Regular strata and moduli spaces of irregular singular connections

Daniel S. Sage

In recent years, there has been extensive interest in meromorphic connections on curves due to their role as Langlands parameters in the geometric Langlands correspondence. In particular, connections with irregular singularities are the geometric analogue of Galois representations with wild ramification.

The classical approach to studying the local behavior of irregular singular meromorphic connections on curves depends on the leading term of the connection matrix being well-behaved. Let V be a trivializable vector bundle over $\mathbb{P}^1$ endowed with a meromorphic (automatically flat) connection ∇. Upon fixing a local parameter z at a singular point y and a local trivialization, one can express the connection near y as

$$d + (M_{-r}t^{-r} + M_{1-r}t^{1-r} + \dots)\frac{dt}{t}, \tag{1}$$

with $M_i \in \mathfrak{gl}_n(\mathbb{C})$, $M_{-r} \neq 0$ and $r \geq 0$. From a more geometric point of view, setting $F = \mathbb{C}((t))$, this formula defines the induced connection $\hat{\nabla}_y$ on the formal punctured disk $\operatorname{Spec}(F)$.

When M_{-r} is well-behaved, this leading term contains important information about the connection. As a first example, if M_{-r} is nonnilpotent, then the expansion of ∇ at y with respect to any local trivialization must begin in degree $-r$ or below. Moreover, if $\hat{\nabla}_y$ is irregular, r is the slope of the connection at y. (The slope is an invariant introduced by Katz [6] that gives one measure of how singular a connection is at a given point.)

Much more can be said in the irregular singular *nonresonant* case when $r > 0$ and M_{-r} is regular semisimple. We assume that $r > 0$ so that we are in the irregular singular case. In this case, asymptotic analysis [9] guarantees that ∇ can be diagonalized at y by an appropriate gauge change so that $\nabla = d + (D_{-r}t^{-r} + D_{1-r}t^{1-r} + \dots D_0)\frac{dt}{t}$, with each D_i diagonal. The

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diagonal 1-form here is called a *formal type* of $\hat{\nabla}_y$. When all of the irregular singularities on a meromorphic connection on $\mathbb{P}^1$ are of this form, Boalch has shown how to construct well-behaved moduli spaces of such connections; he has further realized the isomonodromy equations as an integrable system on an appropriate moduli space [1].

However, many interesting connections do not have regular semisimple leading terms. Consider, for example, the generalized Airy connections:

$$d+\begin{pmatrix} 0 & t^{-(s+1)} \\ t^{-s} & 0 \end{pmatrix}\frac{dt}{t} = d+\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t^{-(s+1)}\frac{dt}{t}+\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} t^{-s}\frac{dt}{t}, \qquad (2)$$

for $s \geq 0$. Note that when $s = 1$, this is the usual Airy connection with the irregular singular point at 0 instead of ∞. Also, when $s = 0$, this is the GL_2 version of the Frenkel-Gross rigid flat G-bundle on $\mathbb{P}^1$ [7]. For the generalized Airy connections, the leading term is nilpotent, and it is no longer the case that one can read off the slope directly from the leading term. Indeed, the slope is $s+\frac{1}{2}$, not $s+1$.

In a recent series of papers joint with Bremer [2, 3, 4, 5], we have generalized these classical results to meromorphic connections on curves (or even flat G-bundles for reductive G) whose leading term is nilpotent. We have introduced a new notion of the "leading term" of a formal connection through a systematic analysis of its behavior in terms of suitable filtrations on the loop algebra. This theory has already proved useful in applications to the geometric Langlands program [8].

In this paper, we will illustrate our theory in the case of rank 2 flat vector bundles, where much of the Lie-theoretic complexity is absent. In this case, up to $\mathrm{GL}_2(F)$-conjugacy, one need only consider two filtrations on $\mathfrak{gl}_2(F)$, the degree filtration and the (standard) *Iwahori filtration.*

Let $\mathfrak{o} = \mathbb{C}[[t]]$ be the ring of formal power series, and let $\omega = \left(\begin{smallmatrix} 0 & 1 \\ t & 0 \end{smallmatrix}\right)$. Then the Iwahori filtration is defined by

$$\mathfrak{i}^r = \begin{pmatrix} t^{\lceil r/2\rceil}\mathfrak{o} & t^{\lfloor r/2\rfloor}\mathfrak{o} \\ t^{\lfloor r/2\rfloor+1}\mathfrak{o} & t^{\lceil r/2\rceil}\mathfrak{o} \end{pmatrix}. \qquad (3)$$

Recalling that the standard Iwahori subgroup $I \subset \mathrm{GL}_2(\mathfrak{o})$ consisting of the invertible matrices which are upper triangular modulo t, one sees that $\mathfrak{i} := \mathrm{Lie}(I)$ is just $\mathfrak{i}^0$; moreover, $\mathfrak{i}^r = \mathfrak{i}\omega^r = \omega^r\mathfrak{i}$. A matrix is homogeneous of degree $2s$ (resp. $2s+1$) with respect to the Iwahori filtration if it is in $\left(\begin{smallmatrix} \mathbb{C}t^s & 0 \\ 0 & \mathbb{C}t^s \end{smallmatrix}\right)$ (resp. $\left(\begin{smallmatrix} 0 & \mathbb{C}t^s \\ \mathbb{C}t^{s+1} & 0 \end{smallmatrix}\right)$). In particular, the matrix of the generalized Airy connection is Iwahori-homogeneous of degree $2s+1$.

The groups $\mathrm{GL}_2(\mathfrak{o})$ and I are examples of "parahoric subgroups". For any parahoric P, there is an associated filtration $\mathfrak{p}^j$ of $\mathfrak{gl}_2(F)$; this filtration satisfies $\mathfrak{p}^{j+e_P} = t\mathfrak{p}^j$ for $e_P \in \{1,2\}$. For $\mathrm{GL}_2(\mathfrak{o})$, the filtered subspaces are $t^j\,\mathfrak{gl}_2(\mathfrak{o})$. For simplicity, we will take P to be I or $\mathrm{GL}_2(\mathfrak{o})$ in this paper. Note $e_{\mathrm{GL}_2(\mathfrak{o})} = 1$ and $e_I = 2$.

It will be convenient to view any one-form $\nu \in \Omega^1(\mathfrak{gl}_2(F))$ as a continuous $\mathbb{C}$-linear functional on (subspaces of) $\mathfrak{gl}_2(F)$ via $Y \mapsto \mathrm{Res}\,\mathrm{Tr}\,Y\nu$. Any such functional on $\mathfrak{p}^r$ can be represented as $X\frac{dt}{t}$ for some $X \in \mathfrak{p}^{-r}$. For our standard examples, a functional $\beta \in (\mathfrak{p}^r/\mathfrak{p}^{r+1})^\vee$ can be written uniquely as $\beta^\flat \frac{dt}{t}$ for $\beta^\flat$ homogeneous.

A GL_2-*stratum* is a triple (P, r, β) with $P \subset \mathrm{GL}_2(F)$ a parahoric subgroup, r a nonnegative integer, and $\beta \in (\mathfrak{p}^r/\mathfrak{p}^{r+1})^\vee$. The stratum is called *fundamental* if $\beta^\flat$ is nonnilpotent. A formal connection $\hat{\nabla}$ contains (P, r, β) if $\hat{\nabla} = d + X\frac{dt}{t}$ with $X \in \mathfrak{p}^{-r}$ and β is induced by $X\frac{dt}{t}$. The following theorem shows that fundamental strata provide the correct notion of the leading term of a connection.

Theorem 1. *Any formal connection $\hat{\nabla}$ contains a fundamental stratum (P, r, β) with $r/e_P = \mathrm{slope}(\hat{\nabla})$; in particular, the connection is irregular singular if and only if $r > 0$. Moreover, if (P', r', β') is any other stratum contained in $\hat{\nabla}$, then $r'/e_{P'} \geq r/e_P$ with equality if (P', r', β') is fundamental. The converse hold if $\hat{\nabla}$ is irregular singular.*

The theorem shows that the classical slope of a connection can also be defined in terms of the fundamental strata contained in it. For flat G-bundles, the analogous result serves to define the slope [4].

Example 1. The connection in (2) (with the M_i's in $\mathfrak{gl}_2(\mathbb{C})$) contains the stratum $(\mathrm{GL}_2(\mathfrak{o}), r, M_{-r}t^{-r}\frac{dt}{t})$; it is fundamental if and only if M_{-r} is nonnilpotent, in which case the slope is r. If M_{-r} is upper triangular with a nonzero diagonal entry, then $\hat{\nabla}$ contains a fundamental stratum of the form $(I, 2r, \beta)$, where β is induced by the diagonal component of $M_{-r}t^{-r}$. Again, one sees that the slope is $2r/2 = r$. On the other hand, if M_{-r} has a nonzero entry below the diagonal, then $\hat{\nabla}$ contains a nonfundamental stratum of the form $(I, 2r+1, \beta')$.

Example 2. The generalized Airy connection with parameter s contains the nonfundamental stratum $(\mathrm{GL}_2(\mathfrak{o}), s+1, \left(\begin{smallmatrix} 0 & t^{-(s+1)} \\ 0 & 0 \end{smallmatrix}\right)\frac{dt}{t})$. It also contains the fundamental stratum $(I, 2s+1, \omega^{-(2s+1)}\frac{dt}{t})$, whence its slope is $s + \frac{1}{2}$.

In order to construct well-behaved moduli spaces, we need a condition on strata that is analogous to the nonresonance condition for diagonalizable connections. This is accomplished through the notion of a *regular stratum*. Let $S \subset \mathrm{GL}_2(F)$ be a (not necessarily split) maximal torus. Up to $\mathrm{GL}_2(F)$-conjugacy, there are two distinct maximal tori: $T(F)$ and $\mathbb{C}((\omega)^\times$ (nonzero Laurent series in ω). For our standard examples, we say that (P, r, β) is S-regular if S is the centralizer of $\beta^\flat$. (See [2, 5] for the general definition.)

Example 3. If M_{-r} is regular semisimple, then the stratum $(\mathrm{GL}_2(\mathfrak{o}), r, M_{-r}t^{-r}\frac{dt}{t})$ is $Z(M_{-r})(F)$-regular.

Example 4. The stratum $(I, 2s+1, \omega^{-(2s+1)}\frac{dt}{t})$ contained in the generalized Airy connection is $\mathbb{C}((\omega))^\times$-regular. On the other hand, if (P, r, β) is $\mathbb{C}((\omega))^\times$-regular, then $r/e_P \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$.

From now on, we assume that S is $T(F)$ or $\mathbb{C}((\omega))^\times$. Note that $\mathfrak{s} = \mathrm{Lie}(S)$ is $\mathfrak{t}(F)$ and $\mathbb{C}((\omega))$ in these two cases, and both are endowed with an obvious filtration by powers of t or ω. We call a connection containing an S-regular stratum *S-toral.* An S-toral connection can be "diagonalized" into $\mathfrak{s} = \mathrm{Lie}(S)$. Again, for simplicity, we will only describe what this means for S equal to $T(F)$ and $\mathbb{C}((\omega))^\times$. For any $r > 0$ such that $\mathfrak{s}^r$ contains a regular semisimple element of homogeneous degree r, one can define a quasiaffine variety $\mathcal{A}(S,r) \subset \mathfrak{s}^r \frac{dt}{t}$ of *S-formal types* of depth r: $\mathcal{A}(T,r) = \{D_{-r}t^{-r} + \cdots + D_0 \mid D_i \in \mathfrak{t}, D_{-r} \text{ regular}\}\frac{dt}{t}$ and $\mathcal{A}(\mathbb{C}((\omega))^\times, 2s+1) = \{p(\omega^{-1})\frac{dt}{t} \mid p \in \mathbb{C}[\omega^{-1}], \deg(p) = 2s+1\}$. We remark that if we set $P_{T(F)} = \mathrm{GL}_n(\mathfrak{o})$ and $P_{\mathbb{C}((\omega))^\times} = I$, then an S-formal type $A_y = X\frac{dt}{t}$ of depth r gives rise to the S-regular stratum $(P_S, r, X\frac{dt}{t})$.

Theorem 2. *If $\hat{\nabla}$ contains the S-regular stratum (P, r, β), then $\hat{\nabla}$ is $P^1 := 1 + \mathfrak{p}^1$-gauge equivalent to a unique connection of the form $d + A$ for $A \in \mathcal{A}(S,r)$ with leading term $\beta^\flat\frac{dt}{t}$.*

Before discussing moduli spaces, we need to define the notion of a *framable* connection. Suppose that ∇ is a flat G-bundle on $\mathbb{P}^1$. Upon fixing a global trivialization ϕ, we can write $\nabla = d + [\nabla]$, where $[\nabla]$ is the matrix of the connection. Assume that the formal connection $\hat{\nabla}_y$ at y has formal type A_y. We say that $g \in \mathrm{GL}_2(\mathbb{C})$ is a *compatible framing* for ∇ at y if $g \cdot \hat{\nabla}_y$ contains the regular stratum determined by A_y. For example, if $A_y = (D_{-r}t^{-r} + \cdots + D_0)\frac{dt}{t}$, then g is a global gauge change such that $g \cdot \hat{\nabla}_y = d + (D_{-r}t^{-r} + Xt^{-r+1})\frac{dt}{t}$ with $X \in \mathfrak{gl}_2(\mathfrak{o})$. The connection ∇ is framable at y if there exists a compatible framing.

We now explain how moduli spaces of connections can be defined for meromorphic connections ∇ on $\mathbb{P}^1$ such that $\hat{\nabla}_y$ is toral at each irregular singularity. We also want to allow for regular singular points. If the residue of a regular singular connection is "nonresonant", in the sense that the eigenvalues do not differ by a nonzero integer, then its formal isomorphism class is determined by the adjoint orbit of the residue. Accordingly, our starting data will consist of:

- A nonempty set $\{x_i\} \subset \mathbb{P}^1$ of irregular singular points;
- $\mathbf{A} = (A_i)$, a set of S_i-formal types with positive depths r_i at the x_i's;
- A set $\{y_j\} \subset \mathbb{P}^1$ of regular singular points disjoint from $\{x_i\}$;
- A corresponding collection $\mathbf{C} = (C_j)$ of nonresonant adjoint orbits.

Let $\mathcal{M}(\mathbf{A}, \mathbf{C})$ be the moduli space classifying meromorphic rank 2 connections (V, ∇) on $\mathbb{P}^1$ with V trivializable such that:

- ∇ has irregular singular points at the x_i's, regular singular points at the y_j's, and no other singular points;
- ∇ is framable and has formal type A_i at x_i;
- ∇ has residue at y_j in C_j.

We will construct this moduli space as the Hamiltonian reduction of a product over the singular points of symplectic manifolds, each of which is endowed with a Hamiltonian action of $\mathrm{GL}_2(\mathbb{C})$. At a regular singular point with adjoint orbit C, the symplectic manifold is C viewed as the coadjoint orbit $C\frac{dt}{t}$.) The symplectic manifold at an irregular singular point with formal type A will be denoted $\mathcal{M}_A$; it is called an extended orbit. To define it, let $\mathcal{O}_A$ be the P_S-coadjoint orbit of $A|_{\mathfrak{p}_S} \in \mathfrak{p}_S^\vee$. If A is a $T(F)$-formal type, then $\mathcal{M}_A = \mathcal{O}_A$. The $\mathrm{GL}_2(\mathbb{C})$-action is the usual coadjoint action, and the moment map μ_A is just restriction of the functional α to $\mathfrak{gl}_2(\mathbb{C})$. The definition is more complicated when A is a $\mathbb{C}((t))^\times$ formal type. In this case, let $B \subset \mathrm{GL}_2(\mathbb{C})$ be the upper triangular subgroup. Then, $\mathcal{M}_A = \{(Bg, \alpha) \mid (\mathrm{Ad}^*(g)(\alpha))|_{\mathfrak{i}} \in \mathcal{O}_A)\} \subset (B\backslash \mathrm{GL}_2(\mathbb{C})) \times \mathfrak{gl}_2(\mathfrak{o})^\vee$. The group $\mathrm{GL}_2(\mathbb{C})$ acts on $\mathcal{M}_A$ via $h(Bg, \alpha) = (Bgh^{-1}, Ad^*(h)\alpha)$ with moment map $\mu_A : (Bg, \alpha) \mapsto \alpha|_{\mathfrak{gl}_2(\mathbb{C})}$.

We can now describe the structure of $\mathcal{M}(\mathbf{A}, \mathbf{C})$.

Theorem 3. *The moduli space $\mathcal{M}(\mathbf{A}, \mathbf{C})$ is obtained as a symplectic reduction of the product of local data:*

$$\mathcal{M}(\mathbf{A}, \mathbf{C}) \cong \left[\left(\prod_i \mathcal{M}_{A_i} \right) \times \left(\prod_j C_j \right) \right] /\!\!/_0 \, \mathrm{GL}_2(\mathbb{C}).$$

Remark 4. For other variants and a realization of the isomonodromy equations as an integrable system, see [2, 3].

Here, $\mathrm{GL}_2(\mathbb{C})$ acts diagonally on the product manifold, so that the moment map μ for the product is the sum of the moment maps of the factors. Since each factor involves a functional on $\mathfrak{gl}_2(\mathfrak{o})$ or $\mathfrak{gl}_2(\mathbb{C})$, the definition of the local moment maps shows that $\mu^{-1}(0)$ is the set of tuples for which the restrictions of these functionals to $\mathfrak{gl}_2(\mathbb{C})$ sum to 0. Writing each functional as a 1-form, this is just the condition that the sum of the residues vanish.

We conclude this paper with two illustrations of the theorem, each with one irregular singular and one regular singular point, say at 0 and ∞. Take $A^s = \mathrm{diag}(a, b)t^{-1}\frac{dt}{t} \in \mathcal{A}(T(F), 1)$ (so $a \neq b$) and $A^e = \omega^{-1}\frac{dt}{t} \in \mathcal{A}(\mathbb{C}((\omega))^\times, 1)$. Also, let C be an arbitrary nonresonant adjoint orbit. Below, we use the identifications $\mathfrak{gl}_2(\mathfrak{o})^\vee = \mathfrak{gl}_2(\mathbb{C})[t^{-1}]\frac{dt}{t}$ and $\mathfrak{i}^\vee = \mathfrak{t}[\omega^{-1}]\frac{dt}{t}$. Under these identifications, the restriction map $\mathfrak{gl}_2(\mathfrak{o})^\vee \to \mathfrak{i}^\vee$ has fiber $\mathbb{C}e_{12}\frac{dt}{t}$.

Example 5 ($\mathcal{M}(A_0^s, C_\infty)$). We first observe that $\mathrm{Ad}^*(1 + t\,\mathfrak{gl}_2(\mathfrak{o}))(A^s) = A^s + \left(\begin{smallmatrix} 0 & u \\ v & 0 \end{smallmatrix}\right)\frac{dt}{t}$ with $u, v \in \mathbb{C}$ arbitrary. Indeed, if $X, Y \in \mathfrak{gl}_2(\mathbb{C})$, then $(1+tX)Y(1+tX)^{-1} \equiv Y + t[X, Y] \pmod{t^2}$, and the claim follows since $\mathrm{ad}(\mathrm{diag}(a, b))(\mathfrak{gl}_2(\mathbb{C}))$ is the off-diagonal matrices. Since $\mathrm{GL}_2(\mathfrak{o}) = \mathrm{GL}_2(\mathbb{C}) \ltimes (1 + t\,\mathfrak{gl}_2(\mathfrak{o}))$, we get

$$\mathrm{Ad}^*(\mathrm{GL}_2(\mathfrak{o}))(A^s) = \mathrm{Ad}^*(\mathrm{GL}_2(\mathbb{C})) \left\{ \begin{pmatrix} at^{-1} & u \\ v & bt^{-1} \end{pmatrix} \frac{dt}{t} \mid u, v \in \mathbb{C} \right\}. \quad (4)$$

The moduli space is the space of $\mathrm{GL}_2(\mathbb{C})$-orbits of pairs (α, Y) with $Y \in C$, and $\mathrm{Res}(\alpha) + Y = 0$. One sees from (4) that every orbit has a representative with α of the form $\left(\begin{smallmatrix} at^{-1} & u \\ v & bt^{-1} \end{smallmatrix}\right)\frac{dt}{t}$ for some $u, v \in \mathbb{C}$. Since T is the stabilizer of the leading term, it follows that the moduli space is the same as the set of T-orbits of pairs (α, Y) with α in this standard form. We claim that

$$|\mathcal{M}(A_0^s, C_\infty)| = \begin{cases} 2, & \text{if } C \text{ is regular nilpotent} \\ 1, & \text{if } C = 0 \text{ or } C \text{ is regular semisimple with trace } 0 \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

To see this, note that there are unique representatives for the T-orbits of standard α's by taking $(u, 1)$ with $u \in \mathbb{C}$, $(1, 0)$, and $(0, 0)$. Each $(u, 1)$ with $u \neq 0$ gives rise to Y regular semisimple with trace 0 and determinant u. The pairs $(1, 0)$ and $(0, 1)$ both lead to regular nilpotent Y's while $(0, 0)$ just gives $Y = 0$.

Example 6 ($\mathcal{M}(A_0^e, C_\infty)$). Here, the moduli space is the space of $\mathrm{GL}_2(\mathbb{C})$-orbits of triples (Bg, α, Y), where $(Bg, \alpha) \in \mathcal{M}_{A^e}$, $Y \in C$, and $\mathrm{Res}(\alpha) + Y = 0$. This is the same as the space of B-orbits of triples (B, α, Y). Using $I = T \ltimes I^1$, an argument similar to the one in the previous example shows that

$$\mathrm{Ad}^*(I)(A^e) = \mathrm{Ad}^*(T) \left\{ \begin{pmatrix} z & t^{-1} \\ 1 & -z \end{pmatrix} \frac{dt}{t} \mid z \in \mathbb{C} \right\}. \tag{6}$$

It follows easily that

$$\alpha = \begin{pmatrix} z & vt^{-1} + w \\ v^{-1} & -z \end{pmatrix} \frac{dt}{t} \tag{7}$$

for some $z, v, w \in \mathbb{C}$ with $v \neq 0$. In fact, each B-orbit has a unique representative with $v = 1$ and $z = 0$. This means that the only adjoint orbits C that give nonempty moduli space are the orbits of $\left(\begin{smallmatrix} 0 & -w \\ -1 & 0 \end{smallmatrix}\right)$. Thus, $\mathcal{M}(A_0^e, C_\infty)$ is a singleton if C is regular nilpotent or a regular semisimple with trace zero; otherwise, it is empty. We remark that in the regular nilpotent case, the unique such connection is the GL_2 version of the Frenkel-Gross rigid connection, and this argument shows that this connection is indeed uniquely determined by its local behavior.

Remark 5. By setting $C = 0$ in these examples, we obtain the corresponding one singularity moduli spaces: $|\mathcal{M}(A_0^s)| = 1$ and $\mathcal{M}(A_0^e) = \emptyset$.

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Daniel S. Sage
Department of Mathematics
Louisiana State University
Baton Rouge, LA 70803
e-mail: `sage@math.lsu.edu`