

A tensor category structure on the equivariant
perverse sheaves over matrices of bounded degree
of nilpotency

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The affine Grassmannian for GL_n

$$F = \mathbb{C}((t)) \supset \mathcal{O} = \mathbb{C}[[t]]$$

Definition

The **affine Grassmannian** (for $GL_n(\mathbb{C})$) is the ind-variety $Gr_n = GL_n(F)/GL_n(\mathcal{O})$.

Lattice interpretation:

Recall that a lattice in F^n is a rank n \mathcal{O} -submodule of F^n .

$$Gr_n \cong \{\text{lattices in } F^n\}$$

Geometric Satake:

$\text{Rep}(GL_n(\mathbb{C}))$ category of complex alg reps of $GL_n(\mathbb{C})$

$\text{Perv}_{GL_n(\mathcal{O})}(Gr_n)$ category of equivariant perverse sheaves on Gr_n .

There exists a convolution product making this into a symmetric monoidal category and

$\text{Perv}_{GL_n(\mathcal{O})}(Gr_n)$ and $\text{Rep}(GL_n(\mathbb{C}))$ are equivalent as tensor categories.

Polynomial representations of $GL_n(\mathbb{C})$

$\text{PolyRep}(GL_n(\mathbb{C})) \subset \text{Rep}(GL_n(\mathbb{C}))$ cat of polynomial representations of $GL_n(\mathbb{C})$

$\{\text{poly irreps}\} \leftrightarrow \{\text{Young diagrams with } \leq n \text{ rows}\}$

If V is an irrep, then $\lambda \text{Id} \in Z(GL_n(\mathbb{C})) \cong \mathbb{C}$ acts on V by λ^m for some $m \in \mathbb{Z}$ called the **degree** of V .

V is a poly irrep iff the degree is nonnegative.

$$\left\{ \begin{array}{l} \text{poly irreps} \\ \text{of degree } m \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Young diagrams with } \leq n \text{ rows} \\ \text{and } m \text{ boxes} \end{array} \right\}$$

Positive part of the affine Grassmannian

The geometric Satake equivalence restricts to give a geometric interpretation of $\text{PolyRep}(\text{GL}_n(\mathbb{C}))$.

Consider lattices intermediate between \mathcal{O}^n and $t^{-m}\mathcal{O}^n$ for fixed m of valuation $-N$:

$$\text{Gr}_n^m(N) = \{L \mid \mathcal{O}^n \subset L \subset t^{-m}\mathcal{O}^n, \dim L/\mathcal{O}^n = N\}$$

These are $\text{GL}_n(\mathcal{O})$ -varieties with orbit structure given by

$$\left\{ \text{GL}_n(\mathcal{O})\text{-orbits in } \text{Gr}_n^m(N) \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Young diagrams with } \leq n \text{ rows} \\ \leq m \text{ cols and } N \text{ boxes} \end{array} \right\}$$

Here, we will only be concerned with the case $N = m$.

Definition

The **positive part of the affine Grassmannian** is the ind-variety

$$\text{Gr}_{n,+} = \bigcup_{m \geq 0} \text{Gr}_n^m(m).$$

Geometric Satake for polynomial representations of $GL_n(\mathbb{C})$

The irreducible $GL_n(\mathcal{O})$ -equivariant perverse sheaves on $Gr_{n,+}$ are again parameterized by the Young diagrams with $\leq n$ boxes. More precisely,

$$\text{Perv}_{GL_n(\mathcal{O})}(Gr_{n,+}) = \bigoplus_{m \geq 0} \text{Perv}_{GL_n(\mathcal{O})}(Gr_n^m(m)).$$

The simple objects in the semisimple cat $\text{Perv}_{GL_n(\mathcal{O})}(Gr_n^m(m))$ are parameterized by the $GL_n(\mathcal{O})$ orbits:

$$\left\{ GL_n(\mathcal{O})\text{-orbits in } Gr_n^m(m) \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Young diagrams with } \leq n \text{ rows} \\ \text{and } m \text{ boxes} \end{array} \right\}$$

$\text{Perv}_{GL_n(\mathcal{O})}(Gr_{n,+}) \simeq \text{PolyRep}(GL_n(\mathbb{C}))$ as tensor categories.

A “transposed” problem

For geometric Satake: product on Young diagrams with a fixed bound n on the number of rows:

$$\begin{aligned} \text{Perv}_{\text{GL}_n(\mathcal{O})}(\text{Gr}_n^m(m)) \times \text{Perv}_{\text{GL}_n(\mathcal{O})}(\text{Gr}_n^{m'}(m')) \\ \rightarrow \text{Perv}_{\text{GL}_n(\mathcal{O})}(\text{Gr}_n^{m+m'}(m+m')). \end{aligned}$$

Problem

Is there a natural transposed version of this product, i.e., a product on Young diagrams with a fixed bound m on the number of columns? More precisely, can one give a geometric definition of a product

$$\begin{aligned} \text{Perv}_{\text{GL}_n(\mathcal{O})}(\text{Gr}_n^m(n)) \times \text{Perv}_{\text{GL}_{n'}(\mathcal{O})}(\text{Gr}_{n'}^{m'}(n')) \\ \rightarrow \text{Perv}_{\text{GL}_{n+n'}(\mathcal{O})}(\text{Gr}_{n+n'}^{m+m'}(n+n')). \end{aligned}$$

which makes $\bigoplus_{n \geq 0} \text{Perv}_{\text{GL}_n(\mathcal{O})}(\text{Gr}_n^m(n))$ into a symmetric

A nilpotent cone version of the transposed problem

Let \mathcal{N}_n denote the set of $n \times n$ nilpotent matrices.

By Jordan canonical form, each $\mathrm{GL}_n(\mathbb{C})$ -orbit corresponds to a partition of n with k th part the number of $k \times k$ Jordan blocks.

Let $\mathcal{N}_n^m = \{x \in \mathfrak{gl}_n(\mathbb{C}) \mid x^m = 0\}$, the matrices with degree of nilpotency $\leq m$.

The simple objects in $\mathrm{Perv}_{\mathrm{GL}_n(\mathbb{C})}(\mathcal{N}_n^m)$ are parameterized by the $\mathrm{GL}_n(\mathbb{C})$ orbits:

$$\left\{ \mathrm{GL}_n(\mathbb{C})\text{-orbits in } \mathcal{N}_n^m \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Young diagrams with } n \text{ boxes} \\ \text{and } \leq m \text{ columns} \end{array} \right\}$$

Main Theorem

Let $\mathcal{P}^m = \bigoplus_{n \geq 0} \text{Perv}_{\text{GL}_n(\mathbb{C})}(\mathcal{N}_n^m)$.

The simple objects are parameterized by Young diagrams with at most m columns.

Theorem (Achar-S. 2017)

Fix $m \geq 0$. There exists a binary operation

$$\begin{aligned} \circledast : \text{Perv}_{\text{GL}_n(\mathbb{C})}(\mathcal{N}_n^m) \times \text{Perv}_{\text{GL}_{n'}(\mathbb{C})}(\mathcal{N}_{n'}^m) \\ \rightarrow \text{Perv}_{\text{GL}_{n+n'}(\mathbb{C})}(\mathcal{N}_{n+n'}^m) \end{aligned}$$

which makes \mathcal{P}^m into a graded symmetric monoidal category.

Construction

We first introduce some notation and some maps.

Fix n, n' , and let $P_{n,n'} \subset \mathrm{GL}_{n+n'}(\mathbb{C})$ be the block-upper triangular parabolic subgroup with diagonal blocks of sizes n and n' .

Let $\mathfrak{p}_{n,n'} = \mathrm{Lie}(P_{n,n'})$.

$$Y_{n,n'} = \left\{ \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \mid x \in \mathcal{N}_n^m, y \in \mathcal{N}_{n'}^m, z \in \mathcal{M}_{n,n'}(\mathbb{C}) \right\} \subset \mathfrak{p}_{n,n'} \cap \mathcal{N}_{n+n'}$$

We have the following maps (all defined in the obvious way):

$$\begin{array}{ccc} Y_{n,n'} \hookrightarrow & \mathrm{GL}_{n+n'}(\mathbb{C}) \times^{P_{n,n'}} Y_{n,n'} & \xrightarrow{\mu} \mathcal{N}_{n+n'} \\ \downarrow \pi & & \\ \mathcal{N}_n \times \mathcal{N}_{n'} & & \end{array}$$

Construction (cont.)

The product \otimes is now defined as the following composition:

$$\begin{aligned} \text{Perv}_{\text{GL}_n(\mathbb{C})}(\mathcal{N}_n^m) \times \text{Perv}_{\text{GL}_{n'}(\mathbb{C})}(\mathcal{N}_{n'}^m) &\xrightarrow{\boxtimes} \text{Perv}_{\text{GL}_n(\mathbb{C}) \times \text{GL}_{n'}(\mathbb{C})}(\mathcal{N}_n^m \times \mathcal{N}_{n'}^m) \\ &\xrightarrow{\pi^*} \text{Perv}_P(Y) \\ &\xrightarrow{\simeq} \text{Perv}_{\text{GL}_{n+n'}(\mathbb{C})}(\text{GL}_{n+n'}(\mathbb{C}) \times^P Y) \\ &\xrightarrow{\mu_*} \text{Perv}_{\text{GL}_{n+n'}(\mathbb{C})}(\mathcal{N}_{n+n'}^m) \\ &\xrightarrow{\text{discard}} \text{Perv}_{\text{GL}_{n+n'}(\mathbb{C})}(\mathcal{N}_{n+n'}^m). \end{aligned}$$

The last step throws away the simple objects corresponding to Young diagrams with more than m columns.

Associativity

We briefly sketch the proof of associativity and commutativity of this product.

For associativity, one shows that the product of three perverse sheaves

$$\begin{aligned} \text{Perv}_{\text{GL}_n(\mathbb{C})}(\mathcal{N}_n^m) \times \text{Perv}_{\text{GL}_{n'}(\mathbb{C})}(\mathcal{N}_{n'}^m) \times \text{Perv}_{\text{GL}_{n''}(\mathbb{C})}(\mathcal{N}_{n''}^m) \\ \rightarrow \text{Perv}_{\text{GL}_{n+n'+n''}(\mathbb{C})}(\mathcal{N}_{n+n'+n''}^m) \end{aligned}$$

with either choice of parenthesis can be computed via a procedure involving a three block parabolic subgroup of $\text{GL}_{n+n'+n''}(\mathbb{C})$.

More precisely, one takes the block upper triangular parabolic $P_{n,n',n''}$ and considers the subspace $Y_{n,n',n''} \subset \text{Lie}(P_{n,n',n''})$ whose block diagonal entries are nilpotent of degree $\leq m$.

The product above may then be realized directly as above, but using $Y_{n,n',n''}$.

Commutativity

This is more difficult; we just mention a few key ingredients.

- ▶ Recall Lusztig's embedding $\mathcal{N}_n \hookrightarrow \mathrm{Gr}_n^n(n)$:
Given $f \in \mathcal{N}_n$, one takes the lattice $\mathcal{O}^n \subset L_f \subset t^{-n}\mathcal{O}^n$ for which $L_f/c\mathcal{O}^n$ is the image of the vector space $\{t^{-1}v + t^{-2}f(v) + \cdots + t^{-n}f^{n-1}(v) \mid v \in \mathbb{C}^n\}$.
This induces an equivalence between $\mathrm{Perv}_{\mathrm{GL}_n(\mathbb{C})}(\mathcal{N})$ and $\mathrm{GL}_n(\mathcal{O})$ -equivariant perverse sheaves on an appropriate piece $\widetilde{\mathrm{Gr}}_n$ of Gr_n .
- ▶ Hyperbolic localization gives rise to a functor $\mathrm{Perv}_{\mathrm{GL}_{n+n'}(\mathcal{O})}(\widetilde{\mathrm{Gr}}_{n+n'}) \rightarrow \mathrm{Perv}_{\mathrm{GL}_n(\mathcal{O}) \times \mathrm{GL}_{n'}(\mathcal{O})}(\widetilde{\mathrm{Gr}}_n \times \widetilde{\mathrm{Gr}}_{n'})$.
- ▶ The equivalence from Lusztig's embedding allows one to compare hyperbolic localization on $\widetilde{\mathrm{Gr}}$ to an appropriate functor involving \mathcal{N} 's.
- ▶ A result of Beilinson and Drinfeld shows that hyperbolic localization is compatible (via geometric Satake) to $\mathrm{rest} : \mathrm{Rep}(\mathrm{GL}_{n+n'}(\mathbb{C})) \rightarrow \mathrm{Rep}(\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_{n'}(\mathbb{C}))$, which is independent of the order of n and n' .

A variant for classical groups

One can apply a similar procedure to construct module categories for \mathcal{P}^m for each classical type. For simplicity, we state the result only for type C.

Let $\mathcal{N}_{\mathrm{Sp}_{2n}}^m = \mathcal{N}_{2n}^m \cap \mathfrak{sp}_{2n}(\mathbb{C})$.

Let $\mathcal{M}^m = \bigoplus_{n \geq 0} \mathrm{Perv}_{\mathrm{Sp}_{2n}(\mathbb{C})}(\mathcal{N}_{\mathrm{Sp}_{2n}}^m)$.

Theorem (Achar-S. 2017)

Fix $m \geq 0$. There exists a binary operation

$$\begin{aligned} \circledast : \mathrm{Perv}_{\mathrm{GL}_n(\mathbb{C})}(\mathcal{N}_n^m) \times \mathrm{Perv}_{\mathrm{Sp}_{2n'}(\mathbb{C})}(\mathcal{N}_{\mathrm{Sp}_{2n'}}^m) \\ \rightarrow \mathrm{Perv}_{\mathrm{Sp}_{2(n+n')}(\mathbb{C})}(\mathcal{N}_{\mathrm{Sp}_{2(n+n')}}^m) \end{aligned}$$

which makes \mathcal{M}^m into a graded module category for \mathcal{P}^m .

The construction uses the fact that $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{Sp}_{2n'}(\mathbb{C})$ is a Levi subgroup of $\mathrm{Sp}_{2(n+n')}(\mathbb{C})$.

Two questions

1. The simple objects of \mathcal{P}^m are parameterized by the same set of Young diagrams (up to transpose) as the simple objects of $\text{PolyRep}(\text{GL}_m(\mathbb{C}))$. Are these two tensor categories equivalent?
2. If this is the case, one obtains various module categories (eg \mathcal{M}^m in type C) over $\text{PolyRep}(\text{GL}_m(\mathbb{C}))$. Can one identify them more explicitly?