial alternating tri-character of G and ω a normalized 3-cocycle of G such that $\psi^*(\omega) = f$. We define the radical B of f by

$$B = \{x \in G | f(x,y,z) = 1 \text{ for any } y, z \in G\}\,.$$

Then, the group of all the group-like elements of $D^\omega(G)_0$, denoted by Γ^ω, is an abelian central extension of B by $\widehat{G}$ (Section 3). In addition, Γ^ω lies in the center of $D^\omega(G)_0$. If $[\omega] \in \ker\psi^*$, then $B = G$. The map $\Lambda : \ker\psi^* \longrightarrow H^2(G, \widehat{G})$, $\Lambda : [\omega] \mapsto \mathbf{b}_\omega$ is a group homomorphism, where $\mathbf{b}_\omega$ is the cohomology class associated to the extension

$$1 \longrightarrow \widehat{G} \longrightarrow \Gamma^\omega \longrightarrow G \longrightarrow 1\,.$$

Though we defined the map Λ in terms of quantum doubles, it turns out that Λ has an purely homological interpretation using the Eilenberg-MacLane cohomology (Section 5).

2 Twisted Quantum Double of a Finite Group

Let G be a finite group and let $\omega : G \times G \times G \longrightarrow \mathbb{C}^*$ be a normalized 3-cocycle; that is a function such that $\omega(x, y, z) = 1$ whenever one of x, y or z is equal to the identity element 1 of G and it satisfies the functional equation

$$\omega(g, x, y)\omega(g, xy, z)\omega(x, y, z) = \omega(gx, y, z)\omega(g, x, yz) \quad \text{for any } g, x, y, z \in G\,.$$

We will denote the group of all normalized 3-cocycles on G by $Z^3(G, \mathbb{C}^*)$. For any $g \in G$, one can define the functions $\theta_g, \gamma_g : G \times G \longrightarrow \mathbb{C}^*$ as follows:

$$\theta_g(x, y) = \frac{\omega(g, x, y)\omega(x, y, (xy)^{-1}gxy)}{\omega(x, x^{-1}gx, y)}\,, \tag{2}$$

$$\gamma_g(x, y) = \frac{\omega(x, y, g)\omega(g, g^{-1}xg, g^{-1}yg)}{\omega(x, g, g^{-1}yg)}\,. \tag{3}$$

Let $\{e(g)|g \in G\}$ is the dual basis of the canonical basis of $\mathbb{C}[G]$. The *twisted quantum double* $D^\omega(G)$ of G with respect to ω is the quasi-triangular quasi-Hopf algebra with underlying vector space $\mathbb{C}[G]^* \otimes \mathbb{C}[G]$. The multiplication, comultiplication, and associator are given, respectively, by

$$(e(g) \otimes x)(e(h) \otimes y) = \theta_g(x, y)e(g)e(xhx^{-1}) \otimes xy\,, \tag{4}$$

$$\Delta(e(g) \otimes x) = \sum_{hk=g} \gamma_x(h, k)e(h) \otimes x \otimes e(k) \otimes x\,, \tag{5}$$

$$\Phi = \sum_{g,h,k\in G} \omega(g, h, k)^{-1}e(g) \otimes 1 \otimes e(h) \otimes 1 \otimes e(k) \otimes 1\,. \tag{6}$$

The counit and antipode are given by

$$\epsilon(e(g) \otimes x) = \delta_{g,1} \quad \text{and} \quad S(e(g) \otimes x) = \theta_{g^{-1}}(x, x^{-1})^{-1}\gamma_x(g, g^{-1})^{-1}e(x^{-1}g^{-1}x) \otimes x^{-1}\,,$$

where $\delta_{g,1}$ is the Kronecker delta. The universal R-matrix is given by

$$R = \sum_{g,h\in G} e(g) \otimes 1 \otimes e(h) \otimes g\,. \tag{7}$$

The corresponding elements α and β are $1_{D^\omega(G)}$ and $\sum_{g\in G} \omega(g, g^{-1}, g)e(g) \otimes 1$ respectively (cf. [DPR92]). For the definition and details about quasi-Hopf algebras, the readers are referred to see [Dri90], [Kas95] or [CP95]. Verification of the details involves the following identities, which result from the 3-cocycle identity for ω:

$$\theta_z(a, b)\theta_z(ab, c) = \theta_{a^{-1}za}(b, c)\theta_z(a, bc)\,, \tag{8}$$

$$\theta_y(a, b)\theta_z(a, b)\gamma_a(y, z)\gamma_b(a^{-1}ya, a^{-1}za) = \theta_{yz}(a, b)\gamma_{ab}(y, z)\,, \tag{9}$$

$$\gamma_z(a, b)\gamma_z(ab, c)\omega(z^{-1}az, z^{-1}bz, z^{-1}cz) = \gamma_z(b, c)\gamma_z(a, bc)\omega(a, b, c)\,, \tag{10}$$

for all $a, b, c, y, z \in G$. We leave the verification for the readers as an exercise.

Remark 2.1 If $\omega = 1$, then the twisted quantum double $D^\omega(G)$ is identical to the Drinfeld double of the group algebra $\mathbb{C}[G]$. However, $D^\omega(G)$ is not a Hopf algebra in general. Moreover, even if ω, ω' are differed by a coboundary, $D^\omega(G)$ and $D^{\omega'}(G)$ are not isomorphic as quasi-bialgebras. Nevertheless, they are *gauge equivalent*.

The dual space, $\mathbb{C}[G]^*$, admits a natural $\mathbb{C}[G]$-module structure given by

$$x \cdot e(g) = e(xgx^{-1})$$

for any $x, g \in G$. Actually, $\mathbb{C}[G]^*$ is a $\mathbb{C}[G]$-module algebra. Moreover, the algebra structure of $D^\omega(G)$ is the cross-product of $\mathbb{C}[G]^* \#_\sigma \mathbb{C}[G]$ where $\sigma \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[G] \otimes \mathbb{C}[G], \mathbb{C}[G]^*)$ is the Sweedler 2-cocycle [Swe68] defined by

$$\sigma(x \otimes y) = \sum_{g \in G} \theta_g(x, y) e(g)$$

for any $x, y \in G$. Since both $\mathbb{C}[G]$ and $\mathbb{C}[G]^*$ are semisimple and so is $D^\omega(G)$ (cf. [Mon93] 7.4.2 or [BM89]).

When G is abelian, $\theta_g = \gamma_g$ for any $g \in G$. We will collectively write ω_g for both θ_g and γ_g in the sequel whenever G is abelian.

Theorem 2.2 [MN01, 2.2]*Let G be a finite abelian group and ω a normalized 3-cocycle ω of G. Then $(D^\omega(G), \cdot, 1_{D^\omega(G)}, \Delta, \epsilon, S)$ is a self-dual Hopf algebra. Moreover, if ω and ω' are normalized 3-cocycles differed by a coboundary, then the Hopf algebras $D^\omega(G)$, $D^{\omega'}(G)$ are isomorphic.*

As a consequence of the above theorem, the Hopf algebra structure on $D^\omega(G)$ depends only on the cohomology class $\mathbf{w} \in H^3(G, \mathbb{C}^*)$ represented by ω but not the representatives of $\mathbf{w}$. In the sequel, we will write $D^\omega(G)_0$ for the Hopf algebra $(D^\omega(G), \cdot, 1_{D^\omega(G)}, \Delta, \epsilon, S)$ whenever G is abelian.

Remark 2.3 The Hopf algebra $D^\omega(G)_0$ is not necessarily quasi-triangular although the quasi-Hopf algebra $D^\omega(G)$ is quasi-triangular. The universal $\mathcal{R}$-matrix for the quasi-Hopf algebra $D^\omega(G)$ given in (7) is failed to be a universal $\mathcal{R}$-matrix for the Hopf algebra $D^\omega(G)_0$.

3 Group-like Elements of $D^\omega(G)_0$

Let G be a finite abelian group and $\omega \in Z^3(G, \mathbb{C}^*)$. A nonzero element u in $D^\omega(G)_0$ is called *group-like* if $\Delta(u) = u \otimes u$. We will denote by Γ^ω the group of all group-like elements of $D^\omega(G)_0$. The elements in Γ^ω can be characterized by the following proposition [MN01, 3.2].

Proposition 3.1 *Let G be a finite abelian group and $\omega \in Z^3(G, \mathbb{C}^*)$. Then $u \in \Gamma^\omega$ if, and only if, $u = \sum_{g \in G} \alpha(g) e(g) \otimes x$ for some $x \in G$ and a function $\alpha : G \longrightarrow \mathbb{C}^*$ such that*

$$\omega_x(g, h) = \frac{\alpha(g)\alpha(h)}{\alpha(gh)}$$

for any $g, h \in G$.

Corollary 3.2 *Let G be a finite abelian group and ω a normalized 3-cocycle of G. Then $\Gamma 6\omega$ lies in the center of $D^\omega(G)_0$.*

Proof. For any $u \in \Gamma^\omega$, by Proposition 3.1,

$$u = \sum_{g\in G} \alpha(g)e(g) \otimes x$$

for some $x \in G$ and $\alpha : G \longrightarrow \mathbb{C}^*$ such that

$$\omega_x(y,z) = \frac{\alpha(y)\alpha(z)}{\alpha(yz)}$$

for any $y, z \in G$. Then for any $h, y \in G$,

$$\begin{aligned} (e(h)\otimes y)\cdot u &= \omega_h(y,x)\alpha(h)\otimes yx \\ u\cdot(e(h)\otimes y) &= \omega_h(x,y)\alpha(h)\otimes yx\,. \end{aligned}$$

Since $\omega_x(h,y) = \omega_x(y,h)$,

$$\frac{\omega_h(x,y)}{\omega_h(y,x)} = \frac{\omega(h,x,y)\omega(y,h,x)\omega(x,y,h)}{\omega(x,h,y)\omega(h,y,x)\omega(y,x,h)} = \frac{\omega_x(y,h)}{\omega_x(h,y)} = 1\,.$$

Therefore, $\omega_h(x,y) = \omega_h(y,x)$ for any $h, y \in G$. Hence, $(e(h)\otimes y)\cdot u = u\cdot(e(h)\otimes y)$ for any $h, y \in G$. $\square$

Since ω is normalized, $\omega_1 \equiv 1$. Therefore, for any character α of G, $\sum\limits_{g\in G} \alpha(g)e(g)\otimes 1 \in \Gamma^\omega$ is a group-like element of $D^\omega(G)_0$. It is easy to see that the map $i : \widehat{G} \longrightarrow \Gamma^\omega$, $i : \alpha \mapsto \sum\limits_{g\in G} \alpha(g)e(g)\otimes 1$ actually defines an injective group homomorphism where $\widehat{G}$ is the character group of G.

The assignment $e(g)\otimes x \mapsto \delta_{g,1}x$ defines an algebra map from $D^\omega(G)$ to $\mathbb{C}[G]$. We write j for the restriction of this map on Γ^ω. Then $j(\Gamma^\omega) \subseteq G$ and $j : \Gamma^\omega \longrightarrow G$ is a group homomorphism. It follows from Proposition 3.1 that $\operatorname{Im} i = \ker j$. Let

$$B^\omega = \{x \in G \,|\, \omega_x \text{ is a 2-coboundary.}\}\,.$$

Then, by Proposition 3.1, $\operatorname{Im} j \subseteq B^\omega$. Conversely, for any $x \in B^\omega$, there is a function $\alpha : G \longrightarrow \mathbb{C}^*$ such that

$$\omega_x(g,h) = \delta(\alpha)(g,h) = \frac{\alpha(g)\alpha(h)}{\alpha(gh)}$$

for any $g, h \in G$. Then, $\sum\limits_{g\in G} \alpha(g)e(g)\otimes x \in \Gamma^\omega$ and $j(u) = x$. Hence, $\operatorname{Im} j = B^\omega$ and B^ω is a subgroup of G. This gives the proof for the first part of following lemma (cf [MN01, 2.3 and 2.4],).

Proposition 3.3 *Let G be a finite abelian group and ω a normalized 3-cocycle of G. Then, Γ^ω is an abelian central extension of B^ω by $\widehat{G}$. If ω and ω' are cohomologous normalized 3-cocycles of G, then $B^\omega = B^{\omega'}$ and the central extensions*

$$1 \longrightarrow \widehat{G} \longrightarrow \Gamma^\omega \longrightarrow B^\omega \longrightarrow 1 \quad \textit{and} \quad 1 \longrightarrow \widehat{G} \longrightarrow \Gamma^{\omega'} \longrightarrow B^{\omega'} \longrightarrow 1$$

are equivalent.

Let $Z^3(G, \mathbb{C}^*)_{ab}$ denote the set of all normalized 3-cocycles w of G such that $B^\omega = G$, and $H^3(G, \mathbb{C}^*)_{ab}$ the set of cohomology classes associated to $Z^3(G, \mathbb{C}^*)_{ab}$. Thus, $\omega \in Z^3(G, \mathbb{C}^*)_{ab}$ if, and only if, ω_g is a 2-coboundary for all $g \in G$. It is fairly easy to show that $H^3(G, \mathbb{C}^*)_{ab}$ is a subgroup of $H^3(G, \mathbb{C}^*)$.

Take any $\omega \in Z^3(G, \mathbb{C}^*)_{ab}$. Then, by Proposition 3.3, Γ^ω is an abelian central extension of G by $\widehat{G}$. Let $\beta_\omega \in Z^2(B^\omega, \widehat{G})$ be a 2-cocycle associated to this extension

$$1 \longrightarrow \widehat{G} \longrightarrow \Gamma^\omega \longrightarrow G \longrightarrow 1\,.$$

Proposition 3.3 also implies that $\omega \mapsto \beta_\omega$ induces a map $\Lambda : H^3(G, \mathbb{C}^*)_{ab} \longrightarrow H^2(G, \widehat{G})$, where $H^3(G, \mathbb{C}^*)_{ab}$. In addition, $\Lambda : H^3(G, \mathbb{C}^*)_{ab} \longrightarrow H^2(G, \widehat{G})$ is a group homomorphism (cf. [MN01, 3.8]). We will discuss this map again in Section 5.

4 Construction of non-commutative, non-cocommutative Hopf algebras

Let G be a finite abelian group and $\omega \in Z^3(G, \mathbb{C}^*)$. Since Γ^ω lies in the center of $D^\omega(G)_0$, $\mathbb{C}[\Gamma^\omega]$ is a commutative normal Hopf subalgebra of $D^\omega(G)_0$. In fact, $D^\omega(G)$ is a commutative algebra if, and only if, $D^\omega(G)$ is identical to the $\mathbb{C}[\Gamma^\omega]$.

Proposition 4.1 [MN01, 3.6] *Let G be a finite abelian group and ω a normalized 3-cocycle of G . Then the following statements are equivalent :*

(i) *$D^\omega(G)$ is spanned by Γ^ω;*

(ii) *$\omega \in Z^3(G, \mathbb{C}^*)_{ab}$;*

(iii) *$D^\omega(G)$ is a commutative algebra.*

In order to obtain a non-commutative, non-cocommutative Hopf algebra from the twisted quantum double $D^\omega(G)$ of a finite abelian group G, the necessary and sufficient condition is that $[\omega] \notin H^3(G, \mathbb{C}^*)_{ab}$, where $[\omega]$ denotes the cohomology class represented by ω.

Proposition 4.1 suggests how to determine whether $D^\omega(G)$ is commutative. However, using the definition of $Z^3(G, \mathbb{C}^*)$ to check whether $\omega \in Z^3(G, \mathbb{C}^*)_{ab}$ is not very practical. Interesting enough, $H^3(G, \mathbb{C}^*)_{ab}$ can be described as the kernel of a map ψ^* from $H^3(G, \mathbb{C}^*)$ to $\mathrm{Hom}(\bigwedge^3 G, \mathbb{C}^*)$.

To see the construction of such a map ψ^*, we firstly consider the homology groups of G. The homology groups $H^*(G,\mathbb{Z})$ is the homology of the standard complex $A_0(G)$ given by

$$\cdots \xrightarrow{\partial} C_3 \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \longrightarrow \mathbb{Z}\,,$$

where C_n is the free abelian group generated by all n-tuples $(x_1,\dots,x_n)$ of elements x_i of G and ∂ is a $\mathbb{Z}$-linear map defined by

$$\partial(x_1,\dots,x_n) = (x_2,\dots,x_n) + \sum_{i=1}^{n-1}(-1)^i(x_1,\dots,x_{i-1},x_ix_{i+1},\dots,x_n) + (-1)^n(x_1,\dots,x_{n-1})\,.$$

We will call $(x_1,\dots,x_n)$ a n-dimensional cell of $A_0(G)$.

For any two cells $(x_1,\dots,x_n)$ and $(y_1,\dots,y_m)$ in $A_0(G)$, we can define a "shuffle" of the n letters $x_1,\dots,x_n$ through the m letters $y_1,\dots,y_m$ to be the $n+m$-tuple in which the order of the x's and the order of y's are preserved. The sign of the shuffle is the sign of the permutation required to bring the shuffled letters back to the standard shuffle $(x_1,\dots,x_n,y_1,\dots,y_m)$. Then, we can define the "star" product $(x_1,\dots,x_n)*(y_1,\dots,y_m)$ to be the signed sum of the shuffles of the letters x through the letters y.

Since G is abelian, $A_0(G)$ is a differential graded ring with respect to $*$. Moreover, $H_*(G,\mathbb{Z})$ is an anti-commutative ring under the induced product of $*$. Since G is also finite, there is a split embedding $\psi : \bigwedge^* G \longrightarrow H_*(G,\mathbb{Z})$ which sends $g_1\wedge\cdots\wedge g_n$ in $\bigwedge^n G$ to the homology class of $\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)(g_{\sigma(1)},\dots,g_{\sigma(n)})$ (cf. [Bro82, V.5]). In particular, $\psi : \bigwedge^3 G \longrightarrow H_3(G,\mathbb{Z})$ is a split monomorphism. Then the dual $\psi^* : \operatorname{Hom}(H_3(G,\mathbb{Z}),\mathbb{C}^*) \longrightarrow \operatorname{Hom}(\bigwedge^3 G,\mathbb{C}^*)$ of ψ is a split epimorphism.

For any $\omega\in Z^3(G,\mathbb{C}^*)$, ω naturally defines a homomorphism E_ω in $\operatorname{Hom}(H_2(G,\mathbb{Z}),\mathbb{C}^*)$ by the evaluation on the cell representatives. By the universal coefficient theorem, the map $E : H^3(G,\mathbb{C}^*) \longrightarrow \operatorname{Hom}(H_2(G,\mathbb{Z}),\mathbb{C}^*)$, $[\omega]\mapsto E_\omega$ is a group isomorphism. We will simply identify $H^3(G,\mathbb{C}^*)$ with $\operatorname{Hom}(H_3(G,\mathbb{Z}),\mathbb{C}^*)$ through E. Then, $\psi^* : H^3(G,\mathbb{C}^*) \longrightarrow \operatorname{Hom}(\bigwedge^3 G,\mathbb{C}^*)$ is given by

$$\psi^*([\omega])(x,y,z) = \frac{\omega(x,y,z)\omega(y,z,x)\omega(z,x,y)}{\omega(y,x,z)\omega(z,y,x)\omega(x,z,y)} \tag{11}$$

for any cohomology class $[\omega]$ represented by a 3-cocycle ω. The group $H^3(G,\mathbb{C}^*)_{ab}$ can be characterized nicely by the kernel of ψ^* [MN01, 7.4]).

Proposition 4.2 *Let G be a finite abelian group. Then $H^3(G,\mathbb{C}^*)_{ab}$ is identical to the kernel of the homomorphism ψ^*.*

The formula of ψ^* given in (11), together with propositions 4.1 and 4.2, suggest an easy way to determine whether $D^\omega(G)_0$ is non-commutative and non-cocommutative.

Theorem 4.3 *Let G be a finite abelian group and ω a normalized 3-cocycle. Then $D^\omega(G)_0$ is a non-commutative, non-cocommutative semisimple Hopf algebra if, and only if $\psi^*(\omega)\neq 1$.*

The radical of $\psi^*([\omega])$ is then given by

$$\begin{aligned} B &= \{x \in G \mid \psi^*([\omega])(x,y,z) = 1 \text{ for all } y,z \in G\} \\ &= \left\{ x \in G \;\middle|\; \frac{\omega(x,y,z)\omega(z,x,y)}{\omega(y,x,z)} = \frac{\omega(x,z,y)\omega(z,y,x)}{\omega(z,x,y)} \text{ for all } y,z \in G \right\} \\ &= \{x \in G \mid \omega_x(y,z) = \omega_x(z,y) \;\text{ for all } y,z \in G\} \end{aligned}$$

Since $\mathbb{C}^*$ is divisible, $\omega_x(y,z) = \omega_x(z,y)$ for all $y,z \in G$ if, and only if, ω_x is a 2-coboundary of G. Hence $B = B^\omega$. This proves the following proposition.

Proposition 4.4 *Let G be a finite abelian group and ω a normalized 3-cocycle of G. Then B^ω is the radical of $\psi^*([\omega])$.*

The set of tri-characters of G is a good source of normalized 3-cocycles of G. By Theorem 4.3, non-commutative and non-cocommutative Hopf algebras of the form $D^\omega(G)$ can be constructed easily.

Example 4.5 Let n be a positive integer and let $G = \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n$. For any $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$, and $c = (c_1, c_2, c_3) \in G$, define $\omega(a,b,c) = \zeta^{a_1 b_2 c_3}$, where ζ is a primitive nth root of unity. One can easily see that ω is a tri-character on G and hence a normalized 3-cocycle on G. Moreover,

$$\psi^*([\omega])(a,b,c) = \zeta^{\det[a,b,c]}$$

where $[a,b,c]$ is the matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

Obviously, $\psi^*([\omega])((1,0,0),(0,1,0),(0,0,1)) = \zeta$. Therefore, $\psi^*([\omega]) \neq 1$. Hence, $D^\omega(G)_0$ is then a non-commutative, non-cocommutative semisimple Hopf algebra of dimension n^6. One can easily see that $\det[a,b,c] = 0$ for all $b,c \in G$ if, and only if, $a = 0$. Hence, the radical of $\psi^*([\omega])$ is trivial and so is B^ω. Therefore, the group Γ^ω is isomorphic to $\widehat{G}$.

Since the map $\psi^* : H^3(G, \mathbb{C}^*) \longrightarrow \mathrm{Hom}(\bigwedge^3 G, \mathbb{C}^*)$ is surjective, for any non-trivial alternating tri-character f of G, there exists a cohomology class $[\omega] \in H^3(G, \mathbb{C}^*)$, represented by a normalized 3-cocycle ω, such that $\psi^*([\omega]) = f$. Then, $D^\omega(G)_0$ is non-commutative, as well as, non-cocommutative. However, if $G \cong C_1 \times C_2$ where C_1, C_2 are finite cyclic groups, then $\bigwedge^3 G$ is trivial. Hence, $D^\omega(G)_0$ is always commutative and cocommutative. When G is a direct sum of more than or equal to three cyclic factors, neither $\bigwedge^3 G$ nor $\mathrm{Hom}(\bigwedge^3 G, \mathbb{C}^*)$ is trivial. Hence, there must be an $[\omega] \in H^3(G, \mathbb{C}^*)$ such that $D^\omega(G)_0$ is non-commutative, and non-cocommutative as well.

5 Relation to the Eilenberg-MacLane Cohomology

Let G be a finite abelian group. The subgroup $H^3(G,\mathbb{C}^*)_{ab}$ of $H^3(G,\mathbb{C}^*)$ is the of cohomology classes $[\omega]$ such that $D^\omega(G)_0$ is commutative. The map $\Lambda : H^3(G,\mathbb{C}^*)_{ab}\longrightarrow H^2(G,\widehat{G})$, defined in the end of section 3, has an interesting relation to the Eilenberg-MacLane cohomology for abelian groups.

Using the multiplicative structure of the complex $A_0(G)$, one can define the complex $A_1(G)$ as follows [Mac52]: The cells of $A_1(G)$ are symbols $\sigma = [\alpha_1|\alpha_2|\cdots|\alpha_p]$, with each α_i a cell of $A_0(G)$. The dimension of σ is $p-1$ plus the sum of the dimensions of the α_i, and the boundary of σ is

$$\partial\sigma = \sum_{i=1}^{p}(-1)^{\epsilon_{i-1}}[\alpha_1|\cdots|\partial\alpha_i|\cdots|\alpha_p] + \sum_{i=1}^{p-1}(-1)^{\epsilon_i}[\alpha_1|\cdots|\alpha_i * \alpha_{i+1}|\cdots|\alpha_p]\,,$$

where $\epsilon_i = 1+\dim[\alpha_1|\cdots|\alpha_i]$. For any abelian group M, we denote by $C^n_{ab}(G,M)$, $Z^n_{ab}(G,M)$, $B^n_{ab}(G,M)$ and $H^n_{ab}(G,M)$ the dimension n-cochains, cocycles, coboundaries, and cohomology classes of $\mathrm{Hom}(A_1(G),M)$ respectively. Note that $H^2_{ab}(G,M)$ is the same as $\mathrm{Ext}_{\mathbb{Z}}(G,M)$.

Notice that $A_0(G)$ is a subcomplex of $A_1(G)$. We have the exact sequence of complexes

$$0\longrightarrow A_0(G)\longrightarrow A_1(G)\longrightarrow B(G)\longrightarrow 0$$

with $B(G)$ the quotient complex $A_1(G)/A_0(G)$. Since $\mathbb{C}^*$ is divisible, the sequence

$$1\longrightarrow \mathrm{Hom}(B(G),\mathbb{C}^*)\longrightarrow \mathrm{Hom}(A_1(G),\mathbb{C}^*)\longrightarrow \mathrm{Hom}(A_0(G),\mathbb{C}^*)\longrightarrow 1$$

is exact and we have the long exact sequence

$$\cdots \xrightarrow{\delta} H^*(B(G),\mathbb{C}^*)\longrightarrow H^*_{ab}(G,\mathbb{C}^*)\longrightarrow H^*(G,\mathbb{C}^*) \xrightarrow{\delta} H^{*+1}(B(G),\mathbb{C}^*)\longrightarrow H^{*+1}_{ab}(G,\mathbb{C}^*)\longrightarrow \cdots$$

where $H^n(B(G),\mathbb{C}^*)$ is the nth cohomology group of the cochain complex $\mathrm{Hom}(B(G),\mathbb{C}^*)$. In particular,

$$H^3(B(G),\mathbb{C}^*)\longrightarrow H^3_{ab}(G,\mathbb{C}^*)\longrightarrow H^3(G,\mathbb{C}^*) \xrightarrow{\delta} H^4(B(G),\mathbb{C}^*)\,. \tag{12}$$

The subgroup $H^3(G,\mathbb{C}^*)_{ab}$ of $H^3(G,\mathbb{C}^*)$ is contained in the kernel of δ and $\mathrm{Im}\,\Lambda \subseteq H^2_{ab}(G,\widehat{G})$. Moreover, there exists a monomorphism $\Xi : H^2_{ab}(G,\widehat{G})\longrightarrow H^4(B(G),\mathbb{C}^*)$ such that the diagram

$$\begin{tikzcd}
H^3_{ab}(G,\mathbb{C}^*) \arrow[r] & H^3(G,\mathbb{C}^*) \arrow[r, "\delta"] & H^4(B(G),\mathbb{C}^*) \\
 & H^3(G,\mathbb{C}^*)_{ab} \arrow[u, hook, "incl"] \arrow[r, "\Lambda"] & H^2_{ab}(G,\widehat{G}) \arrow[u, hook, "\Xi"']
\end{tikzcd}$$

commutes (cf. [MN01, Section 6]). Hence, $\ker\Lambda = \ker\delta$. Using the exact sequence (12), we obtain the following Theorem [MN01, 6.3]:

Theorem 5.1 *Let G be a finite abelian group. Then the kernel of the map*

$$\Lambda : H^3(G, \mathbb{C}^*)_{ab} \longrightarrow H^2(G, \widehat{G})$$

is isomorphic to the maximal elementary 2-subgroup of G.

Recall that $\Lambda[\omega]$ is the cohomology class in $H^2(G, \widehat{G})$ associated to the central extension Γ^ω of G by $\widehat{G}$. Thus, $[\omega] \in \ker \Lambda$ if, and only if, the exact sequence

$$1 \longrightarrow \widehat{G} \longrightarrow \Gamma^\omega \longrightarrow G \longrightarrow 1$$

splits. Hence, $\Gamma^\omega \cong G \times G$ if, and only if, $[\omega] \in \ker \Lambda$. Consequently, if $[\omega] \in \ker \Lambda$ is not trivial, then $D^\omega(G)$ and $D^1(G)$ are not *gauge equivalent* (cf. [MN01, 9.5]).

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