ATOMISTIC SUBSEMIRINGS OF THE LATTICE OF SUBSPACES OF AN ALGEBRA

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Abstract. Let A be an associative algebra with identity over a field k. An atomistic subsemiring R of the lattice of subspaces of A, endowed with the natural product, is a subsemiring which is a closed atomistic sublattice. When R has no zero divisors, the set of atoms of R is endowed with a multivalued product. We introduce an equivalence relation on the set of atoms such that the quotient set with the induced product is a monoid, called the condensation monoid. Under suitable hypotheses on R, we show that this monoid is a group and the class of $k1_A$ is the set of atoms of a subalgebra of A called the focal subalgebra. This construction can be iterated to obtain higher condensation groups and focal subalgebras. We apply these results to G-algebras for G a group; in particular, we use them to define new invariants for finite-dimensional irreducible projective representations.

1. Introduction

Let A be an associative algebra with identity over a field k, and let $S(A)$ be the complete lattice of subspaces of A. The algebra multiplication on A induces a product on $S(A)$ given by $EF = \text{span}\{ef \mid e \in E, f \in F\}$. The lattice S thus becomes an additively idempotent semiring, with $\{0\}$ and $k = k1_A$ (which we will often denote by 0 and 1) as the additive and multiplicative identities.

Let R be a closed sublattice of $S(A)$ which is also a subsemiring, i.e., R contains 0 and k and is closed under arbitrary sums and intersections and finite products. (We do not require the maximum element of R to be A.) A nonzero element $X \in R$ is called decomposable (or join-reducible) if there exists $U, V \subsetneq R$ such that $X = U + V$ and indecomposable otherwise. It is immediate that the multiplication in R is determined by the product of indecomposable elements. In other words, the semiring structure is determined by the structure constants $c^W_{U,V}$ for $U, V, W \in R$ indecomposable, where $c^W_{U,V}$ is 1 if $W \subset UV$ and 0 otherwise.

In this paper, we consider subsemirings R whose product is determined by its minimal nonzero elements–the atoms of the lattice. This means that the indecomposable elements of R are precisely the atoms, so that every nonzero element is a join of atoms, i.e., R is an atomistic lattice[1].

2000 *Mathematics Subject Classification.* Primary: 16Y60, 20N20; Secondary: 16W22.

Key words and phrases. atomistic semiring, hypergroup, G-algebra, subrepresentation semiring.

The author gratefully acknowledges support from NSF grant DMS-0606300 and NSA grant H98230-09-1-0059.

[1] In the usual definition, every nonzero element of an atomistic lattice is a finite join of atoms. In this paper, we allow arbitrary joins of atoms.

Definition 1.1. A subsemiring $R \subset S(A)$ is called an *atomistic subsemiring* of $S(A)$ if it is also a closed atomistic sublattice.

Note that k is always an atom in R.

Example 1.2. For any A, $S(A)$ and $\{0,1\}$ are atomistic subsemirings.

Example 1.3. Let X be any proper subspace with $X + k1_A = A$. Then $R = \{0,1,X,A\}$ is an atomistic subsemiring if and only if $X^2 \in R$. All four possible values for X^2 can occur. Indeed, if we let $X = k\bar{t}$ in the three two-dimensional algebras $k[t]/(t^2)$, $k[t]/(t^2-1)$, and $k[t]/(t^2-t)$, we obtain X^2 equal to 0, 1, and X respectively. On the other hand, if $X = \mathrm{span}(\bar{t}, \bar{t}^2)$ in $A = k[t]/(t^3-1)$, then $X^2 = A$. (Note that there are never any atomistic subsemirings of size 3.)

Example 1.4. Let V be a vector space with $\dim V \geq 2$, and suppose $(\mathrm{char}\, k, \dim V) = 1$. Let $A = \mathrm{End}(V)$, and let $X = \{x \in \mathrm{End}(V) \mid \mathrm{tr}(x) = 0\}$. Then $R = \{0,1,X,A\}$ is atomistic with $X^2 = A$. To see this, simply note that every matrix unit lies in X^2: $E_{ii} = E_{ij}E_{ji}$ and $E_{ij} = E_{ij}(E_{ii} - E_{jj})$ where $i \neq j$.

Our primary motivation for considering atomistic subsemirings comes from representation theory. Let G be a group which acts on A by algebra automorphisms. This means that A is a $k[G]$-module such that $g \cdot 1_A = 1_A$ and $g \cdot (ab) = (g \cdot a)(g \cdot b)$ for all $g \in G$ and $a, b \in A$. We let $S_G(A) \subset S(A)$ be the set of all $k[G]$-submodules of A. This set, called the *subrepresentation semiring* of A, is simultaneously a subsemiring and complete sublattice of $S(A)$; such semirings were introduced and studied in [9, 10]. If A is a completely reducible representation, i.e., a direct sum of irreducible representations, then $S_G(A)$ is an atomistic subsemiring. For example, this occurs when G is finite, A is finite-dimensional, and k has characteristic zero.

When $G = \mathrm{SU}(2)$ (or more generally, G is a *quasi-simply reducible* group), then the subrepresentation semirings for the G-algebras $\mathrm{End}(V)$ (with V a representation of G) have had important applications in materials science and physics [5, 4, 9]. The structure of such semirings is intimately related to the theory of $6j$-coefficients from the quantum theory of angular momentum [9, 10, 11, 6].

Our goal in this paper is to study the set of atoms $\mathfrak{Q}(R)$ of an atomistic subsemiring and to use it to define new invariants for appropriate R–the condensation group, the focus, the focal subalgebra, and higher analogues. Our methods are motivated by the theory of hypergroups.

We now give a brief outline of the contents of the paper. In Section 2, we define a multivalued product on the set $\mathfrak{Q}(R)$ of atoms of an atomistic subsemiring R. In the next section, we introduce an equivalence relation ζ^* on $\mathfrak{Q}(R)$. We show that if R has no zero-divisors, then the quotient set $\mathfrak{Q}(R)/\zeta^*$ is naturally a monoid (called the condensation monoid) while if R is *weakly reproducible*, the condensation monoid is in fact a group. In Section 4, we define the focus $\varpi_R \subset \mathfrak{Q}(R)$ and focal subalgebra $F(R) \subset A$ of R. The main result is Theorem 4.3, which states that if R is weakly reproducible of finite length, then $[0, F(R)]$ is an atomistic subsemiring with the same properties and whose set of atoms is ϖ_R. This allows us to iterate our construction to obtain higher order versions of our invariants. In Section 5, we prove Theorem 4.3 by analyzing *complete subsets* of $\mathfrak{Q}(R)$. We apply our results to G-algebras in the final section. In particular, we show how to associate new invariants to irreducible projective representations.

2. A hyperproduct on the set of atoms

From now on, R will always be an atomistic subsemiring of $S(A)$. Let $\mathcal{Q}(R)$ denote the set of atoms of R. If $R = S_G(A)$ for a G-algebra A, we write $\mathcal{Q}_G(A)$ instead of $\mathcal{Q}(S_G(A))$. We make the notational convention that, unless otherwise specified, capital letters towards the end of the alphabet will denote atoms.

There is a natural operation $\mathcal{Q}(R) \times \mathcal{Q}(R) \to \mathcal{P}(\mathcal{Q}(R))$ given by $X \circ Y = \{Z \in \mathcal{Q}(R) \mid Z \subset XY\}$. Our first goal is to find a natural equivalence relation on $\mathcal{Q}(R)$ (for appropriate R) for which $\circ$ induces a monoid (or group) structure on the set of equivalence classes.

Before proceeding, we need to recall some definitions from the theory of hypergroups. A set $\mathcal{H}$ is called a hypergroupoid if it is endowed with a binary operation $\circ : \mathcal{H} \times \mathcal{H} \to \mathcal{P}^*(\mathcal{H})$, where $\mathcal{P}^*(\mathcal{H})$ is the set of nonempty subsets of $\mathcal{H}$. If this operation is associative, then $\mathcal{H}$ is called a semihypergroup; if $\mathcal{H}$ also satisfies the reproductive law $\mathcal{H} \circ x = \mathcal{H} = x \circ \mathcal{H}$ for all $x \in \mathcal{H}$, then $\mathcal{H}$ is called a hypergroup. (For more details on hypergroups, see the books by Corsini [1] and Vougiouklis [12].)

An element e of the hypergroupoid $\mathcal{H}$ is called a scalar identity if $e \circ x = \{x\} = x \circ e$ for all $x \in \mathcal{H}$; if a scalar identity exists, it is unique. For later use, we introduce a weak version of the reproductive law. A hypergroupoid with scalar identity e satisfies the *weak reproductive law* if for any $x \in \mathcal{H}$, there exists $u, v \in \mathcal{H}$ such that $e \in x \circ u \cap v \circ x$. Note that a semihypergroup that satisfies the weak reproductive law is a hypergroup. Indeed, given $y \in \mathcal{H}$, $y \in y \circ e \subset y \circ (v \circ x) = (y \circ v) \circ x$, so there exists $w \in y \circ v$ such that $y \in w \circ x$. Similarly, there exists w' such that $y \in x \circ w'$.

In general, $\mathcal{Q}(R)$ is not even a hypergroupoid. However, we have the following result:

Proposition 2.1. *Let R be an atomistic subsemiring. Then $(\mathcal{Q}(R), \circ)$ is a hypergroupoid if and only if R is an entire semiring (i.e., R has no left or right zero divisors).*

Proof. Suppose R is entire. If $X, Y \in \mathcal{Q}(R)$, then the nonzero subspace XY must contain an atom, so $X \circ Y \neq \varnothing$. Conversely, if E, F are nonzero elements of R such that $EF = 0$, then choosing $X, Y \in \mathcal{Q}(R)$ such that $X \subset E$ and $Y \subset F$ implies that $XY = 0$, i.e., $X \circ Y = \varnothing$. □

In particular, if A has zero divisors, then $\mathcal{Q}(S(A))$ is not a hypergroupoid. We will only be interested in atomistic subsemirings R for which $\mathcal{Q}(R)$ is a hypergroupoid, so, from now on, we assume that R is entire, unless otherwise specified. Note that k is a scalar identity for $\mathcal{Q}(R)$.

We begin by considering a motivating example. We need to recall some basic properties of semisimple, multiplicity-free representations. This class of G-modules is closed under taking submodules and quotients. Any such representation V is the direct sum of its irreducible submodules, and this is the only way of decomposing V as the internal direct sum of irreducible submodules. Moreover, there is a bijection between the power set of the set of irreducible submodules of V and the set of subrepresentations of V given by $J \mapsto \sum_{X \in J} X$. It follows that if $\{V_i \mid i \in I\}$ is a collection of submodules of V and $W = \sum_{i \in I} V_i$, then for X irreducible, $X \subset W$ if and only if $X \subset V_j$ for some $j \in I$.

Proposition 2.2. *Let A be a multiplicity-free G-algebra with no proper, nontrivial left (or right) invariant ideals. Then $\mathcal{Q}_G(A)$ is a hypergroup.*

Proof. First, we show that the multiplication on $\mathcal{Q}_G(A)$ is associative. Fix $X, Y, Z \in \mathcal{Q}_G(A)$. Since A is multiplicity-free, $XY = \sum_{j\in J} U_j$, where $X \circ Y = \{U_j \mid j \in J\}$. As discussed above, an irreducible submodule W lies in $(XY)Z = \sum U_j Z$ if and only if it is contained in $U_i Z$ for some i, i.e., $W \in U_i \circ Z$. We thus see that $(X\circ Y)\circ Z$ is the set of irreducible submodules of XYZ. A similar argument shows that the same holds for $X \circ (Y \circ Z)$.

Next, we show that $X \circ Y \neq \varnothing$ for any $X, Y \in \mathcal{Q}_G(A)$. It suffices to show that $XY \neq 0$ for all X, Y. Let $Y^{\perp} = \{a \in A \mid ay = 0 \text{ for all } y \in Y\}$. The subspace $Y^{\perp}$ is clearly a left ideal. Moreover, it is a subrepresentation: given $g \in G, u \in Y^{\perp}$, $(g \cdot a)u = g \cdot (a(g^{-1} \cdot u)) = g \cdot 0 = 0$. Since $Y^{\perp} \neq A$, our hypothesis on invariant left ideals implies that $Y^{\perp} = 0$ and $XY \neq 0$ for all X.

Finally, we show that $X \circ \mathcal{Q}_G(A) = \mathcal{Q}_G(A) = \mathcal{Q}_G(A) \circ X$ for any X. The subspace AX is a nonzero left ideal which is obviously a subrepresentation, so $AX = A$. Writing A as a sum of irreducible submodules $A = \sum_{i\in I} U_i$, we have $A = \sum U_i X$. The usual multiplicity-free argument shows that each U_j lies in some $U_{i_j} X$, so $U_j \in U_{i_j} \circ X$. The other equality uses the condition on invariant right ideals. □

Matrix algebras give an important class of examples. If V is a finite-dimensional vector space and $\mathrm{End}(V)$ is a G-algebra, then V is naturally a projective representation of G [8]. It was further shown in [8] that $\mathrm{End}(V)$ for such representations has no proper, nontrivial invariant left or right ideals if and only if V is irreducible. Hence, we obtain:

Corollary 2.3. *If V is a finite-dimensional irreducible projective representation of a group such that* $\mathrm{End}(V)$ *is multiplicity free, then* $\mathcal{Q}_G(\mathrm{End}(V))$ *is a hypergroup.*

This corollary applies, for example, to every irreducible complex representation of $\mathrm{SU}(2)$.

The importance of Proposition 2.2 stems from the fact that there is a group naturally associated to every hypergroup. More generally, let $\mathcal{H}$ be a semihypergroup. Consider the relation β defined by $x \,\beta\, y$ if and only if there exists $z_1, \dots, z_n \in \mathcal{H}$ such that $x, y \in z_1 \circ \cdots \circ z_n$. Koskas showed that if β^* is the transitive closure of β, then the induced multiplication makes $\mathcal{H}/\beta^*$ into a semigroup, and β^* is the largest equivalence relation on $\mathcal{H}$ with this property [7]. If $\mathcal{H}$ is a hypergroup, then Freni proved that β is automatically transitive [2]; thus, $\mathcal{H}/\beta$ is a group.

We are led to the following provisional definition.

Definition 2.4. Let A be a G-algebra satisfying the hypotheses of Proposition 2.2. The group $Q_G(A) = \mathcal{Q}_G(A)/\beta$ is called the *condensation group* of A.

We will generalize this definition to a much broader class of atomistic subsemirings below. However, before continuing we provide a few examples.

Example 2.5. If k denotes the trivial G-algebra, then $Q_G(k)$ is the trivial group.

Example 2.6. If V is any irreducible representation of $\mathrm{SU}(2)$, then $Q_{\mathrm{SU}(2)}(\mathrm{End}(V)) = 1$. The proof is a special case of Theorem 6.6 below.

Example 2.7. Let V be the standard representation of S_3 over the complex numbers. The corresponding S_3-algebra decomposes as $\mathrm{End}(V) = \mathbf{C} \oplus \sigma \oplus V$, where σ is the sign representation. Since $\sigma^2 = \mathbf{C}$, $\sigma V = V\sigma = V$, and $V^2 = \mathbf{C} \oplus \sigma$, we see that the classes of β are $\{\mathbf{C}, \sigma\}$ and $\{V\}$; hence, the condensation group has order 2.

Example 2.8. If F is a finite Galois extension of k with abelian Galois group G, then $\mathcal{Q}_G(F) = G$.

We remark that if the relation β is replaced by Freni's relation γ [3], one gets an abelian group canonically related to any hypergroup. However, we will not attempt to generalize the abelian group $\mathcal{Q}_G(A)/\gamma$ to other atomistic subsemirings in this paper.

3. The equivalence relation ζ^*

It is not true in general that the hypergroupoid $\mathcal{Q}(R)$ is a hypergroup or even a semihypergroup. For example, the binary operation on $\mathcal{Q}_{A_4}(\mathrm{End}(W))$ is not associative, where W is the three-dimensional irreducible representation of A_4. Moreover, the reproductive law is not satisfied. (See Example 6.5 below.) We can thus no longer use the relation β^* to associate a monoid or group to R. Instead, we will do so by introducing a new relation ζ. This relation will coincide with β in the situation of Proposition 2.2.

Definition 3.1. The relation ζ on $\mathcal{Q}(R)$ is defined by $X \,\zeta\, Y$ if and only if there exists $Z_1, \dots, Z_n \in \mathcal{Q}(R)$ such that $X, Y \subset \prod_{i=1}^n Z_i$. We let ζ^* denote the transitive closure of ζ.

It is obvious that ζ^* is an equivalence relation. We will let $\bar{X}$ denote the equivalence class of $X \in \mathcal{Q}(R)$.

Remark 3.2. If Z is an atom contained in the $\circ$ product of $Z_1, \dots, Z_n$ with any choice of parentheses, then $Z \subset \prod_{i=1}^n Z_i$. In fact, the relation β can be defined for hypergroupoids, and this observation just says that $\beta \subset \zeta$. However, the set of β^*-equivalence classes is not necessarily a monoid.

Definition 3.3. Let R be an entire, atomistic subsemiring of $S(A)$.

1. R is called *weakly reproducible* if the hypergroupoid $\mathcal{Q}(R)$ satisfies the weak reproductive law, i.e., for all $X \in \mathcal{Q}(R)$, there exists $Y, Z \in \mathcal{Q}(R)$ such that $k \in X \circ Y \cap Z \circ X$.
2. R is called *reproducible* if $\mathcal{Q}(R)$ satisfies the reproductive law, i.e., for all $X \in \mathcal{Q}(R)$, $\mathcal{Q}(R) \circ X = \mathcal{Q}(R) = X \circ \mathcal{Q}(R)$.

Remark 3.4. One can define an atomistic subsemiring R to be weakly reproducible without the assumption that R is entire. However, R is then entire automatically. Indeed, if $XY = 0$ for $X, Y \in \mathcal{Q}(R)$, then weak reproducibility implies the existence of Z such that $k \subset ZX$, so $Y = kY \subset ZXY = 0$, a contradiction.

Theorem 3.5. *Let R be an entire, atomistic semiring of $S(A)$. Then*

1. *The induced multiplication on classes makes $Q(R) \overset{def}{=} \mathcal{Q}(R)/\zeta^*$ into a monoid.*
2. *If R is weakly reproducible, then $Q(R)$ is a group.*

Definition 3.6. The monoid $Q(R)$ is called the *condensation monoid* (or *group*) of R.

The following lemma shows that this terminology does not conflict with our previous definition.

Lemma 3.7. *If A satisfies the hypotheses of Proposition 2.2, then β and ζ coincide on $\mathcal{Q}_G(A)$.*

Proof. A similar argument to that used to demonstrate the associativity of $\mathcal{Q}_G(A)$ shows that $Z_1 \circ \cdots \circ Z_n$ is the set of irreducible submodules of $\prod_{i=1}^n Z_i$, so $\beta = \zeta$. $\square$

Recall that an equivalence relation $\sim$ on a hypergroupoid $\mathcal{H}$ is called strongly regular if, for any x, y such that $z \sim y$ and any $w \in \mathcal{H}$, then $u \in x \circ w$ and $v \in y \circ w$ (resp. $u \in w \circ x$ and $v \in w \circ y$) implies that $u \sim v$. It is a standard fact that for such $\sim$, $\circ$ induces a binary operation on $\mathcal{H}/\sim$ via $\bar{x} \circ \bar{y} = \bar{z}$, where $z \in x \circ y$ [1]. Indeed, strong regularity implies that the set $\{\bar{z} \mid z \in x' \circ y' \text{ for some } x' \in \bar{x}, y' \in \bar{y}\}$ is a singleton.

Lemma 3.8. *The equivalence relation ζ^* is strongly regular.*

Proof. First, suppose that $X \,\zeta\, Y$, so $X, Y \subset \prod_{i=1}^n Z_i$ for some Z_i's. If $U \in X \circ W$ and $V \in Y \circ W$, then $U \subset XW$ and $V \subset YW$. Thus, $U, V \subset (\prod_{i=1}^n Z_i)W$, i.e., $U \,\zeta\, W$. If $X \,\zeta^*\, Y$, then there exists $X_0, \ldots, X_s \in \mathcal{Q}(R)$ with $X = X_0$, $Y = X_s$, and $X_i \,\zeta\, X_{i+1}$ for all i. Taking $U_i \in X_i \circ W$ with $U = U_0$ and $V = U_s$, the previous case shows that $U_i \,\zeta\, U_{i+1}$ for all i, i.e., $U \,\zeta^*\, V$. The opposite direction in the definition of strong regularity is proved similarly. $\square$

We now verify that the induced binary operation makes $Q(R)$ into a monoid. The identity is given by $\bar{k}$; indeed, this follows immediately from the fact that $k \circ X = X = X \circ k$. Next, we check that $(\bar{X} \circ \bar{Y}) \circ \bar{Z} = \bar{X} \circ (\bar{Y} \circ \bar{Z})$. Choose $U \in X \circ Y$ and $V \in U \circ Z$, so that $\bar{V} = (\bar{X} \circ \bar{Y}) \circ \bar{Z}$. Since $U \subset XY$, $V \subset UZ \subset XYZ$. Similarly, choosing $T \in Y \circ Z$ and $W \in X \circ T$ gives $\bar{W} = \bar{X} \circ (\bar{Y} \circ \bar{Z})$ and $W \subset XT \subset XYZ$. By definition, $V \,\zeta\, W$, so $Q(R)$ is associative.

Remark 3.9. If we allow R to be an atomistic hemiring of $S(A)$, i.e., we do not require that $k \in R$, then the same argument shows that $Q(R)$ is a semigroup.

Finally, assume that R is weakly reproducible. Given $X \in \mathcal{Q}(R)$, choose Y, Z such that $k \subset XY \cap ZX$. By definition of the product on $Q(R)$, we obtain $\bar{X} \circ \bar{Y} = \bar{k} = \bar{Z} \circ \bar{X}$, so $\bar{X}$ is left and right invertible. This shows that $Q(R)$ is a group and finishes the proof of the theorem.

Remark 3.10. Any monoid can be realized as the condensation monoid of an atomistic subsemiring. Indeed, given a monoid M, let kM be the corresponding monoid algebra over k with basis elements $\{e_x | x \in M\}$. Let $R = \{\text{span}\{e_x \mid x \in F\} \mid F \subset M\}$. This is an entire atomistic subsemiring of $S(kM)$ with $\mathcal{Q}(R) = \{ke_x \mid x \in M\}$. It is now easy to see that $Q(R) = M$.

4. The focus and the focal subalgebra

Recall that if $\mathcal{H}$ is a hypergroup, the *heart* $\omega_{\mathcal{H}}$ of $\mathcal{H}$ is the kernel of the canonical homomorphism $\phi : \mathcal{H} \to \mathcal{H}/\beta^*$; it is a subhypergroup of $\mathcal{H}$. Returning to the context of Proposition 2.2, let A be a multiplicity-free G-algebra with no proper, nonzero left or right invariant ideals. We may then use the heart ω of the hypergroup $\mathcal{Q}_G(A)$ to define an invariant subalgebra with the same properties.

Proposition 4.1. *Let A be a multiplicity-free G-algebra with no proper, nontrivial one-sided invariant ideals. Then $B = \sum\{X \mid X \in \omega\}$ is a multiplicity-free G-subalgebra with no proper one-sided invariant ideals.*

Proof. It is trivial that B is a multiplicity-free subrepresentation that contains k. Moreover, if $X, Y \in \omega$ and $Z \subset XY$ is irreducible, then $\phi(Z) = \phi(X)\phi(Y) = 1$, i.e., $Z \in \omega$. This means that Z and hence XY are subspaces of B. It remains to show that $BX = B = XB$ for any $X \in \omega$. Choose $Z \in \omega$. Since $\mathcal{Q}_G(A)$ is a hypergroup, there exists Y irreducible such that $Z \in Y \circ X$. Since $1 = \phi(Z) = \phi(Y)\phi(X) = \phi(Y)$, we see that $Y \in \omega$, so $Z \subset BX$. The proof that $Z \subset XB$ is similar. $\square$

This result allows us to iterate the construction of the condensation group. Indeed, the hypergroup structure on $\mathcal{Q}_G(B) = \omega$ gives rise to the group $Q_G(B)$ and an invariant subalgebra $B' \subset B$ such that $\mathcal{Q}_G(B')$ is again a hypergroup. See Section 6 for examples.

Motivated by this situation, we make the following definitions.

Definition 4.2. Let R be an entire atomistic subsemiring.

(1) The *focus* ϖ_R of R is the kernel of the homomorphism $\psi_R : \mathcal{Q}(R) \to Q(R)$. Equivalently, it is the equivalence class of k.
(2) The subspace $F(R) = \sum\{X \mid X \in \varpi_R\} \in R$ is called the *focal subalgebra* associated to R.

We can now state one of the main results of the paper.

Theorem 4.3. *Let R be an entire atomistic subsemiring of $S(A)$.*

(1) *The focal subspace $F(R)$ is a unital subalgebra of A.*
(2) *The sublattice $[0, F(R)] \subset R$ is an entire atomistic subsemiring of $S(F(A))$ with $\varpi_R \subset \mathcal{Q}([0, F(R)])$.*
(3) *If R is weakly reproducible and has finite length, then $\mathcal{Q}([0, F(R)]) = \varpi_R$.*
(4) *If R is weakly reproducible (resp. reproducible) of finite length, then the same holds for $[0, F(R)]$.*

We remark that part (3) is very useful in computations as it is often easier to calculate $F(R)$ than to compute ϖ_R directly.

We will only prove the first two parts of the theorem now. The proof of the other parts requires a more detailed study of the relation ζ^* and will be given at the end of Section 5.

Proof of parts (1) *and* (2). If $X, Y \in \varpi_R$ and $Z \in X \circ Y$, then $1 = \psi(X)\psi(Y) = \psi(Z)$. This means that $Z \in \varpi$, so $XY \subset F(R)$. Since $k \subset F(R)$, $F(R)$ is a subalgebra. This implies that $F(R)^2 = F(R)$, so if $E, E' \in [0, F(R)]$, then $E + E' \subset F(R)$ and $EE' \subset F(R)$. Thus, the closed sublattice $[0, F(R)] \subset R$ is a subsemiring of R, and it is immediate that it is entire and atomistic. The atoms of $[0, F(R)]$ are precisely the atoms of R which are contained in $F(R)$, so $\varpi_R \subset \mathcal{Q}([0, F(R)])$. $\square$

Corollary 4.4. *If R is weakly reproducible and has finite length, then $Q(R) = 1$ if and only if $F(R)$ is the maximum element of R, i.e., $[0, F(R)] = R$.*

Proof. If $Q(R) = 1$, then $\varpi_R = \mathcal{Q}(R)$. Thus, $F(R)$ contains every atom in R, hence is the maximum element of R. Conversely, if $F(R)$ is the maximum of R, then part (3) of the theorem implies that $\varpi_R = \mathcal{Q}(R)$. This gives $Q(R) = 1$. $\square$

Remark 4.5. The forward implication in the corollary holds for any entire atomistic subsemiring.

The theorem shows that we can iterate the construction of the invariants associated to R.

Definition 4.6. The higher foci, focal subalgebras, and condensation monoids (or groups) for R are defined recursively as follows:

- $\varpi_R^1 = \varpi_R$, $F^1(R) = F(R)$, and $Q^1(R) = Q(R)$;
- $\varpi_R^{n+1} = \varpi_{[0,F^n(R)]}$, $F^{n+1}(R) = F([0, F^n(R)])$, and $Q^{n+1}(R) = Q([0, F^n(R)])$.

We observe that if R is weakly reproducible and has finite length, then $Q^n(R)$ is a group for all n.

5. Complete subsets of $\mathcal{Q}(R)$

In order to prove Theorem 4.3, we need a better understanding of the equivalence relation ζ^*. In this section, we define *complete subsets* of $\mathcal{Q}(R)$ and use them to investigate the ζ^*-equivalence classes. Our analysis of ζ^* follows a similar pattern to that of β^* carried out by Corsini and Freni [1, 2]. In the end, we will show that if R is weakly reproducible, then every element of ϖ_R is ζ-related (and not just ζ^*-related) to k; this will be the key ingredient in the proof of Theorem 4.3.

Definition 5.1.

(1) A subset $E \subset \mathcal{Q}(R)$ is called *complete* if for all $X_1, \dots, X_n \in \mathcal{Q}(R)$, if there exists $X \in E$ such that $X \subset \prod_{i=1}^n X_i$, then for any $Y \subset \prod_{i=1}^n X_i$, $Y \in E$.

(2) If E is a nonempty subset of $\mathcal{Q}(R)$, then the intersection of all complete subsets containing E is denoted by $\mathcal{C}(E)$; it is called the *complete closure* of E.

It is obvious that $\mathcal{C}(E)$ is the smallest closed subset containing E.

Remark 5.2. This is not the usual notion of a complete subset of a semihypergroup [1, 7], though it coincides in the context of Proposition 2.2. In this paper, we only consider completeness in the sense given above.

The basic examples of closed subsets are the ζ^*-equivalence classes.

Proposition 5.3. *Any ζ^*-equivalence class is closed.*

Proof. Consider the class of Z. Suppose that $X \,\zeta^*\, Z$ and $X, Y \subset \prod_{i=1}^n X_i$. Then $Y \,\zeta\, X$, so $Y \,\zeta^*\, Z$. □

The complete closure may be computed inductively. Indeed, given $E \neq \varnothing$, define a sequence of subsets $\kappa_n(E) \subset \mathcal{Q}(R)$ recursively as follows: $\kappa_1(E) = E$ and

$$\kappa_{n+1}(E) = \{X \in \mathcal{Q}(R) \mid \exists Y_1, \dots, Y_s \in \mathcal{Q}(R) \text{ and } Y \in \kappa_n(E) \text{ such that } X, Y \subset \prod_{i=1}^s Y_i\}.$$

Set $\kappa(E) = \cup_{n \geq 1} \kappa_n(E)$.

Proposition 5.4. *For any nonempty $E \subset \mathcal{Q}(R)$, $\mathcal{C}(E) = \kappa(E)$.*

Proof. Suppose $Y \in \kappa(E)$, say $Y \in \kappa_n(E)$, and $X, Y \subset \prod_{i=1}^s Y_i$. Then $X \in \kappa_{n+1}(E) \subset \kappa(E)$, so $\kappa(E)$ is complete. Since $E \subset \kappa(E)$, $\mathcal{C}(E) \subset \kappa(E)$. Conversely, suppose that $F \supset E$ and F is complete. We show inductively that $\kappa_n(E) \subset F$. This is obvious for $n = 1$. Suppose $\kappa_n(E) \subset F$. If $X \in \kappa_{n+1}(E)$, then we can find $Y_1, \dots, Y_s \in \mathcal{Q}(R)$ and $Y \in \kappa_n(E)$ such that $X, Y \subset \prod_{i=1}^s Y_i$. Completeness of F now shows that $X \in F$ as desired. □

We can now give a new characterization of ζ^*. Define a relation κ on $\mathcal{Q}(R)$ by $X \,\kappa\, Y$ if and only if $X \in \mathcal{C}(Y)$, where $\mathcal{C}(Y) = \mathcal{C}(\{Y\})$.

Theorem 5.5. *The relations κ and ζ^* coincide.*

Before beginning the proof, we will need a lemma.

Lemma 5.6.

(1) *For any $X \in \mathcal{Q}(R)$ and $n \geq 2$, $\kappa_{n+1}(E) = \kappa_n(\kappa_2(X))$.*
(2) *For $X, Y \in \mathcal{Q}(R)$, $X \in \kappa_n(Y)$ if and only if $Y \in \kappa_n(X)$.*

Proof. Note that $\kappa_n(\kappa_2(X))$ consists of those atoms Z for which there exists Y_i's and $Y \in \kappa_{n-1}(\kappa_2(X))$ such that $Y, Z \subset \prod_{i=1}^s Y_i$. If $n = 2$, then $\kappa_{n-1}(\kappa_2(X)) = \kappa_2(X)$, and this is precisely the defining property of $\kappa_3(X)$. If $n > 2$, then $\kappa_{n-1}(\kappa_2(X)) = \kappa_n(X)$ by inductive hypothesis, and we see that such atoms are precisely the elements of $\kappa_{n+1}(X)$. This proves part (1).

The second assertion is also proven by induction. Suppose $X \in \kappa_2(Y)$. Then there exist Y_i's such that $X, Y \subset \prod_{i=1}^s Y_i$, so $Y \in \kappa_2(X)$. Next, assume that the statement holds for n. If $X \in \kappa_{n+1}(Y)$, then $X, Z \subset \prod_{i=1}^s Y_i$ for some Y_i's and $Z \in \kappa_n(Y)$. By definition, $Z \in \kappa_2(X)$, and $Y \in \kappa_n(Z)$ by induction. Hence, $Y \in \kappa_n(\kappa_2(X)) = \kappa_{n+1}(X)$. □

Proof of Theorem 5.5. First, we show that κ is an equivalence relation. It is clear that κ is reflexive. If $X \,\kappa\, Y$ and $Y \,\kappa\, Z$, then $X \in \mathcal{C}(Y)$ and $Y \in \mathcal{C}(Z)$. Since $\mathcal{C}(Z)$ is complete and contains Y, $\mathcal{C}(Y) \subset \mathcal{C}(Z)$, so $X \in \mathcal{C}(Z)$, i.e., $X \,\kappa\, Z$. Finally, if $X \,\kappa\, Y$, then Proposition 5.4 implies that $X \in \kappa_n(Y)$ for some n. By the lemma, $Y \in \kappa_n(X) \subset \kappa(X)$, and another application of Proposition 5.4 gives $Y \,\kappa\, X$.

Next, suppose that $X \,\zeta\, Y$. Then $X, Y \subset \prod_{i=1}^s X_i$ for some X_i's, so $X \,\kappa\, Y$. Since κ is an equivalence relation, it follows that $\zeta^* \subset \kappa$.

Conversely, assume that $X \,\kappa\, Y$, say $X \in \kappa_{n+1}(Y)$. Set $X_0 = X$. We recursively construct $X_j \in \kappa_{n+1-j}(Y)$ for $0 \leq j \leq n$ satisfying $X_j \,\kappa\, X_{j+1}$. Choose $X_1 \in \kappa_n(Y)$ such that $X, X_1 \subset \prod_{i=1}^{s_1} Y_{1,i}$ for some $Y_{1,i}$'s. This means that $X \,\zeta\, X_1$. Suppose that we have constructed the desired atoms up through X_r with $r < n$. Again, we can choose $X_{r+1} \in \kappa_{n-r}(Y)$ satisfying $X_r, X_{r+1} \subset \prod_{i=1}^{s_{r+1}} Y_{r+1,i}$ for some $Y_{r+1,i}$'s, and this gives $X_r \,\zeta\, X_{r+1}$. Note that $X_n \in \kappa_1(Y) = \{Y\}$, i.e., $X_n = Y$. We conclude that $X \,\zeta^*\, Y$ as desired.

□

Corollary 5.7. *For any $E \subset \mathcal{Q}(R)$ nonempty, $\psi^{-1}(\psi(E)) = \cup_{X \in E} \mathcal{C}(X) = \mathcal{C}(E)$. In particular, the ζ^*-equivalence class of X is $\mathcal{C}(X)$.*

Proof. The set $\psi^{-1}(\psi(E))$ consists of those atoms equivalent to an atom in E, hence is the union of the $\zeta^* = \kappa$ equivalence classes of atoms in E. This gives the first equality. The second follows immediately from the fact that a union of closed subsets is closed. □

To proceed further, we need to impose additional conditions on R.

Proposition 5.8.

(1) *If R is reproducible, then for all $X \in \mathcal{Q}(R)$, $\mathcal{C}(X) = \varpi_R \circ X = X \circ \varpi_R$. In particular, the subhypergroupoid ϖ_R satisfies the reproductive law.*
(2) *If R is weakly reproducible, then ϖ_R satisfies the weak reproductive law.*

Proof. First, assume that R is reproducible. Suppose that $Y \in \mathcal{C}(X)$, so $Y \zeta^* X$. By reproducibility, there exist U such that $Y \subset XU$, i.e., $Y \in X \circ U$. Hence, $\psi(Y) = \psi(X)\psi(U)$, so $\psi(U) = 1$. This shows that $U \in \varpi_R$, giving $Y \in X \circ \varpi_R$. Conversely, if $Z \in X \circ \varpi_R$, then $\psi(Z) = \psi(X)$. This means that $Z \in \mathcal{C}(X)$. The equality $\mathcal{C}(X) = \varpi_R \circ X$ is proved in the same way. When $X \in \varpi_R$, the first statement says that $\varpi_R = \varpi_R \circ X = X \circ \varpi_R$, which is the reproductive law.

If R is weakly reproducible, then the argument given above (with $Y = k$ and $X \in \varpi_R$) shows that there exists $U, V \in \varpi_R$ such that $k \subset U \circ X \cap X \circ V$ as desired. □

Given $Z \in \mathcal{Q}(R)$, define

$$M(Z) = \{X \in \mathcal{Q}(R) \mid \exists Y_1, \dots, Y_s \in \mathcal{Q}(R) \text{ such that } X, Z \subset \prod_{i=1}^{s} Y_i\}.$$

Lemma 5.9. *If R is reproducible (resp. weakly reproducible), then $M(Z)$ is a complete part for all Z (resp. for $Z = k$).*

Proof. Assume that R is reproducible. Take $Y \in M(Z)$, so $Y, Z \subset \prod_{i=1}^{s} Y_i$ for some Y_i's. Suppose that $Y \subset \prod_{j=1}^{n} Z_j$. By reproducibility, choose V, W such that $Z \subset YV$ and $Z_n \subset WZ$. Now, suppose that $X \subset \prod_{j=1}^{n} Z_j$ also. Then

$$X \subset \prod_{j=1}^{n} Z_j \subset \left(\prod_{j=1}^{n-1} Z_j\right) WZ \subset \left(\prod_{j=1}^{n-1} Z_j\right) WYV \subset \left(\prod_{j=1}^{n-1} Z_j\right) W \left(\prod_{i=1}^{s} Y_i\right) V.$$

On the other hand,

$$Z \subset YV \subset \left(\prod_{j=1}^{n} Z_j\right) V \subset \left(\prod_{j=1}^{n-1} Z_j\right) WZV \subset \left(\prod_{j=1}^{n-1} Z_j\right) W \left(\prod_{i=1}^{s} Y_i\right) V.$$

Thus, $Y \in M(Z)$, so $M(Z)$ is complete.

If R is weakly reproducible, then the same argument works with $Z = k$. Indeed, we need only set $W = Z_n$ and use weak reproducibility to choose V such that $k \subset YV$. □

Corollary 5.10. *If R is reproducible (resp. weakly reproducible), then $M(Z) = \varpi_R$ for any $Z \in \varpi_R$ (resp. for $Z = k$).*

Proof. Suppose that $Z \in \varpi_R$. If $X \in M(Z)$, then $X \zeta Z$ by definition, so $X \in \varpi_R$. Thus, $M(Z) \subset \varpi_R$. Conversely, the lemma shows that, under the hypothesis on R, $M(Z)$ is a complete subset containing Z, so $\mathcal{C}(Z) = \varpi_R \subset M(Z)$. □

Theorem 5.11.

(1) *If R is reproducible, then ζ is transitive.*
(2) *If R is weakly reproducible, then $X \zeta^* k$ implies that $X \zeta k$.*

Proof. Assume that R is reproducible, and take $X\,\zeta^*\,Y$. By Proposition 5.8, there exists $U \in \varpi_R$ such that $Y \in X \circ U$. Since $M(k) = \varpi_R$, Corollary 5.10 implies the existence of Y_i's such that $U, k \subset \prod_{i=1}^s Y_i$. Thus, $Y \subset XU \subset X \prod_{i=1}^s Y_i \supset Xk = X$, so $Y\,\zeta\,X$. If R is weakly reproducible, the same argument works for $Y = k$. □

We are now ready to return to the proof of Theorem 4.3. We first state a proposition.

Proposition 5.12. *Let R be weakly reproducible of finite length. Then there exists $X_1, \dots, X_n \in \mathcal{Q}(R)$ such that $F(R) = \prod_{i=1}^n X_i$.*

Proof. First, note that if $k \subset \prod_{i=1}^n X_i$, then $\prod_{i=1}^n X_i \subset F(R)$. Indeed, if $Z \subset \prod_{i=1}^n X_i$ is an atom, then $Z\,\zeta\,k$, so $Z \subset F(R)$. The claim follows because $\prod_{i=1}^n X_i$ is the sum of the atoms it contains.

Choose $E = \prod_{i=1}^n X_i$ containing k such that $[E, F(R)]$ has minimal length. If this length is 0, then $E = F(R)$, so suppose it is positive, i.e., $E \subsetneq F(R)$. Take $Y \in \varpi_R$ such that $Y \subsetneq E$. By Theorem 5.11, $Y\,\zeta\,1$, so there exist $Y_1, \dots, Y_m \in \mathcal{Q}(R)$ such that $k, Y \subset E' = \prod_{j=1}^m Y_j$. This implies that $E = Ek \subset EE'$ and $Y = kY \subset EE'$, and the previous paragraph shows that $EE' \subset F(R)$. We obtain $E \subsetneq EE' \subset F(R)$, contradicting the minimality of the length of $[E, F(R)]$. □

We apply the proposition to prove part (3) of the theorem. Indeed, if $Z \in \mathcal{Q}(R)$ and $Z \subset F(R) = \prod_{i=1}^n X_i$, then $Z\,\zeta\,1$ by definition. This means that $Z \in \varpi_R$ as desired.

Finally, we prove part (4). Suppose that R is reproducible of finite length, and $X, Y \subset F(R)$ are atoms. By part (3), $X, Y \in \varpi_R$. By hypothesis, there exists $Z \in \mathcal{Q}(R)$ such that $Y \in X \circ Z$. Since $1 = \psi(Y) = \psi(X)\psi(Z) = \psi(Z)$, $Z \subset F(R)$ and so $\mathcal{Q}([0, F(R)]) = X \circ \mathcal{Q}[0, F(R)])$. Similarly, one shows $\mathcal{Q}([0, F(R)]) = \mathcal{Q}[0, F(R)]) \circ X$, so $\mathcal{Q}([0, F(R)])$ is reproducible. The same argument applies when R is weakly reproducible; here, one takes $Y = k$. This completes the proof of Theorem 4.3.

6. Applications to G-algebras

In this section, we apply our results on atomistic semirings to subrepresentation semirings. We assume throughout that A is a G-algebra which is completely reducible as a representation. We write $Q_G(A)$ (resp. $F_G(A)$) instead of $Q(S_G(A))$ (resp. $F(S_G(A))$). We will now be able to generalize our earlier results on multiplicity-free G-algebras.

Proposition 6.1. *Let A be a G-algebra in which the trivial representation has multiplicity one. Then A has no proper, nontrivial one-sided invariant ideals if and only if $S_G(A)$ is weakly reproducible.*

Remark 6.2. Note that both conditions imply that $S_G(A)$ is entire. Indeed, if the condition on invariant ideals holds, then the argument given in the proof of Proposition 2.2 shows that $S_G(A)$ is entire. The analogous statement for weak reproducibility was shown in Remark 3.4.

Proof. Assume that A has no proper, nontrivial invariant ideals. Fix $X \in \mathcal{Q}_G(A)$, and express A as a direct sum of irreducible subrepresentations $A = \bigoplus_{i \in I} Y_i$. The subspace AX is a nonzero invariant left ideal, so we obtain $A = AX = \sum_{i \in I} XY_i$. The trivial representation must accordingly be an irreducible component of some

XY_j. The fact that the trivial representation has multiplicity one in A implies that $k \subset XY_j$ as desired. Similarly, since $A = XA$, there exists Y_l such that $k \subset Y_l X$.

Conversely, suppose that $0 \neq L \neq A$ is an invariant left ideal. Let $X \subset L$ be an irreducible submodule. For any $Y \in \mathcal{Q}_G(A)$, we have $YX \subset L$; since $k \cap L = 0$, $S_G(A)$ is not weakly reproducible. A similar argument works for right ideals. □

Corollary 6.3. *The atomistic semiring $S_G(A)$ is weakly reproducible of finite length if*

1. *A is a finite Galois extension of k and G is the Galois group; or*
2. *$A = \mathrm{End}(V)$ is a finite-dimensional G-algebra whose underlying projective representation V is irreducible.*

Proof. Schur's lemma shows that $\mathrm{End}(V)$ contains the trivial representation with multiplicity one, and the statement about invariant ideals was proved in [8, Theorem 5.2]. The analogous verifications for the other case are obvious. □

We are thus able to define invariants for any G-algebra satisfying the conditions of Proposition 6.1, without our earlier assumption that the G-algebra is multiplicity-free. In particular, our results determine two new sequences of invariants associated to any irreducible projective representation, namely, the condensation groups $Q^n_G(\mathrm{End}(V))$ and the focal subalgebras $F^n_G(\mathrm{End}(V))$.

The focal subalgebras $F^n_G(A)$ are a decreasing sequence of invariant subalgebras (i.e., subalgebras which are also subrepresentations) of A. This is particularly interesting for $A = \mathrm{End}(V)$ with V irreducible and k algebraically closed because in this case, there is a complete classification of such invariant subalgebras in terms of representation-theoretic data [8, Theorem 3.23].

For the rest of the paper, we assume that either G is finite and k is algebraically closed of characteristic zero or G is a compact group and $k = \mathbf{C}$. We let V be an irreducible (linear) representation of G, and set $A = \mathrm{End}(V)$. (We make these assumptions on G and k to guarantee complete reducibility of $\mathrm{End}(V)$; the classification of invariant subalgebras described below holds in general.)

An invariant subalgebra of A is determined by data consisting of a quadruple (H, W, U, U'); here, H is a finite index subgroup of G, W is a linear representation of H such that $V = \mathrm{Ind}_H^G(W)$, and U, U' are a pair of projective representations of H such that $W \cong U \otimes U'$. More precisely, there is a bijection between invariant algebras and equivalence classes of such quadruples under conjugation by G. In particular, there are a finite number of invariant subalgebras.

Given such a quadruple (H, W, U, U'), we construct the corresponding invariant subalgebra as follows: Let $g_1 = e, g_2, \dots, g_n$ be a left transversal for H in G. This gives a direct sum decomposition $V = \bigoplus_{i=1}^n g_i W$ and an associated block diagonal invariant subalgebra $\mathrm{Ind}_H^G(\mathrm{End}(W)) \overset{\mathrm{def}}{=} \bigoplus_{i=1}^n \mathrm{End}(g_i W)$. As an algebra, this is just the direct product of n copies of $\mathrm{End}(W)$. Next, the isomorphism $W \cong U \otimes U'$ shows that the endomorphism algebra factors (as H-algebras) into the tensor product $\mathrm{End}(W) \cong \mathrm{End}(U) \otimes \mathrm{End}(U')$. It is now immediate that $\mathrm{End}(U) \otimes k$ is an H-invariant subalgebra of $\mathrm{End}(W)$. Finally, we obtain the invariant algebra for the quadruple: $\mathrm{Ind}_H^G(\mathrm{End}(U) \otimes k)$. We remark that the two obvious invariant subalgebras k and $\mathrm{End}(V)$ correspond to (G, V, k, V) and (G, V, V, k) respectively.

It now follows that the sequence of focal subalgebras associated to the irreducible representation V gives rise to a sequence of such quadruples.

The classification of invariant subalgebras can be very helpful for computing the $F^n_G(\mathrm{End}(V))$. For example, suppose that $\mathrm{End}(V)$ has no nontrivial invariant subalgebras, so that any irreducible representation generates $\mathrm{End}(V)$. In order to show that $F_G(\mathrm{End}(V)) = \mathrm{End}(V)$, it is only necessary to check that $F_G(\mathrm{End}(V))$ contains a nonscalar matrix. However, it should be noted that computing the invariant subalgebras is not necessarily straightforward. Even when G is finite, it is not determined by the character table of G. In general, one needs to know the character tables of a covering group for every subgroup of G whose index divides $\dim(V)$.

Example 6.4. Let V be the standard representation of S_3. We have already seen that $Q^1_G(\mathrm{End}(V)) = \mathbf{Z}_2$. The focal subalgebra $F^1_G(\mathrm{End}(V)) = \mathbf{C} \oplus \sigma$ is isomorphic to $\mathbf{C} \oplus \mathbf{C}$ as an algebra; it comes from the quadruple $(A_3, \chi, \chi, \mathbf{C})$, where χ is either nontrivial character of A_3. Since $\mathbf{C}$ and σ are not ζ^*-equivalent in $\mathfrak{Q}_G(F^1_G(\mathrm{End}(V)))$, we have $Q^2_G(\mathrm{End}(V)) = \mathbf{Z}_2$ and for $m \geq 2$, $F^m_G(\mathrm{End}(V)) = \mathbf{C}$ (corresponding to $(S_3, V, \mathbf{C}, V)$). Finally, $Q^n_G(\mathrm{End}(V)) = 1$ for $n \geq 3$,

Example 6.5. Let W be the three-dimensional irreducible representation of A_4. We will show that $\mathfrak{Q}_{A_4}(\mathrm{End}(W))$ is not associative and does not satisfy the reproductive law.

We have the direct sum decomposition $\mathrm{End}(W) = \mathbf{C} \oplus Z \oplus Z' \oplus X \oplus Y$, where Z and Z' correspond to the two nontrivial characters of A_4 and X and Y are isomorphic to W. We can choose a basis for W with respect to which Y (resp. X) consists of the skew-symmetric (resp. off-diagonal symmetric) matrices and the diagonal T is the direct sum of $\mathbf{C}$, Z, and Z'. There are an infinite number of atoms isomorphic to W, parameterized by $[a : b] \in \mathbf{P}^1(\mathbf{C})$; we set

$$U_{[a:b]} = \mathrm{span}\{(a+b)E_{23}+(a-b)E_{32}, (a-b)E_{13}+(a+b)E_{31}, (a+b)E_{12}+(a-b)E_{21}\}.$$

In this notation, $X = U_{[1:0]}$ and $Y = U_{[0:1]}$.

Let $P = U_{[1:1]}$. It is easily checked that $P^2 = U_{[1:-1]}$. We now calculate that $P\mathbf{C} = PZ = PZ' = P$, $P(P^2) = T$, and $PU_{[a:b]} = T \oplus P^2$ for $[a : b] \neq [1 : \pm 1]$. We thus see that $\mathfrak{Q}_{A_4}(\mathrm{End}(W))$ does not satisfy the reproductive law; if $[a : b] \neq [1 : \pm 1]$, there is no V for which $U_{[a:b]} \in P \circ V$. To verify that the associative law does not hold, note that $X \in (P \circ P^2) \circ X = \{\mathbf{C}, Z, Z'\} \circ X$. However, $P \circ (P^2 \circ X) = P \circ \{\mathbf{C}, Z, Z', P\} = \{P, P^2\}$ does not contain X.

Since $PX = T \oplus P^2$, we have $\mathbf{C}, Z, Z', P^2 \in \varpi$. Also, $P^2X = T \oplus P$, so $Q \in \varpi$. This implies that $F^n_{A_4}(\mathrm{End}(W)) = \mathrm{End}(W)$ for all n, and by Corollary 4.4, $Q^n_{A_4}(\mathrm{End}(W)) = 1$ for all n.

The only nontrivial invariant subalgebra of $\mathrm{End}(W)$ is T. (It corresponds to $(H, \chi, \chi, \mathbf{C})$, where $H \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ is the subgroup of order 4 and χ is any nontrivial character of H.) Thus, one knows that $F_{A_4}(\mathrm{End}(W)) = \mathrm{End}(W)$ as soon as one know that $P^2 \in \varpi$.

We conclude by computing the condensation groups and focal subalgebras of endomorphism algebras for simple compact Lie groups.

Theorem 6.6. *Let V be an irreducible representation of the simple compact Lie group G. Then $Q^n_G(\mathrm{End}(V)) = 1$ and $F^n_G(\mathrm{End}(V)) = \mathrm{End}(V)$ for all n.*

Proof. If $V = \mathbf{C}$, the statement is trivial. Any other V has dimension at least 2. By Corollary 4.4, it suffices to show that $F_G(\mathrm{End}(V)) = \mathrm{End}(V)$. Moreover, by

[8, Theorem 4.3], the only proper invariant subalgebra of $\mathrm{End}(V)$ is $\mathbf{C}$. Hence, we need only show that $F_G(\mathrm{End}(V))$ contains a nonscalar matrix.

Let λ be the highest weight of V. The highest weight of the dual representation V^* is $-w_0\lambda$, where w_0 is the longest element in the Weyl group. The representation $\mathrm{End}(V) \cong V \otimes V^*$ has a unique irreducible submodule X with highest weight $\lambda - w_0\lambda$. We can write down a highest and lowest weight vector in X explicitly. Let v_λ (resp. w_λ) be a highest (resp. lowest) weight vector in V. (The highest and lowest weights are different since $\dim V \geq 2$.) Extend the set $\{v_\lambda, w_\lambda\}$ to a basis of weight vectors for V, and let v_λ^*, w_λ^* be the corresponding dual basis vectors in V^*. Then w_λ^* (resp. v_λ^*) is a highest (resp. lowest) weight vector in V^*. It follows that $v_\lambda \otimes w_\lambda^*$ (resp. $w_\lambda \otimes v_\lambda^*$) is a highest (resp. lowest) weight vector in X.

Multiplying, we obtain $z = (v_\lambda \otimes w_\lambda^*)(w_\lambda \otimes v_\lambda^*) = v_\lambda \otimes v_\lambda^* \in X^2$. The matrix z has rank one, so is not a scalar matrix. Thus, $X^2 \neq \mathbf{C}$. However, $\mathrm{tr}(z) = 1$, so z is not orthogonal to $\mathbf{C}$. This implies that $\mathbf{C} \subset X^2$. We conclude that ϖ contains at least two elements, so $F_G(\mathrm{End}(V)) \neq \mathbf{C}$. □

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