A GEOMETRIC ANALOGUE OF A CONJECTURE OF GROSS AND REEDER

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Abstract. Let G be a simple complex algebraic group. We prove that the irregularity of the adjoint connection of an irregular flat G-bundle on the formal punctured disk is always greater than or equal to the rank of G. This can be considered as a geometric analogue of a conjecture of Gross and Reeder. We will also show that the irregular connections with minimum adjoint irregularity are precisely the (formal) Frenkel-Gross connections. As a corollary, we establish the de Rham analogue of a conjecture of Heinloth, Ngô, and Yun for $G = \mathrm{SL}_n$.

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1. Introduction

1.1. **The Gross-Reeder Conjecture.** Let k be a p-adic field with residue field $\mathfrak{f}$. The Weil group $\mathcal{W}$ of k is the subgroup of $\mathrm{Gal}(\bar{k}/k)$ which acts on the algebraic closure $\bar{\mathfrak{f}}$ by a power of the Frobenius. More explicitly, $\mathcal{W} = \langle \mathrm{Fr} \rangle \ltimes \mathcal{I}$, where Fr is a geometric Frobenius element and $\mathcal{I}$ is the inertia group, i.e., the subgroup of $\mathcal{W}$ that acts trivially on $\bar{\mathfrak{f}}$. The wild inertia group $\mathcal{I}_+$ is the pro-p-Sylow subgroup of $\mathcal{I}$.

Let G be a connected reductive algebraic group over $\mathbb{C}$. A Langlands parameter (with values in G) is a homomorphism $\phi : \mathcal{W} \times \mathrm{SL}_2(\mathbb{C}) \to G$ such that the restriction to $\mathrm{SL}_2(\mathbb{C})$ is algebraic, the restriction to $\mathcal{I}$ is continuous, and $\phi(\mathrm{Fr})$ is semisimple. The parameter is called discrete if the centralizer of the image is finite; it is called inertially discrete if there are no nonzero invariants of the action of $\phi(\mathcal{I})$ on the Lie algebra $\mathfrak{g}$.

Note that a discrete, inertially discrete parameter ϕ is, in particular, wildly ramified, i.e., the restriction of ϕ to $\mathcal{I}_+$ is nontrivial. A natural question that arises is what is the minimum "wildness" possible for such a parameter. One way to make this precise is through an invariant called the (adjoint) *Swan conductor*. Let $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ denote the adjoint map. Then $\mathrm{Ad}(\phi) : \mathcal{W} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(\mathfrak{g})$ is a Langlands parameter with values in a general linear group. Given a GL_n-parameter, the Swan conductor is a canonical integer

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which measures how wildly ramified it is; in particular, one can consider the Swan conductor of $\mathrm{Ad}(\phi)$. We can now state a special case of the Gross-Reeder Conjecture [GR10]:

Conjecture 1 (Gross-Reeder)**.** *Suppose G is simple. If the Langlands parameter $\phi : \mathcal{W} \times \mathrm{SL}_2(\mathbb{C}) \to G$ is discrete and inertially discrete, then the Swan conductor of $\mathrm{Ad}(\phi)$ is greater than or equal to the rank of G.*

Gross and Reeder proved their conjecture in a number of important cases. In particular, they verified Conjecture 1 under the assumption that the residual characteristic does not divide the order of the Weyl group. (Reeder has recently formulated and proven a modified version of this conjecture [Ree18].) They also analyzed the situation where the Swan conductor equals the rank. This led them to construct *simple wild parameters* where equality is achieved. They also constructed simple supercuspidal representations of a p-adic group with dual group G which correspond under local Langlands to simple wild parameters.

This theory has also had important applications to the global Langlands program. Indeed, Heinloth, Ngô, and Yun used these results to construct Kloosterman sheaves–ℓ-adic local systems on $\mathbb{P}^1 \setminus \{0, \infty\}$ whose single wildly ramified singularity corresponds to a simple wild parameter [HNY13]. These sheaves are *cohomologically rigid*, i.e., they have no infinitesimal deformations which preserve the formal isomorphism classes at 0 and ∞. Moreover, they provide an example of the wild ramification case of the Langlands correspondence between ℓ-adic local systems and Hecke eigensheaves.

1.2. **Translation to geometry.** The goal of this paper is to formulate and prove a geometric analogue of Conjecture 1. This is, therefore, a continuation of our efforts to understand wild ramification in the geometric Langlands program [BS13, BS12, BS18, BS, Sag, Kam16a, Kam15, KS15, CK16].

In the geometric world, formal flat G-bundles play the role of Langlands parameters, cf. the appendix of [Kat87]. Accordingly, we start by reviewing their definition and some of their numerical invariants. For more information, see §2 and [BV83, Kat87, BS18].

1.2.1. *Formal flat G-bundles.* Let $\mathcal{K} = \mathbb{C}(\!(t)\!)$ denote the field of formal Laurent series. Let $\mathcal{D}^\times = \mathrm{Spec}(\mathcal{K})$ be the formal punctured disk. A formal flat G-bundle $(\mathcal{E}, \nabla)$ is a principal G-bundle $\mathcal{E}$ on $\mathcal{D}^\times$ endowed with a connection ∇ (which is automatically flat). Upon choosing a trivialization, the connection may be written in terms of its matrix

$$[\nabla]_\phi \in \Omega^1_F(\mathfrak{g}(\mathcal{K}))$$

via $\nabla = d + [\nabla]_\phi$. If one changes the trivialization by an element $g \in G(\mathcal{K})$, the matrix changes by the *gauge action*:

$$[\nabla]_{g\phi} = g \cdot [\nabla]_\phi = \mathrm{Ad}(g)([\nabla]_\phi) - (dg)g^{-1}. \tag{1}$$

Accordingly, the set of isomorphism classes of flat G-bundles on $\mathcal{D}^\times$ is isomorphic to the quotient $\mathfrak{g}(\mathcal{K})\frac{dt}{t}/G(\mathcal{K})$, where the loop group $G(\mathcal{K})$ acts by the gauge action.

1.2.2. *Irregular Connections.* Recall that a flat G-bundle $(\mathcal{E}, \phi)$ on $\mathcal{D}^\times$ is called *regular singular* if the connection matrix has only simple poles with respect to some trivialization; otherwise, it is called irregular. It is well-known that irregular connections are geometric analogues of wildly ramified Langlands parameters. In this paper, we will be concerned with two invariants which measure "how irregular" a flat G-bundle is: the slope and the irregularity.

1.2.3. *Slope.* There are several equivalent definitions of the slope. The simplest to describe (though not necessarily to compute) uses the fact that there exists a ramified cover $\mathcal{D}_b^\times = \operatorname{Spec}(\mathbb{C}(\!(u)\!))$ with $u = t^{1/b}$ and a trivialization of the pullback bundle such that the pullback connection is of the form

$$d + (X_{-a}u^{-a} + X_{1-a}u^{1-a} + \dots)\frac{du}{u}, \qquad X_i \in \mathfrak{g}, \quad X_{-a} \text{ non-nilpotent}, \quad a \geq 0.$$

It turns out that the quotient a/b is independent of the choice of such an expression, and one calls it the slope of ∇. The slope is positive if and only if the flat G-bundle is irregular, and the smallest possible positive slope is $1/h$, where h is the Coxeter number of G [FG09, CK16, BS18].

1.2.4. *Irregularity.* To start with, suppose $G = \mathrm{GL}_n$. In this case, a flat G-bundle is equivalent to a pair (V, ∇_V) consisting of a vector bundle on $\mathcal{D}^\times$ endowed with a connection. It is a well-known result of Turrittin [Tur55] and Levelt [Lev75] that after passing to a ramified cover $\mathcal{D}_b^\times$, the pullback connection can be decomposed as a finite direct sum

$$\bigoplus (L_i \otimes M_i, \nabla_{L_i} \otimes \nabla_{M_i}),$$

where (L_i, ∇_{L_i}) is rank one and (M_i, ∇_{M_i}) is regular singular. Let s_i denote the slope (in the sense defined above) of the flat connection $(L_i \otimes M_i, \nabla_{L_i} \otimes \nabla_{M_i})$. Then the irregularity $\operatorname{Irr}(\nabla_V)$ is the sum of the slopes where each slope s_i appears with multiplicity $\dim(M_i)$. One can show that the irregularity is a nonnegative integer that is zero if and only if V is regular singular, cf. [Kat87].

Now, suppose G is a connected reductive group. Let $(\mathcal{E}, \nabla)$ be a flat G-bundle on $\mathcal{D}^\times$. We will be interested in the irregularity of the associated adjoint flat vector bundle$(\operatorname{Ad}(\mathcal{E}), \operatorname{Ad}(\nabla))$. Its irregularity $\operatorname{Irr}(\operatorname{Ad}(\nabla))$ is a nonnegative integer which is positive if and only if $(\mathcal{E}, \nabla)$ is irregular. It can be considered as the geometric analogue of the Swan conductor of an adjoint Langlands parameter.

1.2.5. *An inequality for the adjoint irregularity.* We are now ready to state our first main result, which is a geometric analogue of Conjecture 1.

Theorem 2. *Let G be a simple group, and let $(\mathcal{E}, \nabla)$ be an irregular singular formal flat G-bundle. Then* $\operatorname{Irr}(\operatorname{Ad}(\nabla)) \geq \operatorname{rank}(G)$.

Example 3. This inequality is false if G is not simple. For instance, suppose $G = \mathrm{GL}_2$. Note that if $\nabla = d + A\,dt$ with $A \in \mathfrak{gl}_2(\mathcal{K})$, then $\operatorname{Ad}(\nabla) = d + \operatorname{Ad}(A)dt$. Thus, if we take

$$\nabla = d + \operatorname{diag}(t^{-1}, t^{-1})\frac{dt}{t}$$

then ∇ is irregular, but $\operatorname{Ad}(\nabla)$ is regular singular; thus, $\operatorname{Irr}(\operatorname{Ad}(\nabla)) = 0$.

Next, we discuss when equality is achieved.

1.3. **Formal Frenkel-Gross Connections.** Let G be a simple complex algebraic group with Lie algebra $\mathfrak{g}$. Let us fix a maximal torus and a Borel subgroup $H \subset B \subset G$. Let Φ and $\Delta = \{\alpha_1, \dots \alpha_n\}$ be the corresponding sets of roots and simple roots. Also, let α_0 be the highest root of Φ. We denote the root subalgebras of $\mathfrak{g}$ by $\mathfrak{u}_\alpha$. Now, choose nonzero root vectors $x_{-\alpha_i} \in \mathfrak{u}_{-\alpha_i}$ and $x_{\alpha_0} \in \mathfrak{u}_{\alpha_0}$. Note that $N = \sum_{i=1}^n x_{-\alpha_i}$ is principal nilpotent. The global Frenkel-Gross connection associated to these root vectors is the connection on the trivial bundle over $\mathbb{P}^1$ defined by

$$d+(x_{\alpha_0}t^{-1}+\sum_{i=1}^{n}x_{-\alpha_i})\frac{dt}{t}. \tag{2}$$

This connection has a regular singularity at ∞ and an irregular singularity at 0. (The connection in [FG09] has the locations of the irregular and irregular singularities reversed.) It is the de Rham analogue of the Kloosterman sheaf; in particular, it is cohomologically rigid.

We define a formal Frenkel-Gross connection to be one which is isomorphic to the induced formal connection at 0 of the global Frenkel-Gross connection:

Definition 4. *A formal flat G-bundle is called a formal Frenkel-Gross connection if it is isomorphic to $d+(x_{\alpha_0}t^{-1}+\sum_{i=1}^{n}x_{-\alpha_i})\frac{dt}{t}$ for some choice of nonzero vectors $x_{-\alpha_i}\in\mathfrak{u}_{-\alpha_i}$ and $x_{\alpha_0}\in\mathfrak{u}_{\alpha_0}$.*

A priori, we have many different formal Frenkel-Gross connections as we can multiply each x_{-a_i} and x_{α_0} by nonzero complex numbers. However, as we shall see, it is sufficient to fix one such $(n+1)$-tuple and multiply it by nonzero scalars; see Example 18. More precisely, let S be the connected centralizer of the regular element $x_{\alpha_0}t^{-1}+\sum_{i=1}^{n}x_{-\alpha_i}$; it is a *Coxeter torus* (see §3.2). The relative Weyl group $W_S=N(S)/S$ is a cyclic group of order h' dividing h. We will show that any formal Frenkel-Gross connection is isomorphic to $d+\lambda(x_{\alpha_0}t^{-1}+\sum_{i=1}^{n}x_{-\alpha_i})\frac{dt}{t}$ for some $\lambda\in\mathbb{C}^*$. Moreover, the connections associated to λ and λ' are isomorphic if and only if $\lambda'/\lambda\in\mu_{h'}$, the h'^{th} roots of unity. In other words, the set of isomorphism classes of formal Frenkel-Gross connections is isomorphic to $\mathbb{C}^*/\mu_{h'}$. (Of course, this space is homeomorphic to $\mathbb{C}^*$. To get a set of representatives indexed by $\mathbb{C}^*$, one fixes the $x_{-\alpha_i}$'s and multiplies x_{α_0} by a constant.)

We are now ready to state the companion result to Theorem 2:

Theorem 5. *Let G be a simple group, and let $(\mathcal{E},\nabla)$ be an formal flat G-bundle. Then the following are equivalent:*

(1) $\mathrm{Irr}(\mathrm{Ad}(\nabla))=\mathrm{rank}(G)$*;*
(2) $\mathrm{slope}(\nabla)=\frac{1}{h}$*;*
(3) ∇ *is a formal Frenkel-Gross connection.*

We remark that the proofs of the two theorems use quite different methods. The proof of Theorem 2 makes use of the classical Levelt-Turritin theory of Jordan forms for formal flat G-bundles. This allows us to translate the desired inequality into a statement about eigenvectors of elements of finite real reflection groups; see also [Kam16b]. We check this explicitly for each type. The statement makes sense in the context of complex reflection groups as well, and we conjecture that it holds in general. In contrast, our proof of Theorem 5 requires non-classical methods. Indeed, the proof uses the full power of the geometric theory of fundamental and regular strata for formal flat G-bundles developed in [BS18, BS].

1.4. **A de Rham analogue of a conjecture of Heinloth, Ngô, and Yun.** In [HNY13, §7.1], the authors have conjectured that the Kloosterman sheaves are the only ℓ-adic local systems on $\mathbb{P}^1\setminus\{0,\infty\}$ with certain prescribed behaviour at the marked points. They also gave an analogous construction of Kloosterman D-modules and conjectured that they were the same as global Frenkel-Gross connections. This was subsequently proved by Zhu [Zhu17]. Accordingly, one can translate the conjecture of Heinloth, Ngô, and Yun on Kloosterman sheaves into the de Rham setting as follows:

Conjecture 6 (De Rham analogue of Conjecture 7.2 of [HNY13])**.** *Let ∇ be a G-connection on $\mathbb{P}^1$ which is regular away from $\{0,\infty\}$ and satisfies*

- *∇_∞ is regular singular with principal unipotent monodromy.*
- *∇_0 is irregular with irregularity equal to $r = \mathrm{rank}(G)$.*

Then ∇ is isomorphic to a global Frenkel-Gross connection.

Theorem 7. *Conjecture 6 holds for $G = \mathrm{SL}_n$.*

Proof. By Theorem 5, there exists a (global) Frenkel-Gross SL_n-connection ∇^{FG} such that $\nabla_0 \simeq \nabla_0^{\mathrm{FG}}$. Moreover, since ∇ and ∇^{FG} are both regular singular at ∞ and have the same monodromy, we have $\nabla_\infty \simeq \nabla_\infty^{\mathrm{FG}}$. According to [FG09], the Frenkel-Gross connection ∇^{FG} is cohomologically rigid. In fact, Katz's rigidity theorem [Kat90, Theorem 3.7.3] shows that ∇^{FG} is also cohomologically rigid when viewed as a GL_n-connection. To apply this theorem, we need to observe that ∇^{FG} is irreducible as a GL_n-connection and that $\chi(\mathbb{G}_m, \nabla^{\mathrm{FG}}) = -1$, where χ is the Euler characteristic. The first fact is easy; it is even true of the localization ∇_0^{FG}. The second follows from Deligne's formula for the Euler characteristic [Del70, §6.19, (6.21.1)]:

$$\chi(\mathbb{G}_m, \nabla^{\mathrm{FG}}) = \chi(U)\mathrm{rank}(\nabla^{\mathrm{FG}}) - \mathrm{Irr}(\nabla_0^{\mathrm{FG}}) - \mathrm{Irr}(\nabla_\infty^{\mathrm{FG}}) = 0 - 1 - 0 = -1.$$

By the main result of [BE04], ∇^{FG} is physically rigid when viewed as a GL_n-connection. This implies that there exists $g \in \mathrm{GL}_n(\mathbb{C}[t, t^{-1}])$ such that $g.\nabla = \nabla^{\mathrm{FG}}$, where the action is gauge transformation as in (1). It remains to show that g can be chosen to be in SL_n. To this end, let us write

$$g = z^{-1} g', \qquad g' \in \mathrm{SL}_n(\mathbb{C}[t, t^{-1}]), \quad z = \mathrm{diag}(\det(g)^{-1}, 1, 1, \ldots, 1).$$

Then, we have

$$g'.\nabla = z.\nabla^{\mathrm{FG}} = \nabla^{\mathrm{FG}} - (dz)z^{-1} \implies g'.\nabla - \nabla^{\mathrm{FG}} = -(dz)z^{-1}.$$

Now, observe that the matrix of the connection $g'.\nabla - \nabla^{\mathrm{FG}}$ is traceless. This means that the $(dz)z^{-1}$ is traceless and diagonal, hence zero. Thus, $g'.\nabla = \nabla^{\mathrm{FG}}$, so ∇ and ∇^{FG} are equivalent as SL_n-connections. $\square$

1.5. **Further directions.** We observe that our characterization of the formal Frenkel-Gross connection makes explicit the notion that it should be viewed as the geometric version of the simple wild parameters of Gross and Reeder. This perspective also suggests a potential generalization of the results of this paper. Reeder and Yu have constructed *epipelagic representations* of p-adic groups, certain supercuspidal representations which generalize Gross and Reeder's simple supercuspidals [RY14]. This theory has been used by Yun in his construction of *generalized Kloosterman sheaves* [Yun16]. Reeder has very recently shown that the corresponding epipelagic Langlands parameters also are the parameters for which equality holds in a certain inequality involving the Swan conductor [Ree18].

The theory of regular strata suggests that the geometric analogue of epipelagic parameters are elliptic toral connections with minimal (positive) slope. In fact, regular strata can also be used to construct de Rham analogues of generalized Kloosterman sheaves, whose irregular singularity is a formal connection of such a type. (Yun has also accomplished this through his notion of *θ-connections*, cf. [Che15].) We expect that these formal connections can be characterized as those connections for which equality holds in a more complicated inequality involving the adjoint irregularity. We will consider this issue in a future paper.

1.6. **Organization and Notation.** In §2, we review the Jordan form (aka the Levelt-Turrittin form) of a formal flat G-bundles. Using this, we give alternative definitions of slope and irregularity. Properties of formal Frenkel-Gross connections are established in §3. We use the theory of fundamental strata for formal connections [BS, BS18] to prove that every connection with slope $1/h$ is a formal Frenkel-Gross connection. In §4, we prove a result about Weyl groups which will be crucial for our main theorems. Finally, we prove Theorem 2 and 5 in §5.

Throughout the paper, G denotes a connected complex reductive group with Lie algebra $\mathfrak{g}$; unless otherwise specified, we assume that G is simple. We fix a Borel subgroup B with maximal torus H and unipotent radical N; the corresponding Lie algebras are denoted by $\mathfrak{b}$, $\mathfrak{h}$, and $\mathfrak{n}$. As in §1.3, Φ and $\Delta = \{\alpha_1, \dots \alpha_n\}$ are the corresponding sets of roots and simple roots, and W is the Weyl group. When Φ is irreducible, we let α_0 be the highest root of Φ. We denote the root subalgebras of $\mathfrak{g}$ by $\mathfrak{u}_\alpha$.

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2. Jordan decomposition for formal connections

In this section, we recall some basic facts regarding formal flat G-bundles. Here, G is not assumed to be simple.

2.1. **Jordan decomposition.** For each positive integer b, let $\mathcal{K}_b = \mathbb{C}((t^{\frac{1}{b}}))$ denote the unique finite extension of $\mathcal{K}$ of degree b and let $\mathcal{D}_b^\times$ denote the corresponding punctured disk. Let $\pi_b : \mathcal{D}_b^\times \to \mathcal{D}^\times$ denote the canonical covering map. Given a flat G-bundle $(\mathcal{E}, \nabla)$, we denote by $(\pi_b^*\mathcal{E}, \pi_b^*\nabla)$ its pullback to $\mathcal{D}_b^\times$. For ease of notation, we sometimes use the substitution $u = t^{\frac{1}{b}}$.

Theorem 8. *Let $(\mathcal{E}, \nabla)$ be a formal flat G-bundle. Then there exists a positive integer b and a trivialization of $\pi_b^*\mathcal{E}$ in which $\pi_b^*\nabla$ can be written as*

$$\pi_b^*\nabla = d + (h+n)\frac{du}{u}, \qquad h \in \mathfrak{h}[u^{-1}], \quad n \in \mathfrak{n}(\mathbb{C}), \tag{3}$$

and h and n commute. Moreover, the pair h and n satisfying the above properties is unique.

Definition 9. *We call $d + (h+n)\frac{du}{u}$ the* Jordan form *of the formal flat connection $(\mathcal{E}, \nabla)$.*

For $G = \mathrm{GL}_n$, the existence of the trivialization with the properties in the above theorem was first proved by Turrittin [Tur55]. Subsequently, Levelt proved uniqueness [Lev75]. Babbitt and Varadarajan [BV83] have given an alternative proof of this fact. In addition, following a suggestion of Deligne, they showed that the above theorem also holds for G an arbitrary reductive group [BV83, §9].

2.2. **Slope.** Using the Jordan decomposition, we can give an alternative definition of the slope of a formal flat G-bundle. If z is a Laurent series, we let $\mathrm{ord}_{\mathrm{pole}}(z) \in \mathbb{Z}_{\geq 0}$ denote the order of the pole of z. Clearly, z is a power series (i.e., has no singularity) if and only if $\mathrm{ord}_{\mathrm{pole}}(z) = 0$.

Let $(\mathcal{E}, \nabla)$ be a formal flat G-bundle and let $d + (h+n)\frac{du}{u}$ denote its Jordan form, where $u = t^{\frac{1}{b}}$ for some $b \in \mathbb{Z}_{\geq 1}$.

Definition 10. *The non-negative rational number* $s = \max\{0, \frac{\mathrm{ord}_{\mathrm{pole}}(h)}{b}\}$ *is called the* slope *of* ∇*.*

It is immediate that this definition coincides with the one given in the introduction when $h \neq 0$, since the leading term of $h+n$ is evidently non-nilpotent. If $h = 0$, then ∇ is regular singular, and both definitions give 0. We will also need an equivalent definition of the slope given in terms of fundamental strata [BS18]; this will be discussed in §3. For yet another equivalent definition, see [CK16, §2].

2.3. **Irregularity.** Let $G = \mathrm{GL}_n$, and let B (resp. H) denote the upper triangular (resp. diagonal) matrices. Let $(\mathcal{E}, \nabla)$ be a formal flat G-bundle (equivalently, a vector bundle on $\mathcal{D}^{\times}$ equipped with a connection). Let $d + (h+n)\frac{du}{u}$ denote its Jordan form with respect to the above choice of B and H. (Recall that $u = t^{\frac{1}{b}}$ for some $b \in \mathbb{Z}_{\geq 1}$.) Since h is diagonal, we can write

$$h = \mathrm{diag}(h_1, \cdots, h_n), \qquad h_i \in \mathbb{C}[u^{-1}].$$

Definition 11. *The irregularity of* $(\mathcal{E}, \nabla)$ *is defined by*

$$\mathrm{Irr}(\nabla) = \sum_{i=1}^{n} \{\max\{0, \frac{\mathrm{ord}_{\mathrm{pole}}(h_i)}{b}\}\}.$$

One can show that $\mathrm{Irr}(\nabla) \in \mathbb{Z}_{\geq 0}$ and that this integer coincides with the one defined in the introduction.

3. Irregular connections with minimum slope

In §1.3, we introduced the notion of a formal Frenkel-Gross connection. In this section, we will characterize these connections as the irregular flat G-bundles on $\mathcal{D}^{\times}$ with minimum slope, i.e., with slope $1/h$, where h is the Coxeter number of G.

Recall from §1.3 that the connected centralizer of the matrix of a Frenkel-Gross connection is a maximal torus of $G(\mathcal{K})$ called a Coxeter torus; its relative Weyl group is a cyclic group of order h' dividing h.

Proposition 12. *For a flat G-bundle $\nabla = (\mathcal{E}, \nabla)$ the following are equivalent:*

(i) ∇ *is isomorphic to a Frenkel-Gross connection;*
(ii) *The slope of* ∇ *equals* $\frac{1}{h}$*.*

Moreover, the set of isomorphism classes of such flat G-bundles is in bijection with $\mathbb{C}^{*}/\mu_{h'}$. Finally, for each such connection, $\mathrm{Irr}(\mathrm{Ad}(\nabla)) = \mathrm{rank}(G)$*.*

In order to prove the proposition, we will need to recall some of the geometric theory of fundamental strata from [BS18, BS].

3.1. **Fundamental strata.** Let $\mathcal{B}$ be the Bruhat-Tits building of G; it is a simplicial complex whose facets are in bijective correspondence with the parahoric subgroups of the loop group $G(\mathcal{K})$. The standard apartment $\mathcal{A}$ associated to the split maximal torus $H(\mathcal{K})$ is an affine space isomorphic to $X_*(H) \otimes_{\mathbb{Z}} \mathbb{R}$. Given $x \in \mathcal{B}$, we denote by $G(\mathcal{K})_x$ (resp. $\mathfrak{g}(\mathcal{K})_x$) the parahoric subgroup (resp. subalgebra) corresponding to the facet containing x.

For any $x \in \mathcal{B}$, the Moy-Prasad filtration associated to x is a decreasing $\mathbb{R}$-filtration

$$\{\mathfrak{g}(\mathcal{K})_{x,r} \mid r \in \mathbb{R}\}$$

of $\mathfrak{g}(\mathcal{K})$ by $\mathbb{C}[[t]]$-lattices. The filtration satisfies $\mathfrak{g}(\mathcal{K})_{x,0} = \mathfrak{g}(\mathcal{K})_x$ and is periodic in the sense that $\mathfrak{g}(\mathcal{K})_{x,r+1} = t\mathfrak{g}(\mathcal{K})_{x,r}$. Moreover, if we set $\mathfrak{g}(\mathcal{K})_{x,r+} = \bigcup_{s>r} \mathfrak{g}(\mathcal{K})_{x,s}$, then the set of r for which $\mathfrak{g}(\mathcal{K})_{x,r} \neq \mathfrak{g}(\mathcal{K})_{x,r+}$ is a discrete subset of $\mathbb{R}$.

For our purposes, it will suffice to give the explicit definition for $x \in \mathcal{A}$. In this case, the filtration is determined by a grading on $\mathfrak{g}(\mathbb{C}[t, t^{-1}])$, with the graded subspaces given by

$$\mathfrak{g}(\mathcal{K})_x(r) = \begin{cases} \mathfrak{h}t^r \oplus \displaystyle\bigoplus_{\alpha(x)+m=r} \mathfrak{u}_\alpha t^m, & \text{if } r \in \mathbb{Z} \\ \\ \displaystyle\bigoplus_{\alpha(x)+m=r} \mathfrak{u}_\alpha t^m, & \text{otherwise.} \end{cases}$$

Let κ be the Killing form for $\mathfrak{g}$. Any element $X \in \mathfrak{g}(\mathcal{K})$ gives rise to a continuous $\mathbb{C}$-linear functional on $\mathfrak{g}(\mathcal{K})$ via $Y \mapsto \operatorname{Res} \kappa(Y, X)\frac{dt}{t}$. This identification induces an isomorphism

$$(\mathfrak{g}(\mathcal{K})_{x,r}/\mathfrak{g}(\mathcal{K})_{x,r+})^\vee \cong \mathfrak{g}(\mathcal{K})_{x,-r}/\mathfrak{g}(\mathcal{K})_{x,-r+}.$$

The leading term of a connection with respect to a Moy-Prasad filtration is given in terms of *G-strata*. A G-stratum of depth r is a triple (x, r, β) with $x \in \mathcal{B}$, $r \geq 0$, and $\beta \in (\mathfrak{g}(\mathcal{K})_{x,r}/\mathfrak{g}(\mathcal{K})_{x,r+})^\vee$. Any element of the corresponding $\mathfrak{g}(\mathcal{K})_{x,-r+}$-coset is called a representative of β. If $x \in \mathcal{A}$, there is a unique homogeneous representative $\beta^\flat \in \mathfrak{g}(\mathcal{K})_x(-r)$. The stratum is called *fundamental* if every representative is non-nilpotent. When $x \in \mathcal{A}$, it suffices to check that $\beta^\flat$ is non-nilpotent.

Definition 13. *If $x \in \mathcal{A} \cong \mathfrak{h}_\mathbb{R}$, we say that $(\mathcal{E}, \nabla)$ contains the stratum (x, r, β) with respect to the trivialization ϕ if $[\nabla]_\phi - x\frac{dt}{t} \in \mathfrak{g}(\mathcal{K})_{x,-r}\frac{dt}{t}$ and is a representative for β. (See* [BS18] *for the general definition.)*

We recall some of the basic facts about the relationship between flat G-bundles and strata. The following theorem and Theorem 17 hold for reductive G.

Theorem 14 ([BS18, Theorem 2.14])**.** *Every flat G-bundle $(\mathcal{E}, \nabla)$ contains a fundamental stratum (x, r, β), where x is in the fundamental alcove $C \subset \mathcal{A}$ and $r \in \mathbb{Q}$; the depth r is positive if and only if $(\mathcal{E}, \nabla)$ is irregular singular. Moreover, the following statements hold.*

1. *If $(\mathcal{E}, \nabla)$ contains the stratum (y, r', β'), then $r' \geq r$.*
2. *If $(\mathcal{E}, \nabla)$ is irregular singular, a stratum (y, r', β') contained in $(\mathcal{E}, \nabla)$ is fundamental if and only if $r' = r$.*

As a consequence, one can define the slope of ∇ as the depth of any fundamental stratum it contains.

As an example, let x_I be the barycenter of the fundamental alcove in $\mathcal{A}$, which corresponds to the standard Iwahori subgroup. It is immediate from the definition that $\mathfrak{g}(\mathcal{K})_{x_I}(-1/h) = t^{-1}\mathfrak{u}_{\alpha_0} \oplus \bigoplus_{i=1}^n \mathfrak{u}_{-\alpha_i}$. One now sees that a Frenkel-Gross connection contains a stratum of the form $(x_I, \frac{1}{h}, \beta)$. This stratum is fundamental; in fact, $\beta^\flat$ is regular semisimple. In particular, the slope of a formal Frenkel-Gross connection is $\frac{1}{h}$. We note that Frenkel and Gross derived this directly from the definition of slope given in the introduction.

3.2. **Regular strata and toral flat G-bundles.** We will also need some results on flat G-bundles which contain a *regular stratum*, a kind of stratum that satisfies a graded version of regular semisimplicity. For convenience, we will only describe the theory for strata based at points in $\mathcal{A}$.

Let $S \subset G(\mathcal{K})$ be a (in general, non-split) maximal torus, and let $\mathfrak{s} \subset \mathfrak{g}(\mathcal{K})$ be the associated Cartan subalgebra. We denote the unique Moy-Prasad filtration on $\mathfrak{s}$ by $\{\mathfrak{s}_r\}$.

More explicitly, we first observe that if S is split, then this is just the usual degree filtration. In the general case, if $\mathcal{K}_b$ is a splitting field for S, then $\mathfrak{s}_r$ consists of the Galois fixed points of $\mathfrak{s}(\mathcal{K}_b)_r$. Note that $\mathfrak{s}_r \neq \mathfrak{s}_{r+}$ implies that $r \in \mathbb{Z}\frac{1}{b}$.

A point $x \in \mathcal{A}$ is called *compatible* with $\mathfrak{s}$ if $\mathfrak{s}_r = \mathfrak{g}(\mathcal{K})_{x,r} \cap \mathfrak{s}$ for all $r \in \mathbb{R}$.

Definition 15. *A fundamental stratum (x, r, β) with $x \in \mathcal{A}$ and $r > 0$ is an* S-regular stratum *if x is compatible with S and S equals the connected centralizer of $\beta^\flat$.*

In fact, every representative of β will be regular semisimple with connected centralizer a conjugate of S.

Definition 16. *If $(\mathcal{E}, \nabla)$ contains the S-regular stratum (x, r, β), we say that $(\mathcal{E}, \nabla)$ is* S-toral.

Recall that the conjugacy classes of maximal tori in $G(\mathcal{K})$ are in one-to-one correspondence with conjugacy classes in the Weyl group W [KL88]. It turns out that there exists an S-toral flat G-bundle of slope r if and only if S corresponds to a regular conjugacy class of W and $e^{2\pi i r}$ is a regular eigenvalue for this class [BS, Corollary 4.10]. Equivalently, $\mathfrak{s}_{-r} \setminus \mathfrak{s}_{-r+}$ contains a regular semisimple element. For example, a Frenkel-Gross connection is S-toral for S a Coxeter torus, i.e., a maximal torus corresponding to the Coxeter conjugacy class in W. (One way to see this is that $e^{2\pi i/h}$ is a regular eigenvalue for Coxeter elements and for no other elements of W.) Moreover, since the regular eigenvalues of a Coxeter element are the primitive h^{th} roots of 1, the possible slopes for S-toral flat G-bundles are m/h with $m > 0$ relatively prime to h.

An important feature of S-toral flat G-bundles is that they can be "diagonalized" into $\mathfrak{s}$. To be more precise, suppose that there exists an S-regular stratum of depth r. The filtration on $\mathfrak{s}$ can be defined in terms of a grading, whose graded pieces we denote by $\mathfrak{s}(r)$. Let $\mathcal{A}(S, r)$ be the open subset of $\bigoplus_{j \in [-r, 0]} \mathfrak{s}(j)$ whose leading component (i.e., the component in $\mathfrak{s}(-r)$) is regular semisimple. This is called the set of *S-formal types* of depth r. Let $W_S^{\text{aff}} = N(S)/S_0$ be the relative affine Weyl group of S; it is the semidirect product of the relative Weyl group W_S and the free abelian group S/S_0. The group W_S^{aff} acts on $\mathcal{A}(S, r)$. The action of W_S is the restriction of the obvious linear action while S/S_0 acts by translations on $\mathfrak{s}(0)$.

Theorem 17. [BS, Corollary 5.14] *If $(\mathcal{E}, \nabla)$ is S-toral of slope r, then ∇ is gauge-equivalent to a connection with matrix in $\mathcal{A}(S, r)\frac{dt}{t}$. Moreover, the moduli space of S-toral flat G-bundles of slope r is given by $\mathcal{A}(S, r)/W_S^{\text{aff}}$.*

Example 18. Let S be a Coxeter maximal torus. After conjugating, we may assume that it is the connected centralizer of the matrix $\zeta\frac{dt}{t}$ of a Frenkel-Gross connection with $\zeta = x_{\alpha_0} t^{-1} + \sum_{i=1}^{n} x_{-\alpha_i} \in t^{-1}\mathfrak{u}_{\alpha_0} \oplus \bigoplus_{i=1}^{n} \mathfrak{u}_{-\alpha_i}$. In this case, x_I is *graded compatible* with $\mathfrak{s}$, i.e., $\mathfrak{s}(r) = \mathfrak{g}_{x_I}(r) \cap \mathfrak{s}$ for all r. It is easy to see that S is elliptic. Indeed, an element of $\mathfrak{s}(0) \subset \mathfrak{g}_{x_I}(0) = \mathfrak{h}$ would commute with the principal nilpotent element $N = \sum_{i=1}^{n} x_{-\alpha_i}$. However, since G is simple, $\mathfrak{z}(N)$ is a subset of $\bar{\mathfrak{n}}$, the nilpotent radical of the Borel subalgebra opposite to $\mathfrak{b}$, so $\mathfrak{s}(0) = \{0\}$. This means that the action of S/S_0 on $\mathcal{A}(S, \frac{m}{h})$ is trivial, so the moduli space of S-toral flat G-bundles of slope m/h is just $\mathcal{A}(S, \frac{m}{h})/W_S$.

When $m = 1$, we may be entirely explicit. First, we show that $\mathfrak{s}(-1/h)$ is one-dimensional. Suppose that $y_{\alpha_0} t^{-1} + N' \in \mathfrak{s}(-1/h) \subset \mathfrak{g}_{x_I}(-1/h)$; here, $y_{\alpha_0} \in \mathfrak{u}_{\alpha_0}$ and $N' \in \mathfrak{g}_{x_I}(-1/h) \cap \mathfrak{n}$. Since N is regular nilpotent and $[N, N'] = 0$, $N' = \lambda N$ for some $\lambda \in \mathbb{C}$. It follows that $[N, \lambda x_{\alpha_0} - y_{\alpha_0}] = 0$, so $\mathfrak{z}(N) \subset \bar{\mathfrak{n}}$ implies $y_{\alpha_0} = \lambda x_{\alpha_0}$. Next, W_S is isomorphic to a subgroup of the centralizer of a Coxeter element in W [BS, Proposition 5.9], so W_S is a cyclic group of order $h'|h$. Since nonzero elements of $\mathfrak{s}(-1/h)$ are regular semisimple, W_S acts freely on it. Thus, the moduli space $\mathcal{A}(S, \frac{m}{h})/W_S$ is isomorphic to $\mathbb{C}^*/\mu_{h'}$.

3.3. **Proof of Proposition 12.** We will begin the proof of the proposition with two lemmas.

Lemma 19. *If $(\mathcal{E}, \nabla)$ is an S-toral flat G-bundle of slope r, then* $\mathrm{Irr}(\mathrm{Ad}(\nabla)) = r|\Phi|$. *In particular, a Frenkel-Gross connection has irregularity* $r = \mathrm{rank}(G)$.

A consequence of this lemma is that for a toral flat G-bundle of slope r, $r|\Phi|$ is an integer.

Proof. Suppose that S splits over the degree b extension $\mathcal{K}_b$ with uniformizer u such that $u^b = t$. The pullback connection ∇' is toral for a split maximal torus. Accordingly, we can choose a trivialization for which $[\nabla'] = X\frac{dt}{t}$, where $X \in \mathfrak{h}(\mathcal{K}_b)$ with regular semisimple leading term; moreover, for each root α, $\alpha([X])$ has valuation $-rb$. Thus, the adjoint connection of ∇' is the direct sum of $|\Phi|$ flat line bundles of slope rb and n trivial flat line bundles. It follows that the irregularity of $\mathrm{Ad}(\nabla)$ is $|\Phi|rb/b$ as desired.

If ∇ is a Frenkel-Gross connection, it is S-toral with S a Coxeter maximal torus and has slope $1/h$. Since $|\Phi|/h = n$, $\mathrm{Irr}(\mathrm{Ad}(\nabla)) = n$. □

Now, let ∇ be a flat G-bundle of slope $1/h$. We want to show that ∇ must be a formal Frenkel-Gross connection. The fact that the slope of ∇ equals $1/h$ means that ∇ contains a fundamental stratum $(x, 1/h, \beta)$ for some $x \in \mathcal{B}$ with respect to some trivialization. By equivariance, we can assume that x lies in the fundamental alcove C corresponding to the standard Iwahori subgroup I. Let x_I be the barycenter of C.

Lemma 20. *The barycenter x_I is the unique point $x \in C$ for which there exists a fundamental stratum of the form $(x, 1/h, \beta)$.*

Proof. Suppose there is a fundamental stratum of depth $1/h$ based at x. Since $x \in C$, $\alpha_i(x) \geq 0$ for $1 \leq i \leq n$ and $\alpha_0(x) \leq 1$. This implies that if α is positive (resp. negative), then $\alpha(x) \in [0, 1]$ (resp. $\alpha(x) \in [-1, 0]$. As a result,

$$\mathfrak{g}_x(-1/h) = \bigoplus_{\{\alpha<0 \mid \alpha(x)=-1/h\}} \mathfrak{u}_\alpha \oplus \bigoplus_{\{\alpha>0 \mid \alpha(x)=1-1/h\}} t^{-1}\mathfrak{u}_\alpha.$$

Let $I = \{i \in [1, n] \mid \alpha_i(x) \leq 1/h\}$, and let $J = I^c$. Let $\mathfrak{p}_I$ be the standard parabolic subalgebra generated by $\mathfrak{b}$ and those $\mathfrak{u}_{\alpha_i}$ with $i \in I$. We denote its standard Levi decomposition by $\mathfrak{p}_I = \mathfrak{l}_I \oplus \mathfrak{n}_I$.

Since $\mathfrak{g}_x(-1/h)$ contains a non-nilpotent element, there must exist a positive root α with $\alpha(x) = 1 - 1/h$. (Otherwise, $\mathfrak{g}_x(-1/h)$ is contained in $\bar{\mathfrak{n}}$.) Since any positive root is the sum of at most $h-1$ simple roots, either the decomposition of α into simple roots involves α_j for $j \in J$ or else J is empty, $\alpha_i(x) = 1/h$ for all i, and $\alpha = \alpha_0$. The second case is just $x = x_I$.

It remains to show that J cannot be nonempty. If not, then we see that $\mathfrak{g}_x(-1/h) \subset (\mathfrak{l} \cap \bar{\mathfrak{n}}) \oplus \mathfrak{n}_I$. However, if $X \in \mathfrak{p}_I$ is the sum of a nilpotent element of $\mathfrak{l}_I$ and an element in $\mathfrak{n}_I$, it is nilpotent. Thus, every element of $\mathfrak{g}_x(1/h)$ is nilpotent, a contradiction. □

Remark 21. A similar argument gives another proof that $1/h$ is the smallest possible slope of an irregular singular flat G-bundle. Indeed, if this were false, then there would exist a fundamental stratum (x, r, β) with $x \in C$ and $0 < r < 1/h$. Setting $I = \{i \in [1, n] \mid \alpha_i(x) \leq r\}$, we obtain that for any $\alpha > 0$ coming from $\mathfrak{l}_I$, $\alpha(x) \leq (h-1)r < 1 - 1/h < 1 - r$. It follows that $\mathfrak{g}_x(-r) \subset (\mathfrak{l} \cap \bar{\mathfrak{n}}) \oplus \mathfrak{n}_I$ consists entirely of nilpotent elements, a contradiction.

We now know that ∇ contains a fundamental stratum $(x_I, 1/h, \beta)$ so that the leading term of $[\nabla]$ with respect to the x_I filtration has the form $(t^{-1}y_0 + \sum_{i=1}^n y_i)\frac{dt}{t}$ with $y_0 \in \mathfrak{u}_{\alpha_0}$ and $y_i \in \mathfrak{u}_{-\alpha_i}$ for $i \geq 1$. In order for this element to be non-nilpotent, every y_i must be nonzero. Indeed, if some y_i equals 0, then the remaining n roots are a base for a maximal rank reductive subalgebra of $\mathfrak{g}$, so that the leading term is nilpotent, a contradiction.

We thus have each y_i nonzero. It follows that this leading term is regular semisimple with centralizer a Coxeter maximal torus S with Lie algebra $\mathfrak{s}$. The unique Moy-Prasad filtration on $\mathfrak{s}$ is induced by a grading with degrees in $\frac{1}{h}\mathbb{Z}$. By Theorem 17, after applying a gauge change, one can assume that $[\nabla] \in (\mathfrak{s}(-1/h) \oplus \mathfrak{s}(0))\frac{dt}{t}$. However, since G is simple and S is elliptic, $\mathfrak{s}(0) = \{0\}$. Hence, $\nabla = d + (t^{-1}x_0 + \sum_{i=1}^n x_i)\frac{dt}{t}$ in this trivialization. Since $t^{-1}x_0 + \sum_{i=1}^n x_i \in \mathfrak{s}(-1/h)$ is regular semisimple, each x_i is nonzero, i.e., ∇ is a Frenkel-Gross flat G-bundle. In fact, as shown in Example 18, $\mathfrak{s}(-1/h)$ is one-dimensional, and the cyclic group $W_S \cong \mu_{h'}$ acts freely on $\mathfrak{s}(-1/h) \setminus \{0\}$. Thus, the set of isomorphism classes of Frenkel-Gross connection is isomorphic to $\mathbb{C}^*/\mu_{h'}$. This concludes the proof of the proposition. □

4. A key result about Weyl groups

For the remainder of the paper, let G be a simple complex algebraic group of rank r. Recall that $|\Phi| = hr$.

4.1. **Statement of the result.** For $x \in \mathfrak{h}$, define a non-negative integer

$$N(x) := |\{\alpha \in \Phi \mid \alpha(x) \neq 0\}|. \tag{4}$$

The stabilizer W_x of x is a parabolic subgroup generated by the set Φ_x for which $\alpha(x) = 0$, so that $N(x) = |W| - |W_x|$.

For each positive integer b, let $V(b) \subseteq \mathfrak{h}$ denote the union of all the eigenspaces of elements of W with eigenvalue a primitive b^{th} root of unity, i.e.

$$V(b) = \{x \in \mathfrak{h} \mid wx = \zeta x \text{ for some } w \in W \text{ and some primitive } b^{\text{th}} \text{ root of unity } \zeta\}.$$

Let $d_1, \ldots, d_m$ be the degrees of W, and choose $Q_1, \ldots, Q_m \in \mathbb{C}[\mathfrak{h}]^W$ homogeneous polynomials with $\deg(Q_i) = d_i$ that generate $\mathbb{C}[\mathfrak{h}]^W$. It is a result of Springer [Spr74, Proposition 3.2] that

$$V(b) = \bigcap_{b \nmid d_i} \{x \mid Q(x) = 0\}. \tag{5}$$

In particular, $V(b) = \{0\}$ unless b divides an exponent of W, so $V(b)$ nonzero implies that $b \leq h$.

The following result will play a key role for us:

Theorem 22. *Let b be a positive integer, and suppose $x \in V(b) \setminus \{0\}$. Then $N(x) \geq br$. Moreover, equality is achieved if and only if $b = h$, in which case x is a regular eigenvector for a Coxeter element of W.*

For more details about this result, including its relation with previous results on eigenvalues of Coxeter elements, we refer the reader to [Kam16b].

We remark that the case $b = h$ is a theorem of Kostant [Kos59, §9], so in what follows, we assume that $b < h$. We now discuss the proof, which uses a case by case analysis.

4.2. **Proof for classical groups.** The proof in the classical types proceeds as follows. First, we rule out small values of b. Next, we note that x must be a root of a certain class of polynomials. Finally, we analyze the roots of these polynomials and show that their stabilizers in W can never be "too big". To illustrate this, we give the complete proof for $G = \mathrm{SL}_n$. The adaptation to other classical groups is straightforward.

Let V be the subspace of the $\mathbb{R}^n$ consisting of n-tuples $(x_1, \cdots, x_n)$ satisfying $\sum x_i = 0$. Here, the Weyl group is S_n. It will be convenient to take the coefficients of the characteristic polynomial as the invariant polynomials on $\mathfrak{g}$ and on $\mathfrak{h}$. Up to sign, these are the elementary symmetric polynomials: if $x = \mathrm{diag}(x_1, \cdots, x_n) \in \mathfrak{h} \cong V_{\mathbb{C}}$, then the characteristic polynomial of x is

$$P(X) = \sum c_i X^i,$$

where $c_1 = -\sum x_i = 0, \cdots, c_n = (-1)^n x_1 \cdots x_n$.

Lemma 23. *Suppose $x \in V(b)$. Then, there exists $1 \leq k \leq \frac{n}{b}$ and complex numbers a_i such that the x_i's are the roots of the polynomial*

$$P(X) := X^n + a_1 X^{n-b} + \cdots + a_k X^{n-bk}, \qquad a_k \neq 0. \tag{6}$$

Proof. Indeed, suppose $x = (x_1, \cdots, x_n) \in V(b)$. Then by (5), $c_i = 0$ for all i with $b \nmid i$. Now take $a_i = c_{n-bi}$. □

Let $x \in V_{\mathbb{C}}$ be a non-zero element, so that W_x is a proper parabolic subgroup. It is easy to check that the proper parabolic of W with the largest number of roots is isomorphic to S_{n-1}. Thus, for all non-zero $x \in V_{\mathbb{C}}$, we have

$$|\Phi| - |\Phi_x| \geq n(n-1) - (n-1)(n-2) = 2(n-1) > (n-1).$$

Therefore, the theorem is evident for $b = 1$.

Henceforth, we assume $1 < b < n$, so $n \geq 3$. Let $P(X)$ denote the polynomial in the above lemma. Let $\gamma_1, \cdots, \gamma_k$ denote the roots of the polynomial

$$Q(Y) := Y^k + a_1 Y^{k-1} + \cdots + a_k.$$

Since $a_k \neq 0$, we have that $\gamma_i \neq 0$ for all i. Let ζ be a primitive b^{th} root of unity. Then, the roots of $P(X)$ equal

$$\zeta^i \gamma_j, \qquad i \in \{1, 2, \cdots, b\}, \quad j \in \{1, 2, \cdots, k\}$$

together with $n - kb$ copies of 0.

Note that the largest possible stabilizer for x (for fixed b) is achieved if and only if $\zeta^{i_1}\gamma_1 = \zeta^{i_2}\gamma_2 = \cdots = \zeta^{i_k}\gamma_k$ for some integers $i_1, \cdots, i_k$. In this case,

$$W_x \simeq (S_k)^b \times S_{n-bk}.$$

Thus, for every non-zero $x \in V_{\mathbb{C}}$, we have

$$\begin{aligned} N(x) = |\Phi| - |\Phi_x| &\geq |\Phi_{S_n}| - |\Phi_{(S_k)^b \times S_{n-bk}}| \\ &= n(n-1) - [bk(k-1) + (n-kb)(n-kb-1)] \\ &= 2kbn - k^2b^2 - bk^2. \end{aligned}$$

We claim that $N(x) > b(n-1)$. Indeed, if $k = 1$, then since $b < n$,

$$N(x) = 2bn - b^2 - b > b(n-1).$$

On the other hand, if $k > 1$, then

$$N(x) = 2kbn - k^2b^2 - bk^2 \geq b(2kn - kn - k^2) > b(n-1).$$

Here, the first inequality follows from the fact that $bk \leq n$. The second inequality is equivalent to

$$n(k-1) > (k-1)(k+1) \iff n > k+1,$$

which is true because $k \leq \frac{n}{2}$ and $n \geq 3$. This completes the proof for $G = \mathrm{SL}_n$. $\square$

4.3. **Proof in the exceptional cases.** Next, we turn our attention to the proof in the exceptional cases. We provide the complete proof for E_6. The proof for the other exceptional types is similar, but easier.

Recall that E_6 has 72 roots and its degrees are 2, 5, 6, 8, 9, and 12. It is easy to check that the proper parabolic of E_6 with the largest number of roots is D_5 with 40 roots. Thus, for all non-zero $x \in V_{\mathbb{C}}$, we have

$$N(x) \geq 72 - 40 = 32 > 5 \times 6,$$

so the claim holds for $b \leq 5$.

Now, assume $b > 5$. Let Q denote the unique, up to scalar, homogeneous quadratic invariant polynomial. If x is an eigenvector with eigenvalue a primitive b^{th} root of unity, then $Q(x) = 0$, because $b \nmid 2 = \deg(Q)$. Using this fact, it is easy to show that the stabilizer W_x must be a parabolic subgroup of rank ≤ 4, cf. [Kam16b, Corollary 5]. The maximum number of roots in a parabolic subgroup of E_6 of rank ≤ 4 is 24 (for D_4). Thus,

$$N(x) \geq 72 - 24 = 48 > 6 \times 6, \tag{7}$$

so the result is also true for $b = 6$.

The remaining cases are $b = 8$ and $b = 9$. Let $x \in V(9)$ be a non-zero element. One can show that if x is not regular, then there exists a proper parabolic subgroup of W with a degree divisible by 9. But there is no such parabolic subgroup of E_6. Thus, x must be regular, and so the theorem is immediate.

It remains to treat the case $b = 8$. Suppose x is a non-regular non-zero element of $V(8)$. One can show (cf. [Spr74, Lemma 4.12]) that there exists a proper parabolic subgroup $P < W$ and $w \in P$ such that

$$w \cdot x = \zeta x,$$

where ζ is a primitive 8^{th} root of unity.

Now, the parabolic P must have a degree divisible by 8. The only possibility is $P \simeq D_5$. In this case, however, 8 is the *highest* degree of P, and so, by a theorem of Kostant [Kos59, §9], x is *regular* for the reflection action of P. In particular,

$$\alpha(x) \neq 0, \qquad \text{for all roots } \alpha \text{ of } P.$$

It follows that $\alpha(x)$ is zero for at most one simple root of W. Hence, either x is regular or $W_x \simeq A_1$. In both cases, we have $N(x) > 6 \times 8$. $\square$

4.4. **Conjugacy classes over Laurent series.** We record an application of Theorem 22 to the study of rationality of conjugacy classes over Laurent series.

Corollary 24. *Suppose that $Y \in \mathfrak{g}(\bar{\mathcal{K}})$ is conjugate to an element of $\mathfrak{g}(\mathcal{K})$ and that the semisimple part X of Y lies in $\mathfrak{h}(\bar{\mathcal{K}})$. Suppose*

$$X = xt^{a/b} + \textit{higher order terms}, \qquad x \in \mathfrak{h} \setminus \{0\}, \quad \gcd(a,b) = 1.$$

Then, $N(x) \geq br$. Moreover, equality is achieved if and only if $b = h$ is the Coxeter number, $Y = X$, and x is a regular eigenvector of a Coxeter element of W.

Proof. We first observe that it suffices to assume that Y is semisimple. Indeed, if $\mathrm{Ad}(g)(Y) \in \mathfrak{g}(\mathcal{K})$, then the semisimple part of $\mathrm{Ad}(g)(Y)$ is $\mathrm{Ad}(g)(X)$. Since $\mathcal{K}$ is perfect, $\mathrm{Ad}(g)(X) \in \mathfrak{g}(\mathcal{K})$ [Bor91, Theorem 4.4(2)].

In view of Theorem 22, to obtain the inequality, it is sufficient to show that $x \in V(b)$, i.e., x is an eigenvector for some element of W with eigenvalue a primitive b^{th} root of unity.

Let $Q \in \mathbb{C}[\mathfrak{g}]^G$ be an invariant homogeneous polynomial of degree d. Note that Q is also an invariant polynomial on $\mathfrak{g}(\mathcal{K})$ and $\mathfrak{g}(\bar{\mathcal{K}})$. Choosing $g \in G(\bar{\mathcal{K}})$ such that $\mathrm{Ad}(g)(X) \in \mathfrak{g}(\mathcal{K})$, we see that $Q(X) = Q(\mathrm{Ad}(g)(X)) \in \mathcal{K}$.

Since $xt^{\frac{a}{b}}$ is the leading term of X, we have have

$$Q(X) = Q(xt^{\frac{a}{b}}) + \text{higher order terms} = t^{\frac{da}{b}}Q(x) + \text{higher order terms}.$$

Then, for the leading term of the above expression to be in $\mathcal{K}$, we must have

$$Q(x) = 0 \qquad \text{whenever } b \text{ does not divide } d.$$

Here, we are using the fact that $(a, b) = 1$. Since $\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{h}]^W$, it now follows from (5) that $x \in V(b)$; thus, the inequality is established.

It is immediate from Theorem 22 that $N(x) = br$ if and only if $b = h$ and x is a regular eigenvector of a Coxeter element. This means that the leading term of X is regular semisimple, which implies that X itself is regular semisimple. In particular, $Y = X$. □

Remark 25. Take $X \in \mathfrak{g}(\bar{\mathcal{K}})$, and let $\mathcal{O}_X \subset \mathfrak{g}(\bar{\mathcal{K}})$ denote the $G(\bar{\mathcal{K}})$-orbit of X. Then X is $G(\bar{\mathcal{K}})$-conjugate to an element of $\mathfrak{g}(\mathcal{K})$ if and only if $\mathcal{O}_X$ is closed under the action of $\mathrm{Gal}(\bar{\mathcal{K}}/\mathcal{K})$, i.e., $\mathcal{O}_X$ is defined over $\mathcal{K}$. To see this, note that the forward implication is trivial. The converse follows from a theorem of Steinberg, stating that any homogeneous space defined over a perfect field of cohomological dimension ≤ 1 has a rational point [Ste65, Theorem 1.9].

4.5. **Gauge classes over Laurent series.** We record a version of the above corollary for gauge equivalence classes over Laurent series.

Proposition 26. *Let ∇ be an irregular singular connection on $\mathcal{D}^\times$ with Jordan form $d + (X+N)\frac{du}{u}$. Let $xt^{a/b}$ be the leading term of X, with $x \in \mathfrak{h} \setminus \{0\}$ and* $\gcd(a, b) = 1$*. Then, $N(x) \geq br$. Moreover, equality is achieved if and only if $b = h$, $N = 0$, and x is a regular eigenvector of a Coxeter element of W.*

Proof. We will show that X is conjugate to an element of $\mathfrak{g}(\mathcal{K})$; the desired inequality will then follow by Corollary 24. Let us write

$$X = \sum_i x_i t^{r_i}, \qquad x_i \in \mathfrak{h} \setminus \{0\}, \quad r_i \in \mathbb{Q}_{\leq 0}.$$

Since $d + (X+N)\frac{du}{u}$ is $G(\bar{\mathcal{K}})$-gauge equivalent to the pullback of a connection on $\mathcal{D}^\times$, the proposition in §9.8 of [BV83] implies that there exists an integer $c \geq 1$ and an element $\theta \in G$ such that $\theta^c = 1$, $cr_i = s_i \in \mathbb{Z}$, and

$$\mathrm{Ad}(\theta)(X) = \sum_i x_i t^{r_i}\omega_c^{s_i} = \sum_i x_i t^{s_i/c}\omega_c^{s_i} = \sigma_c(X).$$

Here, $\omega_c = e^{2\pi i/c}$ and σ_c is the generator of $\mathrm{Gal}(\mathcal{K}_c/\mathcal{K})$ defined by $t^{1/c} \mapsto \omega_c t^{1/c}$. Applying θ repeatedly, we see that

$$\mathrm{Ad}(\theta^j)(X) = \sigma_c^j(X), \qquad j = 0, \ldots, c-1.$$

Since the action of $\mathrm{Gal}(\bar{\mathcal{K}}/\mathcal{K})$ on X factors through $\mathrm{Gal}(\mathcal{K}_c/\mathcal{K})$, we have shown that the Galois orbit of X is contained in $\mathrm{Ad}(G(\bar{\mathcal{K}}))(X)$. It is now immediate that this adjoint orbit is defined over $\mathcal{K}$, so by Remark 25, X is conjugate to an element of $\mathfrak{g}(\mathcal{K})$.

As in the proof of Corollary 24, $N(x) = br$ holds if and only if $b = h$ and x is a regular eigenvector of a Coxeter element, and this implies that X is regular semisimple. Since N commutes with X, we obtain $N = 0$. □

Remark 27. The corollary in §9.8 of [BV83] gives a gauge version of Remark 25. Let $\nabla = d + (X+N)\frac{du}{u}$ be a G-connection in Jordan form on $\mathcal{D}_b^\times$ for some b; here, $u^b = t$. Then ∇ is $G(\bar{\mathcal{K}})$-gauge equivalent to the pullback of a connection on $\mathcal{D}^\times$ if and only if the gauge class of ∇ is closed under the action of $\mathrm{Gal}(\bar{\mathcal{K}}/\mathcal{K})$.

5. Proofs of the main theorems

5.1. **Proof of Theorem 2.** Let $(\mathcal{E},\nabla)$ be an irregular singular formal flat G-bundle. Let $d+(X+N)\frac{du}{u}$ denote the Jordan form of ∇, where $u = t^{1/b}$, $X \in \mathfrak{h}(\mathbb{C}[t^{-\frac{1}{b}}])$, and $N \in \mathfrak{n}(\mathbb{C})$. Let $xt^{-k/b}$ be the leading term of X. Since ∇ is irregular singular, $k > 0$, so by Proposition 26, $N(x) \geq br$.

Now, we consider the adjoint connection. By uniqueness of the Jordan form, $d+(\mathrm{Ad}(X)+\mathrm{Ad}(N))\frac{du}{u}$ is the Jordan form for $\mathrm{Ad}(\nabla)$. We view $\mathrm{Ad}(X)$ as a matrix in terms of a basis for $\mathfrak{g}$ consisting of weight vectors. In this basis, $\mathrm{Ad}(X)$ is clearly diagonal, and the irregularity is the sum of the order of the poles of the diagonal entries. If $\alpha(x) \neq 0$, then the corresponding diagonal element has a pole of order k/b, thereby contributing $k/b \geq 1/b$ to the irregularity. We thus obtain

$$\mathrm{Irr}(\mathrm{Ad}(\nabla)) \geq N(x)\frac{k}{b} \geq \frac{N(x)}{b} \geq r. \tag{8}$$

□

5.2. **Proof of Theorem 5.** The equivalence (2) $\iff$ (3) and the implication (3) $\implies$ (1) have been established in Proposition 12. We will complete the proof of the theorem by showing that (1) $\implies$ (2). Suppose ∇ is a connection with $\mathrm{Irr}(\mathrm{Ad}(\nabla)) = r$. This means that all the inequalities in (8) are equalities. In particular, we have $N(x) = br$, implying that $b = h$ and $k = 1$. Thus, ∇ has slope $1/h$. □

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