Moduli spaces of irregular singular connections

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ICRA, Syracuse, August 2016

Overview

New approach to the local theory of flat G-bundles over curves, i.e. formal flat G-bundles, using methods from representation theory: Systematic study of the "leading terms" of the flat structures with respect to Moy-Prasad filtrations

Two main motivations:

- Moduli spaces and the isomonodromy problem for meromorphic flat G-bundles with nondiagonalizable irregular singularities; connections with quiver varieties
- The wild ramification case of the geometric Langlands program

In this talk, for simplicity $G = \mathrm{GL}_n$.

Flat GL_n-bundles

$\mathcal{O}$ structure sheaf of $\mathbb{P}^1(\mathbb{C})$, K function field (meromorphic functions)
$\Omega^1_{K/\mathbb{C}}$ meromorphic 1-forms

Definition
A flat GL_n-bundle on $\mathbb{P}^1(\mathbb{C})$ is a rank n trivializable vector bundle with a meromorphic connection, i.e., a $\mathbb{C}$-derivation $\nabla : V \to V \otimes_{\mathcal{O}} \Omega^1_{K/\mathbb{C}}$.

If one fixes a trivialization $\phi : V \to V^{\mathrm{triv}}$, then

$$\nabla = d + [\nabla]_\phi, \text{ where } [\nabla]_\phi \in M_n(\Omega^1_{K/\mathbb{C}}).$$

$$\{\text{Meromorphic connections}\} \longleftrightarrow \{\text{1st order linear diff eqs}\}$$

Localization

∇ merom. connection induces formal connections at each $y \in \mathbb{P}^1$
Let z be a parameter at y
$\mathfrak{o} = \mathbb{C}[[z]]$ completion of local ring at y, $F = \mathbb{C}((z))$ fraction field,
$\Delta_y^\times = \mathrm{Spec}(F)$ is a formal punctured disk at y
One obtains an induced formal connection $(\hat{V}_y, \hat{\nabla}_y)$ on $\Delta_y^\times$. Note that $[\hat{\nabla}_y] \in \mathfrak{gl}_n(F)\frac{dz}{z}$.
(Formal) change of trivialization gives rise to gauge change on the local connection matrix:

$$g \cdot [\hat{\nabla}_y]_\phi = g([\hat{\nabla}_y]_\phi)g^{-1} - (dg)g^{-1}, \text{ where } g \in \mathfrak{gl}_n(F).$$

Gauge change is the correct notion of equivalence in categories of flat G-bundles.
If $[\hat{\nabla}_y]_\phi$ has a simple pole for some trivialization ϕ, then y is a regular singular point. Otherwise, it is irregular.

Some problems on moduli spaces of flat GL_n-bundles

If the singular points are $y_1, \ldots, y_m$, one gets a localization functor $L : \nabla \mapsto (\hat{\nabla}_{y_i})$.
($\hat{\nabla}_y$ is trivial except at the singularities.)

$$\left\{ \begin{matrix} \text{flat } GL_n\text{-bundles with} \\ \text{singularities at } y_1, \ldots, y_m \end{matrix} \right\} \xrightarrow{L = \prod L_i} \prod_i \left\{ \begin{matrix} \text{formal flat} \\ GL_n\text{-bundles on } \Delta^\times_{y_i} \end{matrix} \right\}$$

Want to study these categories via the geometry of the moduli spaces. In general, these moduli spaces are stacks; to understand, look for better-behaved subcategories of flat GL_n-bundles.

1. Find classes of formal isomorphism types for which $L^{-1}((\hat{\nabla}_i))$ are well-behaved moduli spaces.
2. When are such moduli spaces nonempty (Deligne-Simpson problem)? Reduced to a singleton (a version of rigidity)?

Nonresonant case for GL_n (reg semisimple leading term)

$[\hat{\nabla}_y] = (M_{-r}z^{-r} + M_{1-r}z^{1-r} + \dots)\frac{dz}{z}$, $M_i \in \mathfrak{gl}_n(\mathbb{C})$, $M_{-r} \neq 0$.
If M_{-r} is regular semisimple, then $[\hat{\nabla}_y]$ is gauge equivalent to an element of $\mathcal{A}(r)\frac{dz}{z} = \{D_{-r}z^{-r} + \cdots + D_0 \mid D_i \text{ diag}, D_{-r} \text{ reg}\}\frac{dz}{z}$.
($\mathcal{A}(r)$ is the set of "formal types").
Consider connections with only nonresonant singularities.

$$\widetilde{\mathcal{M}}^{\mathrm{nr}}(\mathbf{r}) \xrightarrow{L=\prod L_i} \prod_i \mathcal{A}(r_i)$$

Boalch has shown

- The moduli space $\widetilde{\mathcal{M}}^{\mathrm{nr}}(\mathbf{r})$ is a Poisson manifold.
- Its symplectic leaves are the connected components of the fibers of L.
- In certain case, the fibers of L are quiver varieties.

Generalized Airy connections

For $s \geq 1$, consider the nondiagonalizable connection

$$d + \begin{pmatrix} 0 & z^{-s} \\ z^{-s+1} & 0 \end{pmatrix} \frac{dz}{z} = d + \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) z^{-s} \frac{dz}{z} + \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) z^{-s+1} \frac{dz}{z}.$$

Irregular singular at 0, regular singular at ∞

$s = 2$ classical Airy, $s = 1$ GL_2-version of Frenkel-Gross rigid connection

Leading term is nilpotent; classical techniques do not apply.

However, there are filtrations of $\mathfrak{gl}_2(F)$ better adapted to it than $(z^k\, \mathfrak{gl}_2(\mathfrak{o}))_{k \in \mathbb{Z}}$.

Take filtration wrt complete lattice chain

$\supset L_{-1} = z^{-1} L_1 \supset L_0 = \mathfrak{o}^2 \supset L_1 = \mathfrak{o} e_1 \oplus z \mathfrak{o} e_2 \supset L_2 = z L_0 \supset$

$$\mathfrak{I}^k = \{x \in \mathfrak{gl}_2(F) \mid x(L_i) \subset L_{i+k} \forall i\}$$

Eg $\mathfrak{I}^0 \subset \mathfrak{gl}_2(\mathfrak{o})$ are upper triang matrices mod z, Iwahori subalg.

Now, $\begin{pmatrix} 0 & z^{-s} \\ z^{-s+1} & 0 \end{pmatrix} \in \mathfrak{I}^{1-2s} \setminus \mathfrak{I}^{2-2s}$; can be viewed as its own leading term wrt this filtration (nonnilp; in fact, reg semisimple)

Moy-Prasad filtrations

$\mathfrak{B}$ the Bruhat-Tits building for GL_n–simplicial complex

cells $\leftrightarrow$ 'parahoric subgroups' points $\leftrightarrow$ 'periodic filtrations'

$x \in \mathfrak{B}$, $\hat{\mathfrak{g}}_x$ corresponding parahoric subalgebra. There is a decreasing filtration $(\hat{\mathfrak{g}}_{x,r})_{r\in\mathbb{R}}$ of the loop algebra $\hat{\mathfrak{g}} = \mathfrak{gl}_n(F)$ by $\mathfrak{o}$-lattices with $\hat{\mathfrak{g}}_{x,0} = \hat{\mathfrak{g}}_x$, $\hat{\mathfrak{g}}_{x,r+1} = z\hat{\mathfrak{g}}_{x,r}$, only a finite number of jumps in $[0,1]$. Compatible filtration of parahoric subgroup $\hat{G}_x$.

Examples

- The degree filtration $(z^k\mathfrak{g}(\mathfrak{o}))$ is the MP filtration associated to the "origin" o of $\mathfrak{B}$–the vertex associated to the maximal parahoric $\mathrm{GL}_n(\mathfrak{o})$.
- The standard Iwahori filtration for GL_2 is the MP filtration coming from the barycenter x_I of the edge corresponding to the standard Iwahori subgroup (upper triangular mod z), except the jumps here are at $\frac{1}{2}\mathbb{Z}$.

Fundamental strata

Definition

- A stratum (x, r, β) consists of $x \in \mathfrak{B}$, a real number $r \geq 0$, and a coset $\beta \in \hat{\mathfrak{g}}_{x,-r}/\hat{\mathfrak{g}}_{x,-r+}$.
- (x, r, β) is fundamental if every elt of β is nonnilpotent.

One can define what it means for a flat G-bundle to contain a stratum; the stratum should be viewed as the leading term of the connection wrt the given filtration.
Fundamental strata give the most useful leading terms of a formal flat G-bundle.

Examples

- $[\hat{\nabla}] = \left(z^{-r}M_{-r} + z^{-r+1}M_{1-r} + \text{h.o.t.}\right)\frac{dz}{z}$ with $M_i \in \mathfrak{gl}_n(\mathbb{C})$. $\hat{\nabla}$ contains the stratum $(o, r, z^{-r}M_{-r} + z^{-r+1}\mathfrak{gl}_n(\mathfrak{o}))$, fundamental if M_{-r} is non-nilpotent.
- $\hat{V} = F^2$, $[\hat{\nabla}] = \begin{pmatrix} 0 & z^{-s} \\ z^{-s+1} & 0 \end{pmatrix}\frac{dz}{z}$.
  Here, $\hat{\nabla}$ contains a nonfundamental stratum of depth s at o and the fundamental stratum $(x_I, s - \frac{1}{2}, \beta)$, where $I \subset \mathrm{GL}_2(\mathfrak{o})$ is the standard Iwahori subgroup, $\beta \in \mathfrak{I}^{1-2s}/\mathfrak{I}^{2-2s}$.

Theorem (Bremer-S. 2014)

Every formal flat G-bundle $\hat{\nabla}$ contains a fundamental stratum (x, r, β) with $r \in \mathbb{Q}$; the depth r is positive iff $\hat{\nabla}$ is irregular singular. Moreover,

- *If $\hat{\nabla}$ contains a stratum (x', r', β'), then $r' \geq r$.*
- *If $r > 0$, (x', r', β') is fundamental if and only if $r' = r$.*

For GL_n, this minimal depth is the slope of the connection defined by Katz. For reductive G, it allows one to define the slope.

Regular strata

One needs a stronger condition on strata to get nice moduli spaces. Let $S \subset \mathrm{GL}_n(F)$ be a (possibly non-split) maximal torus. There is a unique Moy-Prasad filtration $\{\mathfrak{s}_r\}$ on $\mathfrak{s} = \mathrm{Lie}(S)$.

Definition

A point $x \in \mathfrak{B}$ is compatible with $\mathfrak{s}$ if $\mathfrak{s}_r = \hat{\mathfrak{g}}_{x,r} \cap \mathfrak{s}$ for all r.

A stratum (x, r, β) is *S-regular* if x is compatible with $\mathfrak{s}$ and it satisfies a graded version of regular semisimplicity.

Examples

- If M_{-r} is reg. semisimple, then $(o, r, z^{-r}M_{-r} + z^{-r+1}\,\mathfrak{gl}_n(\mathfrak{o}))$ is $T = Z_{\mathrm{GL}_n(F)}(M_{-r})$-regular (split torus).
- Let $\omega = \left(\begin{smallmatrix} 0 & z \\ 1 & 0 \end{smallmatrix}\right)$, so $S = \mathbb{C}((\omega))^*$ is a non-split maximal torus. $(x_I, s - \frac{1}{2}, \begin{pmatrix} 0 & z^{-s} \\ z^{-s+1} & 0 \end{pmatrix} + \mathfrak{I}^{2-2s})$ is S-regular.

Toral connections and formal types

If $\hat{\nabla}$ contains an S-regular stratum, it can be "diagonalized" into S (and we call it S-toral). More precisely, one can define an affine variety $\mathcal{A}(S,r) \subset \mathfrak{s}_{-r}\frac{dz}{z}$ of S-formal types of depth r.

Examples

- T diagonal, $\mathcal{A}(T,r) = \{D_{-r}z_{-r} + \cdots + D_0 \mid D_i \text{ diag}, D_{-r} \text{ reg}\}\frac{dz}{z}$.
- $S = \mathbb{C}((\omega))$, $\mathcal{A}(S, s-1/2) = \{\text{deg } 2s-1 \text{ polys in } \omega^{-1}\}\frac{dz}{z}$.

Theorem (Bremer-S. 2015)

If $\hat{\nabla}$ contains the S-regular stratum (x,r,β), then $[\hat{\nabla}]$ is $\hat{G}_{x+}$-gauge equivalent to a unique elt of $\mathcal{A}(S,r)$ with "leading term" β.

The set of formal types is a W_S^{aff}-torsor over the set of formal isomorphism classes, with W_S^{aff} the relative affine Weyl group.

Framable connections ($G = \mathrm{GL}_n$)

(V, ∇) global connection; fix a trivialization ϕ.
Assume $\hat{\nabla}_y$ has formal type A_y.

Definition

$g \in \mathrm{GL}_n(\mathbb{C})$ is a compatible framing for ∇ at y if $g \cdot [\hat{\nabla}_y]$ has the same leading term as $A_y \frac{dz}{z}$. If such a g exists, ∇ is framable at y.

$g \circ \phi$ is a global trivialization which makes the leading term of $[\hat{\nabla}_y]$ match the leading term of $A_y \frac{dz}{z}$.

Example

$P = \mathrm{GL}_n(\mathfrak{o})$, $A_y = (D_{-r} z^{-r} + \cdots + D_0) \frac{dz}{z}$
$g \cdot [\hat{\nabla}_y] = (D_{-r} z^{-r} + M_{1-r} z^{1-r} + \mathrm{h.o.t.}) \frac{dz}{z}$.

Moduli spaces

Starting data

- $\{y_i\}$ *irregular singular points*
- $\mathbf{A} = (A_i)$ *collection of formal types at* y_i *(which determine regular strata* (x_i, r_i, β_i) *at each* y_i*).*

Let $\mathcal{M}(\mathbf{A})$ be the moduli space of iso classes (V, ∇), where

- V is a trivializable rank n vector bundle on $\mathbb{P}^1$;
- ∇ is a meromorphic connection on V with singular points only at $\{y_i\}$;
- ∇ is framable and has formal type A_i at y_i.

$\mathcal{M}(\mathbf{A})$ is the moduli space of framable connections with formal types $\mathbf{A}$.
There are also moduli spaces $\widetilde{\mathcal{M}}(\mathbf{A})$ (resp. $\widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{r})$) of framed connections with fixed formal types (resp. fixed regular combinatorics), which include data of compatible framings.
Can also include some regular singular points (formal types are coadjoint orbits of the residue).

Symplectic and Poisson reduction

We will construct these moduli spaces via symplectic (or Poisson) reduction of a symplectic (Poisson) manifold which is a direct product of local pieces. This is a result of Boalch (2001) in the case of regular diagonalizable leading terms.

Setup

- *X symplectic mfld with Hamiltonian action of the group G*
- *$\mu : X \to \mathfrak{g}^\vee$ the moment map*
- *$\alpha \in \mathfrak{g}^\vee$ is a singleton coadjoint orbit.*

Definition

The symplectic reduction $X /\!\!/_\alpha G$ is defined to be the quotient $\mu^{-1}(\alpha)/G$.

Fact

If $\mu^{-1}(\alpha)/G$ is smooth, then the symplectic structure on X descends to $X /\!\!/_\alpha G$.

Poisson reduction is analogous.

Structure of the moduli spaces

For each i, one can define a GL_n-symplectic variety $\mathcal{M}(A_i) \subset$ partial flag var $\times$ affine space; similar local manifolds $\widetilde{\mathcal{M}}(A_i)$ and $\widetilde{\mathcal{M}}(x_i, r_i)$ (symplectic and Poisson respectively). $\mathcal{M}(A_i)$ encodes the local data of $\nabla \in \mathcal{M}(\mathbf{A})$ at y_i.

Theorem (Bremer-S.)

1. *The moduli space $\widetilde{\mathcal{M}}(\mathbf{A})$ is a symplectic manifold obtained as a symplectic reduction of the product of local data:*

$$\widetilde{\mathcal{M}}(\mathbf{A}) \cong \left(\prod_i \widetilde{\mathcal{M}}(A_i)\right) /\!\!/_0 \, \mathrm{GL}_n(\mathbb{C}).$$

2. *The moduli space $\mathcal{M}(\mathbf{A})$ may be constructed in a similar way. Moreover, it is the symplectic reduction of $\widetilde{\mathcal{M}}(\mathbf{A})$ via a torus action.*

The condition that the moment map take value 0 just says that the sum of the residues over all singular points is 0.

Theorem (Bremer-S.)

1. *The space* $\widetilde{\mathcal{M}}(\mathbf{x},\mathbf{r})$ *is a Poisson manifold obtained by Poisson reduction of the product of local pieces.*
2. *The fibers of the localization map* L *are the* $\widetilde{\mathcal{M}}(\mathbf{A})$*.*
3. *The symplectic leaves are the connected components of the* $\widetilde{\mathcal{M}}(\mathbf{A})$*'s.*

These results and those on the previous slide are due to Boalch (2001) in the case where all irregular formal types have regular diagonalizable leading term.

Some quiver varieties

Let Q be a quiver with nodes J, dimension vector $\vec{d}$, and $\lambda \in \bigoplus \mathfrak{gl}(\mathbb{C}^{d_i})$ such that $\vec{d} \cdot \lambda = 0$.
The corresponding quiver variety is

$$\bigoplus T^*(\mathrm{Hom}(\mathbb{C}^{d_{t(e)}}, \mathbb{C}^{d_{h(e)}})) /\!\!/_\lambda \left(\prod \mathrm{GL}(\mathbb{C}^{d_i})\right)/\mathbb{C}^*.$$

Take D_{-2}, D_{-1} diagonal, D_{-2} regular.
Consider the moduli space of connections with formal type $(D_{-2}z^{-2} + D_{-1}z^{-1})\frac{dz}{z}$ at 0 and no other singular points.
Boalch has shown this moduli space is the quiver variety on K_n with $d_i = 1$, $\lambda_i = 0$ for all i.
Modifying the D_i's and adding a residue term allow one to get quivers on complete k-partite graphs and to change $\vec{d}$ and λ.
Moduli spaces involving nondiagonalizable connections can give rise to the fixed point subvariety of a quiver variety endowed with a cyclic group action.