ISOMONODROMIC DEFORMATIONS OF CONNECTIONS WITH SINGULARITIES OF PARAHORIC FORMAL TYPE

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ABSTRACT. In previous work, the authors have developed a geometric theory of fundamental strata to study connections on the projective line with irregular singularities of parahoric formal type. In this paper, the moduli space of connections that contain regular fundamental strata with fixed combinatorics at each singular point is constructed as a smooth Poisson reduction. The authors then explicitly compute the isomonodromy equations as an integrable system. This result generalizes work of Jimbo, Miwa, and Ueno to connections whose singularities have parahoric formal type.

1. INTRODUCTION

The study of transcendental solutions to differential equations has a long pedigree in mathematics. An important innovation in this field from the turn of the century was Schlesinger's observation that families of solutions to *linear* differential equations often satisfy interesting nonlinear equations. His work on monodromy-preserving deformations of Fuchsian differential equations produced a remarkable family of nonlinear differential equations which satisfy the Painlevé property, namely, that the only movable singularities are simple poles [16].

For example, consider a family of regular singular differential equations on $\mathbb{P}^1$ of the form

$$(1.1) \qquad \frac{d}{dz}\Phi = \sum_{i=1}^{p} \frac{1}{z - x_i} A_i \Phi,$$

where $x_1, \ldots, x_p \in \mathbb{P}^1$ and A_i is a matrix valued holomorphic function in the coordinates $x_1, \ldots, x_p$. A family of fundamental solutions to (1.1) has constant monodromy if the A_i's satisfy the Schlesinger equations:

$$dA_i = \sum_{j \neq i} [A_i, A_j] \frac{d(x_i - x_j)}{x_i - x_j}.$$

(See [14, IV.1] for a contemporary exposition.) In general, we will refer to the differential equations satisfied by a family of linear differential equations with constant monodromy as isomonodromy equations.

It took almost seventy years for progress to be made on the isomonodromy problem for irregular singular differential equations. In 1981, Jimbo, Miwa, and Ueno characterized the isomonodromy equations for certain generic families of irregular

singular differential equations [10]. One explanation for the long delay is that the monodromy map for irregular singular point differential equations is significantly more complicated; it involves the asymptotic behavior of solutions along Stokes sectors at each singular point, so it is less explicitly topological than the monodromy map in the regular singular case. The proof of Jimbo, Miwa and Ueno required a much clearer geometric picture of the monodromy map.

In this paper, we will think of linear differential equations in terms of meromorphic connections ∇ on a trivial vector bundle V on $\mathbb{P}^1$. After fixing a basis for V, we may write $\nabla = d + \alpha$, where d is the usual exterior derivative and α is an $\mathrm{End}(V)$-valued meromorphic one-form. Roughly speaking, if one considers the moduli stack $\mathcal{M}_{\mathrm{DR}}$ of meromorphic connections on $\mathbb{P}^1$, there is a Riemann-Hilbert map to the moduli stack $\mathcal{M}_{\mathrm{B}}$ of irregular monodromy representations.[1] An explicit description of the irregular Riemann-Hilbert correspondence may be found in [11].

If one can find a smooth family $\mathcal{M}'$ that maps to $\mathcal{M}_{\mathrm{DR}}$, the Malgrange-Sibuya theorem [11] implies that the fibers of the monodromy map are a foliation of $\mathcal{M}'$. In particular, the isomonodromy equations should correspond to an integrable distribution on $\mathcal{M}'$. By an observation of Boalch [2, Appendix], $\mathcal{L} \subset \mathcal{M}'$ is a leaf of this foliation if and only if the family of connections corresponding to $\mathcal{L}$ is integrable, i.e., there exists a connection $\bar{\nabla}$ on $\mathcal{L} \times \mathbb{P}^1$ with the property that $\bar{\nabla}|_{\{x\} \times \mathbb{P}^1}$ is a representative of the isomorphism class $x \in \mathcal{M}_{\mathrm{DR}}$. Throughout the paper, we will suppress the monodromy point of view in favor of this integrability condition. As an example, we describe the smooth family of framed connections $\widetilde{\mathcal{M}}_{\mathrm{JMU}}$ which was constructed by Boalch, building on work of Jimbo, Miwa and Ueno.

Fix a finite collection of points $(x_i)_{i \in I} \subset \mathbb{P}^1$ and a vector of non-negative integers $(r_i)_{i \in I}$.[2] Points in $\widetilde{\mathcal{M}}_{\mathrm{JMU}}$ correspond to isomorphism classes of connections $\nabla = d + \alpha$, singular at x_i, with the following additional data: there is a collection of framings $g_i \in \mathrm{GL}_n(\mathbb{C})$ such that the Laurent expansion of $\mathrm{Ad}(g_i)\alpha$ at x_i has the form

$$\mathrm{Ad}(g_i)\alpha = (A_{r_i} \frac{1}{(z - x_i)^{r_i}} + \ldots + A_1 \frac{1}{z - x_i} + A_0)\frac{dz}{z - x_i},$$

where $A_i \in \mathfrak{gl}_n(\mathbb{C})$ and the leading term A_{r_i} is a regular diagonal matrix. The moduli space $\widetilde{\mathcal{M}}_{\mathrm{JMU}}$ is, in fact, smooth. The isomonodromy equations may be expressed in terms of a Pfaffian system involving terms Θ_i, which control the dynamics of the framings, and terms Ξ_i, which are essentially the principal parts of the curvature of $\bar{\nabla}$ [10].

In [3], the authors describe smooth moduli spaces of framed connections with arbitrary slope, generalizing a construction of [2]. The goal of this paper is to study isomonodromic deformations of such connections. The primary technical tool is a local invariant of meromorphic connections called the fundamental stratum, which plays the role of the leading term. A stratum is a triple (P, r, β), consisting of a parahoric subgroup $P \subset \mathrm{GL}_n(\mathbb{C}[[z]])$, a non-negative integer r, and a 'non-degenerate' linear functional on the rth graded piece associated to the canonical filtration on the Lie algebra of P. The relevant condition on connections which

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[1]Here, DR stands for the "DeRham" theory of meromorphic connections, and B stands for the "Betti" theory of irregular monodromy representations.

[2]There is a slight simplification here: Jimbo, Miwa, and Ueno and Boalch allow x_i to vary in $\mathbb{P}^1$.

assures smoothness of the moduli space is that ∇ must contain a 'regular' stratum. This approach is described in detail in Section 2.

The primary motivation behind the introduction of fundamental strata into the study of connections comes from the geometric Langlands program, which in this case suggests an analogy between wildly ramified adelic representations of GL_n and irregular monodromy representations of rank n on $\mathbb{P}^1$ (see [6] or, for a more physical interpretation, [17]). The fundamental stratum (alternatively, the minimal K-type) was originally used as a tool for classifying wildly ramified representations of a reductive group over a p-adic field [4, 13]. Thus, one hopes that a dictionary between fundamental strata and families of monodromy representations will better illuminate the wild ramification case of the geometric Langlands correspondence.

We conclude with a short overview of the results in this paper. Suppose that $\mathbf{x} = (x_i)_{i \in I} \subset \mathbb{P}^1$ is a collection of singular points, $\mathbf{T}$ is a collection of uniform maximal tori $T_i \subset \mathrm{GL}_n(\mathbb{C}((z - x_i)))$, $\mathbf{P} = (P_i)_{i \in I}$ is the associated collection of uniform parahoric subgroups $P_i \subset \mathrm{GL}_n(\mathbb{C}[[z - x_i]])$, and $\mathbf{r} = (r_i)_{i \in I}$ is a vector of non-negative integers. In Section 3, we give a construction of the moduli space $\widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$ consisting of isomorphism classes of connections with compatible framings on $\mathbb{P}^1$ that contain regular strata of the form (P_i, r_i, β_i) at x_i. By Proposition 3.15 and Theorem 3.16, $\widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$ is a Poisson manifold; moreover, the symplectic leaves correspond to connected components of moduli spaces of connections with a fixed formal isomorphism class at each singular point. When all of the tori are split (so the parahoric subgroups are maximal, i.e., $P_i = \mathrm{GL}_n(\mathbb{C}[[z - x_i]])$), $\widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r}) = \widetilde{\mathcal{M}}_{\mathrm{JMU}}$.

In Section 4, we calculate the isomonodromy equations for connections corresponding to points of $\widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$ (Theorem 4.12). In this context, the framing data is given by a coset $U \backslash \mathrm{GL}_n(\mathbb{C})$, where $U \subset \mathrm{GL}_n(\mathbb{C})$ is the unipotent subgroup determined by P. In particular, part of the isomonodromy data is a time dependent flow on an affine bundle over a partial flag manifold. Since U is trivial when P is a maximal parahoric subgroup, this phenomenon is absent in the isomonodromy equations of Jimbo, Miwa and Ueno. We remark that Bertola and Mo have also investigated generalizations of the isomonodromy equations using a very different approach; there is some overlap between the families of connections considered in this paper and in [1].

In Section 5, we describe a differential ideal $\overline{\mathcal{I}}$ on the moduli space $\widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$ corresponding to the isomonodromy equations. To be precise, we construct a differential ideal $\mathcal{I}$ on a principal $\mathrm{GL}_n(\mathbb{C})$-bundle over $\widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$ and then show that it descends. The main result of the paper, Theorem 5.1, states that both $\overline{\mathcal{I}}$ and $\mathcal{I}$ are integrable Pfaffian systems. Moreover, there is a natural correspondence between the leaves of the foliation determined by $\overline{\mathcal{I}}$ and isomonodromic deformations of framed connections. The proof is deferred to Section 6.

Finally, in Section 7, we compute an explicit example of the isomonodromy equations in the case where T_i is the group of units of a field, P_i is an Iwahori subgroup, and $r_i = 1$ for all i. To the authors' knowledge, this is a completely new example of an integrable system on a Poisson manifold.

2. Formal Types

In this section, we review some results from the geometric theory of fundamental strata and recall how they may be used to associate formal types to irregular singular connections. Let $F = k((z))$ be field of formal Laurent series with coefficients in an algebraically closed field k in characteristic zero, and let $\mathfrak{o} \subset F$ be the corresponding power series ring. Let $\hat{V}$ be an n-dimensional F vector space. A lattice chain $\mathcal{L} = \{L^i\}_{i \in \mathbb{Z}}$ is a collection of $\mathfrak{o}$-lattices in $\hat{V}$ satisfying the following properties: $L^i \supset L^{i+1}$, and $zL^i = L^{i+e}$ with a fixed period $e > 0$. We say $\mathcal{L}$ is *uniform* if $\dim_k L^i/L^{i+1} = n/e$ for all i; the lattice chain is *complete* if $e = n$.

Definition 2.1. A uniform parahoric subgroup $P \subset \mathrm{GL}(\hat{V})$ is the stabilizer of a uniform lattice chain $\mathcal{L}$, i.e., $P = \{g \in \mathrm{GL}(\hat{V}) \mid gL^i = L^i \text{ for all } i\}$. The Lie algebra of P is the parahoric subalgebra $\mathfrak{P} \subset \mathfrak{gl}(\hat{V})$ consisting of $\mathfrak{P} = \{p \in \mathfrak{gl}(\hat{V}) \mid pL^i \subset L^i \text{ for all } i\}$. Note that $\mathfrak{P}$ is in fact an associative subalgebra of $\mathfrak{gl}(\hat{V})$. An Iwahori subgroup I is the stabilizer of a complete lattice chain, and an Iwahori subalgebra $\mathfrak{I}$ is the Lie algebra of I.

For more details on lattice chains and the corresponding parahoric subgroups and subalgebras, see for example [15, 3].

There are natural filtrations on P (resp. $\mathfrak{P}$) by congruence subgroups (resp. ideals). For $r \in \mathbb{Z}$, define the $\mathfrak{P}$-module $\mathfrak{P}^r$ to consist of $X \in \mathfrak{P}$ such that $XL^i \subset L^{i+r}$ for all i; it is an ideal of $\mathfrak{P}$ for $r \geq 0$ and a fractional ideal otherwise. The congruence subgroup $P^r \subset P$ is then defined by $P^0 = P$ and $P^r = \mathrm{I}_n + \mathfrak{P}^r$ for $r > 0$. Note that these ideals are multiplicative, in the sense that $\mathfrak{P}^r\mathfrak{P}^s = \mathfrak{P}^{r+s}$. If we fix a form $\nu \in \Omega^1_{F/k}$ of order -1, the pairing

$$(2.1) \qquad\qquad \langle X, Y \rangle_\nu = \mathrm{Res}\,\mathrm{Tr}(XY\nu)$$

identifies $\mathfrak{P}^{-r}$ with $(\mathfrak{P}^{r+1})^\perp$ and $(\mathfrak{P}^{r+1}/\mathfrak{P}^{s+1})^\vee$ with $\mathfrak{P}^{-s}/\mathfrak{P}^{-r}$. There are similar formulas for ν of arbitrary order, for example, $(\mathfrak{P}^{r+1})^\perp \cong \mathfrak{P}^{-r+(1+\mathrm{ord}(\nu))e}$. Throughout this section, we will assume for simplicity that $\mathrm{ord}(\nu) = -1$, but all definitions and results can be stated for other ν.

Definition 2.2. Let $\hat{V}$ be an F-vector space, and let (P, r, β) be a triple consisting of

- $P \subset \mathrm{GL}(\hat{V})$ a uniform parahoric subgroup;
- $r \in \mathbb{Z}_{\geq 0}$, with $\gcd(r, e) = 1$;
- $\beta \in (\mathfrak{P}^r/\mathfrak{P}^{r+1})^\vee$.

After fixing ν as above, we may identify β with a coset $\beta_\nu + \mathfrak{P}^{-r+1} \subset \mathfrak{P}^{-r}/\mathfrak{P}^{-r+1}$. Thus, we may choose a representative $\beta_\nu \in \mathfrak{P}^{-r}$ for β. We say that (P, r, β) is a *uniform stratum* if $\beta_\nu + \mathfrak{P}^{-r+1}$ contains no nilpotent elements.

In this paper, we are interested in strata that satisfy a 'graded' version of regular semisimplicity. We say that a maximal torus T is *uniform* if it isomorphic to $(E^\times)^\ell$ for some field extension E of F. If $[E : F] = e$, then

$$(E^\times)^{n/e} \cong T \subset \mathrm{GL}(\hat{V}),$$

with corresponding Cartan subalgebra $\mathfrak{t} \subset \mathfrak{gl}(\hat{V})$ isomorphic to $E^{n/e}$. Let $\mathfrak{o}_E$ be the ring of integers of E, and let χ_j be the identity elements of the jth Wedderburn component of $\mathfrak{t}$. We set $T(\mathfrak{o}) = \prod_j (\mathfrak{o}_E)^\times \chi_j \subset T$ and $\mathfrak{t}(\mathfrak{o}) = \bigoplus_j \mathfrak{o}_E \chi_j \subset \mathfrak{t}$. Moreover, we write $\mathfrak{t}^\flat$ for the k-span of the χ_j and $T^\flat$ for $(\mathfrak{t}^\flat)^\times$.

Now, suppose that (P, r, β) is a uniform stratum. There is a map $\partial_{\beta,s} : \mathfrak{P}^s/\mathfrak{P}^{s+1} \to \mathfrak{P}^{s-r}/\mathfrak{P}^{s-r+1}$ given by $\partial_{\beta,s}(X + \mathfrak{P}^s) = \mathrm{ad}(X)(\beta_\nu) + \mathfrak{P}^{s-r+1}$.

Definition 2.3. We say that (P, r, β) is a regular stratum centralized by T if it satisfies the following conditions:

(1) $T(\mathfrak{o}) \subset P$;
(2) $\ker(\partial_{\beta,s}) = \mathfrak{t} \cap \mathfrak{P}^s + \mathfrak{P}^{s+1}$ for all s;
(3) $y_\beta = z^r \beta_\nu^e + \mathfrak{P}^1$ is a semi-simple element of the algebra $\mathfrak{P}/\mathfrak{P}^1$;
(4) when $r = 0$ (and hence, $e_P = 1$), the eigenvalues of y_β are distinct modulo $\mathbb{Z}$.

In fact, by [3, Theorem 3.8], $\mathcal{L}$ induces a complete lattice chain on each $\chi_j(\hat{V})$ (with period e). Write I_j for the corresponding Iwahori subgroup. It follows from [3, Lemma 2.4] that it is the unique lattice chain $\mathcal{L}_j$ with $\mathfrak{o}_E^\times \subset I_j$. Moreover, if we fix a uniformizer ϖ_E for $\mathfrak{o}_E$, then $\varpi_E \mathfrak{I}_j = \mathfrak{I}_j^1$. By [4, Proposition 1.18], we deduce that the matrix $\varpi_T = (\varpi_E, \dots, \varpi_E) \in \mathfrak{t}$ satisfies the property $\varpi_T \mathfrak{P} = \mathfrak{P} \varpi_T = \mathfrak{P}^1$.

Let T be a uniform torus, and suppose that $L \subset \hat{V}$ is an $\mathfrak{o}$-lattice such that $T(\mathfrak{o}) \subset \mathrm{GL}(L)$. Then, [3, Proposition 3.12] shows that there is a unique parahoric subgroup $P_{T,L}$ such that $P_{T,L} \subset \mathrm{GL}(L)$ and there exists a regular stratum $(P_{T,L}, r, \beta)$ centralized by T. Moreover, there is a canonical $\mathbb{Z}$-filtration on $\mathfrak{t}$ (determined by the degree of the eigenvalues of an element), and $\mathfrak{t}^i = \mathfrak{P}_{T,L}^i \cap \mathfrak{t}$.

Recall the following result:

Proposition 2.4 ([3, Proposition 2.11]). *Let (P, r, β) be a regular stratum centralized by T, and let $\beta_\nu \in \mathfrak{P}^{-r}$ be a representative for β. There is a morphism of $\mathfrak{t}$-bimodules $\pi_\mathfrak{t} : \mathfrak{gl}(\hat{V}) \to \mathfrak{t}$ satisfying the following properties:*

(1) *$\pi_\mathfrak{t}$ restricts to the identity on $\mathfrak{t}$;*
(2) *$\pi_\mathfrak{t}(\mathfrak{P}^\ell) \subset \mathfrak{P}^\ell$;*
(3) *the kernel of the induced map*

$$\bar{\pi}_\mathfrak{t} : (\mathfrak{t} + \mathfrak{P}^{\ell-r})/\mathfrak{P}^{\ell-r+1} \to \mathfrak{t}/(\mathfrak{t}^{\ell-r+1})$$

is given by the image of $\mathrm{ad}(\mathfrak{P}^\ell)(\beta_\nu)$ modulo $\mathfrak{P}^{\ell-r+1}$;
(4) *if $y \in \mathfrak{t}$ and $X \in \mathfrak{gl}(\hat{V})$, then $\langle y, X \rangle_\nu = \langle y, \pi_\mathfrak{t}(X) \rangle_\nu$.*

An immediate corollary of part (3) is the following.

Corollary 2.5. *Let $\bar{\pi}_{\mathfrak{t},\ell} : \mathfrak{P}^\ell/\mathfrak{P}^{\ell+1} \to \mathfrak{t}/\mathfrak{t}^{\ell+1}$ be the induced map, and set $W_\ell = \ker(\bar{\pi}_{\mathfrak{t},\ell})$. Then, the induced map $\mathrm{ad}(\beta_\nu) : W_\ell \to W_{\ell-r}$ is an isomorphism.*

A connection on $\hat{V}$ is a k-derivation $\nabla : \hat{V} \to \hat{V} \otimes \Omega^1_{F/k}$. If τ is a k-derivation on F, we write ∇_τ for the composition of ∇ with the inner derivation associated to $\tau : \nabla_\tau(v) = i_\tau(\nabla(v))$. In particular, let τ_ν be the derivation with the property that $i_{\tau_\nu}(\nu) = 1$.

Definition 2.6. Let (P, r, β) be a uniform stratum. When $r \geq 1$, we say that $(\hat{V}, \nabla)$ contains (P, r, β) if $\nabla_{\tau_\nu}(L^i) \subset L^{i-r}$ and $(\nabla_{\tau_\nu} - \beta_\nu)(L^i) \subset L^{i-r+1}$ for all i. When $r = 0$, and thus $e = 1$, we say that $(\hat{V}, \nabla)$ contains $(P, 0, \beta)$ if $(\nabla_{\tau_\nu} - \beta_\nu)(L^i) \subset L^{i+1}$ for some lattice L^i.

Given a trivialization $\phi : \hat{V} \xrightarrow{\sim} F^n$, we write $[\nabla]_\phi$ for the matrix of ∇ with respect to the standard basis of F^n. By the Leibniz rule, $\nabla = d_z + [\nabla]_\phi$ where d_z is the

usual exterior k-differential on F. The group $\mathrm{GL}_n(F)$ acts transitively on the space of trivializations for V, and

$$(2.2) \qquad\qquad [\nabla]_{g\phi} = g \cdot [\nabla]_\phi := \mathrm{Ad}(g)[\nabla]_\phi - (d_z g)g^{-1}.$$

If we have fixed a base trivialization, we will shorten $[\nabla]_{g\phi}$ to $[\nabla]_g$ and $[\nabla]_\phi$ to $[\nabla]$. In general, the left action $g\cdot$ on $\mathfrak{gl}_n(F) \otimes \Omega^1_{F/k}$ is called a gauge transformation, and we say that two matrices $X, Y \in \mathfrak{gl}_n(F) \otimes \Omega^1_{F/k}$ are gauge equivalent if there exists $g \in \mathrm{GL}_n(F)$ such that $g \cdot X = Y$. Thus, if ∇ and ∇' are connections on $\hat{V}$, and $[\nabla]_\phi$ is gauge equivalent to $[\nabla']_\phi$, then $(\hat{V}, \nabla)$ and $(\hat{V}, \nabla')$ are isomorphic.

Suppose that (P, r, β) is a regular stratum in $\mathrm{GL}_n(F)$ with $P \subset \mathrm{GL}_n(\mathfrak{o})$ centralized by a torus T. We denote the pullback of P and β to $\mathrm{GL}(\hat{V})$ by P^ϕ and β^ϕ, respectively.

Theorem 2.7. [3, Theorem 4.13] *If $(\hat{V}, \nabla)$ contains the stratum (P^ϕ, r, β^ϕ), then there exists $p \in P^1$ and a regular element $A_\nu \in \mathfrak{t}^{-r}$ such that $p \cdot [\nabla]_\phi = A_\nu \nu$. Furthermore, the orbit of $A_\nu \nu$ under P^1-gauge transformations contains $(A_\nu + \mathfrak{P}^1)\nu$.*

Remark 2.8. By [3, Lemma 4.4], the map $\mathfrak{P}^{-r} \to \mathfrak{P}^{-r}/\mathfrak{P}^1$ intertwines the gauge and adjoint actions of P. This implies that the functional induced on $\mathfrak{P}^r/\mathfrak{P}^{r+1}$ by A_ν coincides with β.

We now define the "formal types" of a connection. Let $T \subset \mathrm{GL}_n(F)$ be a uniform maximal torus such that $T(\mathfrak{o}) \subset \mathrm{GL}_n(\mathfrak{o})$. We set $P = P_{T, \mathfrak{o}^n}$ as defined before Proposition 2.4.

Definition 2.9. A functional $A \in (\mathfrak{t}^0)^\vee$ is called a *T-formal type of depth r* if
  (1) $\mathfrak{t}^{r+1}$ is the smallest congruence ideal contained in $A^\perp$; and
  (2) the stratum (P, r, β) is regular and centralized by T, where β is the functional on $\mathfrak{P}^r/\mathfrak{P}^{r+1}$ induced by $\pi_\mathfrak{t}^*(A) \in \mathfrak{P}^\vee$.

We always use the notation A_ν for a representative of A in $\mathfrak{t}^{-r}$. Observe that A_ν is also a representative for $\pi_\mathfrak{t}^*(A)$.

Definition 2.10. The set of T-formal types associated to the formal connection $(\hat{V}, \nabla)$ consists of those A for which there is a trivialization $\phi : \hat{V} \to F^n$ such that $(\hat{V}, \nabla)$ contains the stratum (P^ϕ, r, β^ϕ) and $[\nabla]_\phi$ is formally gauge equivalent to an element of $(A_\nu + \mathfrak{P}^1)\nu$ by an element of P^1.

This notion is independent of ν and only depends on the choice of torus up to $\mathrm{GL}_n(F)$-conjugacy. (For a precise statement, see [3, Proposition 4.20].) Moreover, [3, Theorem 4.28] shows that this conjugacy class and the depth (or equivalently, the slope) are uniquely determined by the connection. We say that two connections that contain regular strata are *combinatorially equivalent* if they correspond to the same class of uniform torus and have the same slope.

By [3, Corollary 4.22], any two connections with formal type A are formally isomorphic. The converse is false. Indeed, there is an action of the relative affine Weyl group $W_T^{\mathrm{aff}} = N(T)/T(\mathfrak{o})$ on the set of T-formal types, and the possible formal types of $(\hat{V}, \nabla)$ are a single W_T^{aff}-orbit. However, fixing a compatible lattice and a regular stratum uniquely determines the formal type.

Proposition 2.11. *Fix a lattice $L \subset \hat{V}$. If $(\hat{V}, \nabla)$ contains a regular stratum (P^ϕ, r, β^ϕ) with $\phi(L) = \mathfrak{o}^n$, then there exists a unique formal type A such that*

$\pi_{\mathfrak{t}}^*(A)|_{\mathfrak{P}^r} = \beta$ and $[\nabla]_\phi$ is gauge equivalent to an element of $(A_\nu + \mathfrak{P}^1)\nu$ by an element of $\mathrm{GL}_n(\mathfrak{o})$.

Proof. This follows directly from [3, Corollary 4.29(2)]. $\qquad\square$

Finally, throughout the paper we will need to consider a slight variation on the formal type. Choose a uniformizer ϖ_E for E such that $\varpi_E^e = z$; under a suitable embedding $E \hookrightarrow \mathfrak{gl}_{n/e}(F)$,

$$(2.3) \qquad \varpi_E = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & 1 \\ z & 0 & \cdots & 0 \end{pmatrix}.$$

We choose a basis for each $\chi_j \hat{V}$ (and hence for $\hat{V}$) such that ϖ_T is block diagonal with these blocks. Using this basis, we define $H_T = (H_E, \ldots, H_E) \in \mathfrak{P}$ as the block diagonal matrix with blocks given by the diagonal matrix $H_E = \mathrm{diag}(\frac{e-1}{2e}, \frac{e-3}{2e}, \ldots, \frac{1-e}{2e})$. Let $H_T' \in \mathfrak{P}^\vee$ to be the corresponding functional $H_T'(X) = \mathrm{Tr}(H_T X)|_{z=0}$.

Definition 2.12. The *normalized formal types* of $(\hat{V}, \nabla)$ are the functionals $\tilde{A} = \pi_{\mathfrak{t}}^*(A) + H_T' \in \mathfrak{P}^\vee$, where A is a formal type of $(\hat{V}, \nabla)$.

Note that if $e = 1$, then $\tilde{A} = \pi_{\mathfrak{t}}^*(A)$.

Proposition 2.13. *Suppose that $(\hat{V}, \nabla)$ contains a regular stratum and has normalized formal type $\tilde{A}$. If $\tilde{A}_\nu \in \mathfrak{P}^{-r}$ is a representative for $\tilde{A}$, then there exists $\hat{p} \in P^1$ such that $\hat{p} \cdot [\nabla]_\phi = \tilde{A}_\nu \nu$.*

Proof. If $r = 0$ (so $e = 1$), this is just Theorem 2.7, so assume that $r \geq 1$. Note that $H_T' \in (\mathfrak{t}^\flat)^\perp$. Therefore, part 4 of Proposition 2.4 imply that $\pi_{\mathfrak{t}}(H_T) \in \mathfrak{P}^1$. Part 3 of the same proposition shows that there exists $X \in \mathfrak{P}^r$ such that $\mathrm{ad}(X)\tilde{A}_\nu \in H_T \frac{1}{\mathrm{Res}(\nu)} + \mathfrak{P}^1$, so there exists $p \in P^1$ such that $p \cdot [\nabla]_\phi = A_\nu \nu$ by Theorem 2.7. Moreover, $\exp(-X) \cdot \tilde{A}_\nu \nu \in (A_\nu + \mathfrak{P}^1)\nu$, so there exists $p' \in P^1$ such that $p' \exp(-X) \cdot \tilde{A}_\nu \nu = A_\nu \nu$. It follows that $(\exp(X)(p')^{-1}p) \cdot [\nabla]_\phi = \tilde{A}_\nu \nu$. $\qquad\square$

Proposition 2.14. *Any normalized formal type $\tilde{A}$ is stabilized by $T \cap P^1$.*

Proof. Since $\mathrm{ad}(\mathfrak{P}^1)H_T \in \mathfrak{P}^1$, P^1 stabilizes $H_T + \mathfrak{P}^1$. Moreover, the corresponding formal type A is stabilized by T. $\qquad\square$

3. Moduli spaces of connections

In this section, we will describe the moduli space of connections on $\mathbb{P}^1(\mathbb{C})$ with compatible framings and fixed combinatorics at each singular point.

First, we recall the construction of the moduli space of framed connections on $\mathbb{P}^1(\mathbb{C})$ with fixed formal type at each singular point [3]. Throughout, I will be a finite indexing set and $\mathbf{x} = \{x_i\}_{i \in I}$ a collection of distinct points in $\mathbb{P}^1$. We denote the completion of the function field of $\mathbb{P}^1$ at x_i by F_i and the corresponding power series ring by $\mathfrak{o}_i$.

Let V be a trivial rank n vector bundle on $\mathbb{P}^1$, i.e., we have fixed a trivialization $V \cong \mathcal{O}_{\mathbb{P}^1}^n$. Accordingly, we may identify the space of all global trivializations of V

with $\mathrm{GL}_n(\mathbb{C})$. Note that a global trivialization determines a local trivialization of $V_i := V \otimes_{\mathcal{O}_{\mathbb{P}^1}} F_i$, so there is a natural inclusion $\mathrm{GL}_n(\mathbb{C}) \hookrightarrow \mathrm{GL}_n(F_i)$; moreover, the global sections of V generate a distinguished lattice $L_i \subset V_i$. Suppose that ∇ is a connection on V with the property that the induced connection on V_i has formal type A_i. Therefore, by [3, Proposition 3.12], A_i determines a unique regular stratum (P_i, r, β_i) in $\mathrm{GL}_n(F_i)$ centralized by T_i. We also set $\varpi_i = \varpi_{T_i}$.

Throughout this section, U_i will denote the unipotent subgroup $P_i^1 \cap \mathrm{GL}_n(\mathbb{C})$ with Lie algebra $\mathfrak{u}_i = \mathfrak{P}_i^1 \cap \mathfrak{gl}_n(\mathbb{C})$. For simplicity, we write $\mathfrak{g}_i$ for the parahoric subalgebra $\mathfrak{gl}_n(\mathfrak{o}_i)$; its radical is $\mathfrak{g}_i^1 = t\mathfrak{g}_i$. Note that $U_i \cong P_i^1/(1+\mathfrak{g}_i^1)$ and $\mathfrak{u}_i \cong \mathfrak{P}_i^1/\mathfrak{g}^1$.

Definition 3.1. A *compatible framing* for ∇ at x_i is a global trivialization $g \in \mathrm{GL}_n(\mathbb{C})$ with the property that ∇ contains the $\mathrm{GL}(V_i)$-stratum (P_i^g, r, β_i^g) defined above. We say that ∇ is *framed* at x_i if there exists such a g.

We now define the moduli space of connections with fixed formal type and a specified framing at each singular point. Set $\bar{A}_i = A_i|_{\mathfrak{t}^1}$.

Definition 3.2. Let $\widetilde{\mathcal{M}}(\mathbf{A})$ be the the moduli space of isomorphism classes of triples $(V, \nabla, \mathbf{g})$, where

- ∇ is a meromorphic connection on the trivial bundle V with singularities at $\{x_i\}_{i \in I}$;
- $\mathbf{g} = \{U_i g_i\}_{i \in I}$, with g_i a compatible framing for ∇ at x_i;
- the formal type A_i' of ∇ at x_i satisfies $\bar{A}_i' = \bar{A}_i$.

The moduli space $\widetilde{\mathcal{M}}(\mathbf{A})$ is built out of "extended" coadjoint orbits $\widetilde{\mathcal{M}}(A_i)$ determined locally by each formal type A_i. In the following, let A be a formal type. We define $\pi_{\mathfrak{P}} : \mathfrak{g}^\vee \to \mathfrak{P}^\vee$ and $\pi_{\mathfrak{P}^1} : \mathfrak{g}^\vee \to (\mathfrak{P}^1)^\vee$ to be the restriction maps and $\mathcal{O}^1$ to be the orbit of $\bar{A}$ under the coadjoint action of P^1. Also, when $r = 0$ (so $e = 1$), we take $(\mathfrak{t}^\flat)' \subset \mathfrak{gl}_n(\mathbb{C})^\vee$ to be the set of functionals of the form $\phi(X) = \mathrm{Tr}(DX)$, where $D \in \mathfrak{t}^\flat$ is a diagonal matrix with distinct eigenvalues modulo $\mathbb{Z}$.

Definition 3.3. When $r > 0$, we define the extended orbit $\widetilde{\mathcal{M}}(A) \subset (U \backslash \mathrm{GL}_n(\mathbb{C})) \times \mathfrak{g}^\vee$ to be
$$\widetilde{\mathcal{M}}(A) = \{(Ug, \alpha) \mid \pi_{\mathfrak{P}^1}(\mathrm{Ad}^*(g)(\alpha)) \in \mathcal{O}^1)\}.$$
When $r = 0$, we define $\widetilde{\mathcal{M}}(A)$ by
$$\widetilde{\mathcal{M}}(A) = \{(g, \alpha) \in \mathrm{GL}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})^\vee \mid \mathrm{Ad}^*(g)\alpha \in (\mathfrak{t}^\flat)'\}.$$

There is a natural action ρ of $\mathrm{GL}_n(\mathbb{C})$ on $\widetilde{\mathcal{M}}(A)$ given by $\rho(h)(Ug, \alpha) = (Ugh^{-1}, \mathrm{Ad}^*(h)\alpha)$.

Note that when $r = 0$, $\widetilde{\mathcal{M}}(A)$ is independent of A.

Proposition 3.4. [3, Proposition 5.11] *The space* $\widetilde{\mathcal{M}}(A)$ *is a symplectic manifold, and ρ is a Hamiltonian action. The moment map for ρ is given by* $\mu_\rho(Ug, \alpha) = \mathrm{res}(\alpha)$, *where* $\mathrm{res}(\alpha) = \alpha|_{\mathfrak{gl}_n(\mathbb{C})}$.

Remark 3.5. If α_ν is any representative of α, then $\mathrm{res}(\alpha) = \mathrm{Res}(\alpha_\nu \nu)$.

Theorem 3.6. [3, Theorem 5.5] *The moduli space* $\widetilde{\mathcal{M}}(\mathbf{A})$ *is the symplectic reduction of* $\prod_{i \in I} \widetilde{\mathcal{M}}(A_i)$ *by the diagonal action of* $\mathrm{GL}_n(\mathbb{C})$:
$$\widetilde{\mathcal{M}}(\mathbf{A}) \cong (\prod_{i \in I} \widetilde{\mathcal{M}}(A_i)) /\!\!/_0 \mathrm{GL}_n(\mathbb{C}).$$

Moreover, $\widetilde{\mathcal{M}}(\mathbf{A})$ *is a symplectic manifold.*

Specifically, $\prod_{i \in I} \widetilde{\mathcal{M}}(A_i)$ is a symplectic manifold, and the diagonal action of $\mathrm{GL}_n(\mathbb{C})$ has moment map $\mu_{\mathrm{GL}_n} = \sum_{i \in I} \mathrm{res}_i$. Thus,

$$(\prod_{i \in I} \widetilde{\mathcal{M}}(A_i)) \,/\!\!/_0\, \mathrm{GL}_n(\mathbb{C}) = \mu_{\mathrm{GL}_n}^{-1}(0)/\mathrm{GL}_n(\mathbb{C}).$$

In the remainder of this section, we describe a larger moduli space in which we fix only the combinatorics of (V_i, ∇_i) at x_i, and not the formal type. Again, it will be constructed as a reduction of the product of local pieces.

First, we discuss these local pieces. Let $T \subset \mathrm{GL}_n(F)$ be a uniform maximum torus with splitting field of degree e such that $T(\mathfrak{o}) \subset \mathrm{GL}_n(\mathfrak{o})$. Let $P = P_{T,\mathfrak{o}^n}$ be the associated uniform parahoric as in Section 2, and let $r \geq 0$ be an integer relatively prime to e. We let $(\mathfrak{P}/\mathfrak{P}^{r+1})_{\mathrm{reg}}^{\vee}$ be the set of $\gamma \in (\mathfrak{P}/\mathfrak{P}^{r+1})^{\vee}$ such that $(P, r, \gamma|_{\mathfrak{P}^r/\mathfrak{P}^{r+1}})$ is a regular stratum centralized by T. We also let $\mathcal{A}(T, r)$ be the subset of $(\mathfrak{P}/\mathfrak{P}^{r+1})_{\mathrm{reg}}^{\vee}$ consisting of normalized formal types. (This notation differs from that of [3], where $\mathcal{A}(T, r)$ was the set of (non-normalized) formal types.) Note that $\mathcal{A}(T, r)$ is the subset of $(\mathfrak{P}/\mathfrak{P}^{r+1})_{\mathrm{reg}}^{\vee}$ consisting of $X + H_T$, where X is stabilized by the coadjoint action of $T(\mathfrak{o})$. We further define $(\mathfrak{P}^1/\mathfrak{P}^{r+1})_{\mathrm{reg}}^{\vee}$ to be the projection of $(\mathfrak{P}/\mathfrak{P}^{r+1})_{\mathrm{reg}}^{\vee}$ onto $(\mathfrak{P}^1/\mathfrak{P}^{r+1})^{\vee}$.

Definition 3.7. Let $P = P_{T,\mathfrak{o}^n} \subset \mathrm{GL}_n(\mathfrak{o})$ as in Section 2. If $r = 0$, define $\widetilde{\mathcal{M}}(T, r) = \widetilde{\mathcal{M}}(A)$ for any T-formal type A of depth 0. If $r > 0$, define $\widetilde{\mathcal{M}}(T, r) \subset (U \backslash \mathrm{GL}_n(\mathbb{C})) \times \mathfrak{gl}_n(\mathfrak{o})^{\vee}$ by

$$\widetilde{\mathcal{M}}(T, r) = \{(Ug, \alpha) | \pi_{\mathfrak{P}}(\mathrm{Ad}^*(g)\alpha) \in (\mathfrak{P}/\mathfrak{P}^{r+1})_{\mathrm{reg}}^{\vee}\}.$$

The space $\widetilde{\mathcal{M}}(T, r)$ is a manifold; the argument is similar to the proof that $\widetilde{\mathcal{M}}(A)$ is smooth given in [3].

Set $\mathfrak{V} = \pi_{\mathfrak{P}}^{-1}(\mathfrak{P}/\mathfrak{P}^{r+1})_{\mathrm{reg}}^{\vee} \subset (\mathfrak{g}/\mathfrak{P}^{r+1})^{\vee}$. In the notation of Corollary 2.5, let $\widetilde{W}_r \subset \mathfrak{P}^r$ be the subset of elements that map to W_r modulo $\mathfrak{P}^{r+1}$. We define $Z = (\mathfrak{g}/\widetilde{W}_r)^{\vee}$. Note that in the case $r = 0$, $\mathfrak{V} = (\mathfrak{g}/\mathfrak{g}^1)_{\mathrm{reg}}^{\vee} \cong (\mathfrak{t}^{\flat})' = \mathcal{A}(T, 0)$ and $Z \cong \mathfrak{t}^{\flat}$.

Lemma 3.8. *There is an isomorphism $\mathfrak{V} \times_U \mathrm{GL}_n(\mathbb{C}) \cong \widetilde{\mathcal{M}}(T, r)$. Furthermore, there are open dense inclusions $(\mathfrak{P}/\mathfrak{P}^{r+1})_{\mathrm{reg}}^{\vee} \hookrightarrow (\mathfrak{P}/\widetilde{W}_r)^{\vee}$, $\mathfrak{V} \hookrightarrow Z$, and $(\mathfrak{P}^1/\mathfrak{P}^{r+1})_{\mathrm{reg}}^{\vee} \hookrightarrow (\mathfrak{P}^1/\widetilde{W}_r)^{\vee}$ in the case $r \geq 1$.*

Proof. The first isomorphism is given by the map $(v, g) \mapsto (Ug, \mathrm{Ad}^*(g^{-1})v)$. Next, we observe that $(\mathfrak{P}/\mathfrak{P}^{r+1})_{\mathrm{reg}}^{\vee}$ may be identified with an open subset of $(\mathfrak{t}^{-r} + \mathfrak{P}^{-r+1})/\mathfrak{P}^1$. By part (4) of Proposition 2.4, $\widetilde{W}_r = (\mathfrak{t}^{-r} + \mathfrak{P}^{-r+1})^{\perp}$. It follows that $(\mathfrak{t}^{-r} + \mathfrak{P}^{-r+1})/\mathfrak{P}^1 \cong (\mathfrak{P}/\widetilde{W}_r)^{\vee}$. The other inclusions are proved similarly. $\square$

Proposition 3.9. *There is a smooth map $\Gamma : \widetilde{\mathcal{M}}(T, r) \to \mathcal{A}(T, r)$ which assigns to (Ug, α) the unique normalized formal type in the P^1-orbit of $\pi_{\mathfrak{P}}(\mathrm{Ad}(g)(\alpha))$. The fiber $\Gamma^{-1}(\tilde{A})$ is isomorphic to $\widetilde{\mathcal{M}}(A)$.*

Proof. Using the description of $\widetilde{\mathcal{M}}(T, r)$ in Lemma 3.8, we will construct a smooth map $\zeta : \mathfrak{V} \times \mathrm{GL}_n(\mathbb{C}) \to \mathcal{A}(T, r)$ and then show that it U-equivariant.

First, we show that

$$(3.1) \qquad\qquad (\mathfrak{P}/\mathfrak{P}^{r+1})_{\mathrm{reg}}^{\vee} \cong \mathcal{A}(T, r) \times_{T \cap P^1} P^1/P^{r+1},$$

where $T \cap P^1$ acts on the right factor by left translation, and on the left factor by the coadjoint action. There is a natural map $\mathcal{A}(T, r) \times_{T \cap P^1} P^1/P^{r+1} \to (\mathfrak{P}/\mathfrak{P}^{r+1})^\vee_{\mathrm{reg}}$ given by $(Y, \bar{p}) \mapsto \mathrm{Ad}^*(p^{-1})Y$. The inverse map takes a regular functional γ to the class of $(A, \bar{p})$, where $\tilde{A}$ is the unique normalized formal type in the P^1-orbit of $\tilde{A}$ and $\tilde{A} = \mathrm{Ad}^*(p)\gamma$.

We now define $\zeta' : (\mathfrak{P}/\mathfrak{P}^{r+1})^\vee_{\mathrm{reg}} \to \mathcal{A}(T, r)$ as the projection onto the left factor of $\mathcal{A}(T, r) \times_{T \cap P^1} P^1/P^{r+1}$; this makes sense since the coadjoint action of $T \cap P^1$ on $\mathcal{A}(T, r)$ is trivial by Proposition 2.14. In particular, it is clear that $\zeta'(\mathrm{Ad}^*(u)X) = X$ for any $u \in U$. It follows that the map $\zeta(v, g) = \zeta'(\pi_{\mathfrak{P}}(v))$ is U-equivariant, where U acts trivially on $\mathcal{A}(T, r)$. We define Γ to be the map induced by ζ on $\mathfrak{V} \times_U \mathrm{GL}_n(\mathbb{C})$.

Finally, we see that $\widetilde{\mathcal{M}}(A)$ embeds into $\Gamma^{-1}(\tilde{A})$ by comparing Definitions 3.3 and 3.7. Moreover, if $(Ug, \alpha) \in \Gamma^{-1}(\tilde{A})$, then $\pi_{\mathfrak{P}^1}(\mathrm{Ad}^*(g)(\alpha)) \in \mathcal{O}^1$. Therefore, $\widetilde{\mathcal{M}}(A) \cong \Gamma^{-1}(\tilde{A})$.

$\square$

Before proceeding, we need to recall some facts about Poisson reduction. Recall that if a Lie group G acts on a Poisson manifold M via a canonical Poisson action, then there is a corresponding moment map $\mu_M : M \to \mathfrak{g}^\vee$. The following result appears in [12].

Proposition 3.10 ([12, Examples 3.B, 3.F]). *Let M be a Poisson manifold, and suppose that G is a linear algebraic group with a free canonical Poisson action on M. If 0 is a regular value for μ_M, then the Poisson structure on M induces a Poisson structure on $M \mathbin{/\!\!/}_0 G \stackrel{\mathrm{def}}{=} \mu^{-1}(0)/G$ called the Poisson reduction of M. If the symplectic leaves of M intersect the G-orbits cleanly (in the terminology of [7, II.25, p. 180]), then the symplectic leaves of $M \mathbin{/\!\!/}_0 G$ are the connected components of the symplectic reductions of those symplectic leaves of M that intersect $\mu^{-1}(0)$.*

Lemma 3.11. *When $r \geq 1$, $(\mathfrak{P}^1/\mathfrak{P}^{r+1})^\vee_{\mathrm{reg}}$ has a natural Poisson structure.*

Proof. By Lemma 3.8, $(\mathfrak{P}^1/\mathfrak{P}^{r+1})^\vee_{\mathrm{reg}}$ is naturally isomorphic to an open dense subset of $(\mathfrak{P}^1/\widetilde{W}_r)^\vee$. It suffices to show that $(\mathfrak{P}^1/\widetilde{W}_r)^\vee$ is a Poisson manifold. Since $[\mathfrak{P}^1, \widetilde{W}_r] \subset \mathfrak{P}^{r+1}$, and the Poisson bracket restricted to linear functions on $(\mathfrak{P}^1/\mathfrak{P}^{r+1})^\vee$ is just the usual Lie bracket on $\mathfrak{P}^1/\mathfrak{P}^{r+1}$, this implies that the ideal generated by $W_r \subset \mathfrak{P}^1/\mathfrak{P}^{r+1}$ is a Poisson ideal. It follows that $(\mathfrak{P}^1/\widetilde{W}_r)^\vee \cong W_r^\perp \subset (\mathfrak{P}^1/\mathfrak{P}^{r+1})^\vee$ is a Poisson space. Therefore $(\mathfrak{P}^1/\mathfrak{P}^{r+1})^\vee_{\mathrm{reg}}$ is Poisson. $\square$

Proposition 3.12. *The manifold $\widetilde{\mathcal{M}}(T, r)$ has a Poisson structure. When $r \geq 1$, the manifold $\widetilde{\mathcal{M}}(T, r)$ is isomorphic to a Poisson reduction:*

$$\widetilde{\mathcal{M}}(T, r) \cong ((\mathfrak{P}^1/\mathfrak{P}^{r+1})^\vee_{reg} \times T^* \mathrm{GL}_n(\mathbb{C})) \mathbin{/\!\!/}_0 U.$$

The symplectic leaves of $\widetilde{\mathcal{M}}(T, r)$ are the fibers of the map Γ.

Proof. In the case $r = 0$, $\widetilde{\mathcal{M}}(T, r) \cong \widetilde{\mathcal{M}}(A)$ by the remark following Definition 3.7. Therefore, $\widetilde{\mathcal{M}}(T, r)$ is in fact symplectic.

We now suppose that $r \geq 1$. The space $(\mathfrak{P}^1/\mathfrak{P}^{r+1})^\vee_{reg} \times T^* \mathrm{GL}_n(\mathbb{C}))$ is a Poisson manifold using the Poisson structure of Lemma 3.11 on the first factor and the usual symplectic structure of a cotangent bundle on the second.

The group U acts on $(\mathfrak{P}^1/\mathfrak{P}^{r+1})^\vee_{\mathrm{reg}}$ and $T^* \mathrm{GL}_n(\mathbb{C})$ via the coadjoint action and the free action induced by left multiplication on $\mathrm{GL}_n(\mathbb{C})$ respectively; these actions are canonical Poisson.

The moment map of the diagonal action is given by

$$(3.2) \qquad \tilde{\mu}(Y, (g, X)) = -\mathrm{Ad}^*(g)(X)|_\mathfrak{u} + Y|_\mathfrak{u}.$$

It is clear that $\tilde{\mu}$ is a submersion.

If $\tilde{\mu}(Y, (g, X)) = 0$, then $\mathrm{Ad}^*(g)(X)|_\mathfrak{u} = Y|_\mathfrak{u}$. Therefore, we may glue $\mathrm{Ad}^*(g)(X)$ and Y together to obtain a functional $\psi_{Y,X} \in (\mathfrak{g}/\mathfrak{P}^{r+1})^\vee$; note that $\psi_{Y,X} \in \mathfrak{V}$ if and only if $Y \in (\mathfrak{P}^1/\mathfrak{P}^{r+1})^\vee_{reg}$. Using the description of $\widetilde{\mathcal{M}}(T, r)$ in Lemma 3.8, we define a map $p : \tilde{\mu}^{-1}(0) \to \widetilde{\mathcal{M}}(T, r)$ by

$$(Y, (g, X)) \mapsto (\psi_{Y,X}, g) \in (\mathfrak{g}/\mathfrak{P}^{r+1})^\vee \times_U \mathrm{GL}_n(\mathbb{C}).$$

The map is surjective, and the fibers of p are U-orbits.

The symplectic leaves of $(\mathfrak{P}^1/\mathfrak{P}^{r+1})^\vee_{reg} \times T^* \mathrm{GL}_n(\mathbb{C})$ are given by $\mathcal{O} \times T^* \mathrm{GL}_n(\mathbb{C})$, where $\mathcal{O}$ is any coadjoint orbit in $(\mathfrak{P}^1/\mathfrak{P}^{r+1})^\vee_{reg}$. It is obvious that the U-orbits intersect the leaves cleanly. It now follows from Definition 3.3 and the fact that the $\widetilde{\mathcal{M}}(A)$ are connected that the symplectic leaves of $\widetilde{\mathcal{M}}(T, r)$ are given by $\widetilde{\mathcal{M}}(A)$ for A a formal type corresponding to P and r.

$\square$

Lemma 3.13. *The $\mathrm{GL}_n(\mathbb{C})$-action on $\widetilde{\mathcal{M}}(T, r)$ defined by $h(Ug, \alpha) = (Ugh^{-1}, \mathrm{Ad}^*(h)(\alpha))$ is free canonical Poisson with submersive moment map $\mu(Ug, \alpha) = \mathrm{res}(\alpha)$.*

Proof. By [3, Lemma 5.13], this action restricts to a free action on each symplectic leaf $\widetilde{\mathcal{M}}(A)$. To see that it is canonical Poisson, first observe that the $\mathrm{GL}_n(\mathbb{C})$-action on $(\mathfrak{P}^1/\mathfrak{P}^{r+1})^\vee_{reg} \times T^* \mathrm{GL}_n(\mathbb{C})$ given by $h \cdot (Y, (g, X)) = (Y, gh^{-1}, \mathrm{Ad}^*(h)X)$ is canonical Poisson with moment map $(Y, (g, X)) \mapsto X$. Since it commutes with the action of U, it induces a canonical Poisson action on the Poisson reduction with moment map given by the same formula. It is easy to check that this action corresponds to the given action on $\widetilde{\mathcal{M}}(T, r)$ under the isomorphism of Proposition 3.12. Since $(Y, (g, X))$ corresponds to $(\psi_{Y,X}, g)$ and $\mathrm{res}(\psi_{Y,X}) = X$, we obtain the desired expression for the moment map. Finally, by [3, Lemma 5.15], μ is even a submersion when restricted to any symplectic leaf. $\square$

We are now ready to construct the moduli space of framed connections on $\mathbb{P}^1(\mathbb{C})$ with fixed combinatorics. Recall that $\mathbf{x}$ is a finite set of points in $\mathbb{P}^1(\mathbb{C})$ indexed by I. Let $\mathbf{T} = \{T_i\}_{i \in I}$ be a collection of uniform tori with splitting fields of degree e_i such that $T_i(\mathfrak{o}_i) \subset \mathrm{GL}_n(\mathfrak{o}_i)$. We write $\mathbf{P} = \{P_{T_i, \mathfrak{o}_i^n}\}$, and let $\mathbf{r} = (r_i)_{i \in I}$ with $r_i \geq 0$ and $\gcd(r_i, e_i) = 1$.

It is immediate from Lemma 3.13 that the diagonal action of $\mathrm{GL}_n(\mathbb{C})$ on $\prod_{i \in I} \widetilde{\mathcal{M}}(T_i, r_i)$ is free canonical Poisson and that its moment map $\mu = \sum_{i \in I} \mathrm{res}_i$ is a submersion.

Definition 3.14. Define $\widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$ as the Poisson reduction

$$\widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r}) = \prod_{i \in I} \widetilde{\mathcal{M}}(T_i, r_i) /\!/_0 \mathrm{GL}_n(\mathbb{C}).$$

We also set $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r}) = \mu^{-1}(0)$ and $\mathcal{A}(\mathbf{x}, \mathbf{T}, \mathbf{r}) = \prod_{i \in I} \mathcal{A}(T_i, r_i)$.

Proposition 3.15. *There is a smooth map $\mathbf{\Gamma} : \widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r}) \to \mathcal{A}(\mathbf{x}, \mathbf{T}, \mathbf{r})$ which assigns a normalized formal type $\tilde{A}_i$ to ∇ at each pole x_i. The fiber $\mathbf{\Gamma}^{-1}(\tilde{\mathbf{A}})$ is isomorphic to $\widetilde{\mathcal{M}}(\mathbf{A})$, and the symplectic leaves of $\widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$ are the connected components of these fibers.*

Proof. Since the maps $\Gamma_i : \widetilde{\mathcal{M}}(T_i, r_i) \to \mathcal{A}(T_i, r_i)$ are $\mathrm{GL}_n(\mathbb{C})$-equivariant (where $\mathrm{GL}_n(\mathbb{C})$ acts trivially on $\mathcal{A}(T_i, r_i)$), $\prod_{i \in I} \Gamma_i$ induces the desired map $\mathbf{\Gamma}$. The statement about the fibers of $\mathbf{\Gamma}$ follows from Proposition 3.9 and the construction of $\widetilde{\mathcal{M}}(\mathbf{A})$ in Theorem 3.6. Finally, the symplectic leaves of $\prod_i \widetilde{\mathcal{M}}(T_i, r_i)$ are given by $\prod_i \widetilde{\mathcal{M}}(A_i)$, and they intersect the $\mathrm{GL}_n(\mathbb{C})$-orbits cleanly. By Proposition 3.10, the symplectic leaves of $\widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$ are the connected components of $\widetilde{\mathcal{M}}(\mathbf{A})$. $\qquad\square$

Theorem 3.16. *The Poisson manifold $\widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$ is isomorphic to the moduli space of isomorphism classes of triples $(V, \nabla, \mathbf{g})$, where $(V, \nabla, \mathbf{g})$ satisfies the first two conditions of Definition 3.2, and (V_i, ∇_i) contains a regular stratum of the form (P_i, r_i, β). The manifold $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$ is isomorphic to the moduli space of isomorphism classes $(V, \nabla, \mathbf{g})$ satisfying the conditions above, with a fixed global trivialization ϕ.*

Proof. At each singular point, $V \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathfrak{o}_i$ determines a fixed lattice in V_i. By Proposition 2.11, we may associate a unique formal type A to every formal connection that contains a regular stratum (P, r, β). Therefore, if $(V, \nabla, \mathbf{g})$ satisfies the conditions above, there is a unique element $\mathbf{A} \in \mathcal{A}(T_i, r_i)$ given by the formal type of $(V, \nabla, \mathbf{g})$ at each singular point. In particular, by Theorem 3.6, $(V, \nabla, \mathbf{g})$ corresponds to a unique point in $\widetilde{\mathcal{M}}(\mathbf{A})$. However, by Proposition 3.15, $\widetilde{\mathcal{M}}(\mathbf{A}) \cong \mathbf{\Gamma}^{-1}(\mathbf{A})$. On the other hand, every point $p \in \widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$ corresponds to a unique connection with formal type $\mathbf{\Gamma}(p)$.

Now, suppose $m = (U_i g_i, \alpha_i)_{i \in I} \in \widehat{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$. Fix a global form $\nu \in \Omega^1_{\mathbb{P}^1}$. By (2.1), we may associate to α_i a unique meromorphic form $\alpha_{i\nu}\nu$ with coefficients in $\mathfrak{gl}_n(F_i)$. Since $\mu(m) = 0$, $\sum_{i \in I} \mathrm{Res}_i(\alpha_{i\nu}\nu) = 0$. It follows that q determines a global form $\alpha\nu$, and thus a global meromorphic connection $\nabla = d + \alpha\nu$ on the trivial rank n vector bundle over $\mathbb{P}^1$. It is easily checked that this gives a bijection between points in $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$ and triples $(V, \nabla, \mathbf{g})$.

$\qquad\square$

4. Integrable deformations

Let $X = \mathbb{P}^1(\mathbb{C})$, and let V be an n-dimensional trivial vector bundle on X. In this section, we will consider the deformations of a connection (V, ∇) which contains a regular stratum at each singularity.

4.1. **Formal Deformations.** Without loss of generality, take $x_1 = 0$ and fix a parameter z at 0. We will suppress the subscripts on (P_i, r_i, β_i), etc. when we work locally at 0. Let F be the field of Laurent series at 0 and $\mathfrak{o} \subset F$ the ring of power series. Define $\hat{V}$ to be the formal completion of V at 0, and let $\hat{\nabla}$ be the induced formal connection.

Let $D = \mathrm{Spec}(\mathfrak{o})$. Fix a standard uniform torus $T \subset \mathrm{GL}_n(F)$ such that $T(\mathfrak{o}) \subset \mathrm{GL}_n(\mathfrak{o})$ and an integer r with $\gcd(r, e) = 1$. We also set $P = P_{T, \mathfrak{o}^n}$. Let Δ be an analytic polydisk; we denote its ring of functions by R. A *formal flat deformation* is a flat, meromorphic connection $(\bar{V}, \bar{\nabla})$ on $D \times \Delta$ satisfying the following properties:

- the vector bundle $\bar{V}$ is isomorphic to the trivial rank n vector bundle, and
- the restriction of $\bar{\nabla}$ to $D \times \{y\}$, denoted by $\hat{\nabla}_y$, contains a regular stratum (P_y, r, β_y).

Fix a trivialization ϕ of $\bar{V}$ so that we may identify all other trivializations with elements of $\mathrm{GL}_n(\mathfrak{o} \otimes R)$. We say that the deformation $(\bar{V}, \bar{\nabla})$ is *framed* if there exists a trivialization $g \in \mathrm{GL}_n(R)$ with the property that $P_y = P^{g(y)}$, $\beta_y = \beta^{g(y)}$, and the regular stratum (P_y, r, β_y) is centralized by $T_y := T^{g(y)}$. In particular, any representative $(\beta_y)_\nu \in \mathfrak{P}_y^{-r}$ for β_y lies in $\mathfrak{t}_y + \mathfrak{P}_y^{-r+1}$ by [3, Remark 3.6] (with $\mathrm{ord}(\nu) = -1$).

We denote $\mathfrak{P}_\Delta^\ell = \mathfrak{P}^\ell \otimes R$, $\bar{\mathfrak{P}}_\Delta^\ell = \mathfrak{P}_\Delta^\ell / \mathfrak{P}_\Delta^{\ell+1}$, and $\mathfrak{t}_\Delta = \mathfrak{t} \otimes R$. We define P_Δ and T_Δ similarly. Suppose that $(\bar{V}, \bar{\nabla})$ is framed by g, and fix a nonzero one-form ν at 0. If $\nu = u \frac{dz}{z}$, write $\frac{z\nu}{dz} = u$. Let $A(y)$ be the formal type of $\bar{\nabla}$ at y. Using the pairing in (2.1), we may choose a representative $A_\nu(y)$ for $A(y)$ of the form

$$A_\nu(y) = \left(\frac{-r}{n} t_{-r}(y) \varpi_T^{-r} + \ldots + \frac{-1}{n} t_{-1}(y) \varpi_T^{-1} + t_0(y) \right) \frac{dz}{z\nu},$$

with $t_i(y) \in \mathfrak{t}_\Delta^\flat$. For example, $A_{\frac{dz}{z}} = \left(\frac{-r}{n} t_{-r}(y) \varpi_T^{-r} + \ldots + \frac{-1}{n} t_{-1}(y) \varpi_T^{-1} + t_0(y) \right)$.

Recall that any element of $\mathfrak{t}$ can be written as a Laurent series $t = \sum_{i=-N}^\infty t_j \varpi_T^j$ with $t_i \in \mathfrak{t}^\flat$. We define an endomorphism δ_e of $\mathfrak{t}$ via $\delta_e(t) = \sum_{i=-N}^\infty \frac{i}{e} t_i \varpi_T^i$.

Lemma 4.1. *Suppose $t \in \mathfrak{t}_\Delta$. Then, $z\partial_z t - [t, H_T] = \delta_e t$. Moreover, any solution $B \in \Omega_\Delta^1((\mathfrak{t} + \mathfrak{P}^\ell)/\mathfrak{P}^\ell)$ to the differential equation $z\partial_z B - [B, H_T] = \delta_e t + t_0 + \mathfrak{P}^\ell$ has the form $B = t + f + \mathfrak{P}^\ell$, where $f \in \mathfrak{t}_\Delta^\flat$. There is no solution when $\ell \geq 0$ and $t_0 \neq 0$.*

Proof. Since the equations are block diagonal in $\mathfrak{t}$, we may immediately reduce to the case where $T = E^\times$. A direct calculation shows that $z\partial_z \varpi_T^i - [\varpi_T, H_E] = \frac{i}{e} \varpi_T^i = \delta_e(\varpi_T^i)$. This proves the first statement. The second follows by applying the same calculation to each term of B up to ϖ_T^ℓ. Note that $z\partial_z \varpi_T^0 - [\varpi_T^0, H_E] = 0$, so there is no solution when $t_0 \neq 0$. $\qquad\square$

As in the previous section, we will let $p \cdot [\bar{\nabla}]_g$ denote the d_z part of the gauge transformation formula: $p \cdot [\bar{\nabla}]_g = \mathrm{Ad}(p)([\bar{\nabla}]_g) - (d_z p) p^{-1}$. Similarly, upon fixing $\nu \in \Omega_F^1(R)$, we write $p \cdot M = \mathrm{Ad}(p)(M) - (\tau_\nu p) p^{-1}$ when $M \in \mathfrak{gl}_n(F \otimes_\mathbb{C} R)$. We will use d_Δ and $\bar{d}$ to denote the exterior differential on Δ and $\mathbb{P}^1 \times \Delta$, respectively. Suppose that $\tilde{A}_\nu$ is the normalized formal type associated to A_ν. By Proposition 2.13, there exists an element $p \in P_\Delta$ such that

$$p(y) \cdot [\hat{\nabla}_y]_g = \tilde{A}_\nu(y) \nu.$$

Let d_Δ be the exterior differential on Δ. Define

$$A_\Delta(y) = \sum_{i=-r}^{-1} \varpi_T^{-i} d_\Delta t_i,$$

so that $d_\Delta A_{\frac{dz}{z}} = \delta_e A_\Delta + d_\Delta t_0$.

Proposition 4.2. *Let p be as above. Then, $p \cdot [\bar{\nabla}]_g - (d_\Delta p) p^{-1} = \tilde{A}_\nu \nu + A_\Delta + f$, where $f \in \Omega_\Delta^1(\mathfrak{t}^\flat)$ is closed. Moreover, $t_0(y)$ must be constant. Conversely, if $q \in \mathrm{GL}_n(F \otimes R)$, f is closed, and $t_0(y)$ is constant, then $[\bar{\nabla}'] = q^{-1} \cdot \left(\tilde{A}_\nu \nu + A_\Delta + f \right) + q^{-1}(d_\Delta q)$ determines a flat meromorphic connection $\bar{\nabla}'$ on $\bar{V}$.*

Remark 4.3. The case $e = 1$ is proved in the Appendix of [2].

Proof. Note that the connection determined by $p \cdot [\bar{\nabla}]_g - (d_\Delta p)p$ is flat, since $\bar{\nabla}$ is flat and $-(d_\Delta p)p$ is simply the d_Δ part of the gauge transformation formula. Conversely, if the connection determined by $\tilde{A}_\nu \nu + A_\Delta + f$ is flat, then $\bar{\nabla}'$ is flat by the same argument.

Without loss of generality, set $\nu = \frac{dz}{z}$. It suffices to show that whenever $[\bar{\nabla}] = \tilde{A}_{\frac{dz}{z}} \frac{dz}{z} + B$ for some $B \in \Omega_\Delta^1(\mathfrak{gl}_n(F))$, then $\bar{\nabla}$ is flat if and only if B has the form $A_\Delta + f$. We observe that $\bar{\nabla}$ is flat if and only if it satisfies the conditions

(4.1) $$d_\Delta(A_{\frac{dz}{z}}) - z\partial_z B + [B, A_{\frac{dz}{z}} + H_T] = 0$$

(4.2) $$\text{and } d_\Delta B + B \wedge B = 0.$$

If $B = A_\Delta + f$, (4.2) holds trivially while (4.1) follows from the first part of Lemma 4.1, so $\bar{\nabla}$ is flat.

We now prove the converse. If $r = 0$, then $e = 1$ and $H_T = 0$. In this case, we may take $A_{\frac{dz}{z}} = t_0(y)$ to be a regular diagonal matrix with entries in R; moreover, no two eigenvalues of $A_{\frac{dz}{z}}$ differ by an integer. Setting $B = \sum B_\ell z^\ell$ with $B_\ell \in \mathfrak{gl}_n(\mathbb{C})$, (4.2) reduces to $[B_\ell, A_{\frac{dz}{z}}] = \ell B_\ell$ when $\ell \neq 0$. The eigenvalue condition now implies that $B_\ell = 0$. whenever $\ell \neq 0$, by the condition on the eigenvalues of $A_{\frac{dz}{z}}$. On the other hand, $[B_0, A_{\frac{dz}{z}}] = -d_\Delta(A_{\frac{dz}{z}})$. Since the right hand side is a diagonal matrix, both sides must be identically 0. It follows that $d_\Delta(A_{\frac{dz}{z}}) = 0$ and $B_0 \in \Omega_\Delta^1(\mathfrak{t}^\flat)$ while the fact that B_0 is closed follows from (4.2).

We now consider the case $r \geq 1$. In the following, let $t_0(y)$ be the constant term of $A_{\frac{dz}{z}}$. Suppose, by induction, that $B \in A_\Delta + f + \Omega_\Delta^1(\mathfrak{t}^\flat + \mathfrak{P}^\ell)$. (Note that $B \in \Omega^1(\mathfrak{P}^L)$ for some $L \leq -r$, so the first inductive step is trivially satisfied for $\ell = L$). Applying the first part of Lemma 4.1 with $t = A_\Delta$ gives

$$d_\Delta(A_{\frac{dz}{z}}) + [B, H_T] - z\partial_z B \in d_\Delta t_0 + \Omega_\Delta^1(\mathfrak{P}^\ell).$$

We deduce from (4.1) that $[B, A_{\frac{dz}{z}}] \in d_\Delta t_0 + \Omega_\Delta^1(\mathfrak{P}^\ell)$.

When $\ell < 1$, $d_\Delta t_0 \in \Omega_\Delta^1(\mathfrak{P}^\ell)$. This implies that $[B, A_{\frac{dz}{z}}] \in \Omega_\Delta^1(\mathfrak{P}^\ell)$, so $B \in \Omega_\Delta^1(\mathfrak{t} + \mathfrak{P}^{\ell+r})$. Next, consider $\ell = 1$. By the $\ell = 0$ step, we know that $B \in \Omega_\Delta^1(\mathfrak{t} + \mathfrak{P}^r)$. Part (3) of Proposition 2.4 shows that $[B, A_{\frac{dz}{z}}] + \Omega_\Delta^1(\mathfrak{P}^1) \in \ker(\bar{\pi}_\mathfrak{t} \otimes I_{\Omega_\Delta^1})$. Since $d_\Delta t_0 \in \Omega_\Delta^1(\mathfrak{t}^\flat)$, part (1) of the same proposition gives $d_\Delta t_0 \in \Omega_\Delta^1(\mathfrak{P}^1)$. Since $\mathfrak{t}^\flat \cap \mathfrak{P}^1 = \{0\}$, we see that $d_\Delta t_0 = 0$. Thus, we may conclude that the inductive hypothesis implies that $[B, A_{\frac{dz}{z}}] \in \Omega_\Delta^1(\mathfrak{P}^\ell)$ for $\ell \geq 1$ as well. As before, $B \in \Omega_\Delta^1(\mathfrak{t} + \mathfrak{P}^{\ell+r}) \subset \Omega_\Delta^1(\mathfrak{t} + \mathfrak{P}^{\ell+1})$.

To complete the inductive step, we apply the second part of Lemma 4.1 to obtain $B \in A_\Delta + \Omega_\Delta^1(\mathfrak{t}^\flat + \mathfrak{P}^{\ell+1})$. Using the fact that the sum $\mathfrak{t}^\flat + \mathfrak{P}^\ell$ is direct for $\ell \geq 1$, we actually obtain $f \in \Omega_\Delta^1(\mathfrak{t}^\flat)$ such that $B \in A_\Delta + f + \Omega_\Delta^1(\mathfrak{P}^\ell)$ for all ℓ. The result now follows since $\bigcap_\ell \mathfrak{P}^\ell = 0$. $\square$

Definition 4.4. We say that a compatible framing g for $\bar{\nabla}$ is *good* if there exists $p \in P_\Delta^1$ such that $p \cdot [\bar{\nabla}]_g - (d_\Delta p)p^{-1} = \tilde{A}_\nu \nu + A_\Delta$.

Proposition 4.5. *Every framed flat deformation has a good compatible framing.*

Proof. If g is a compatible framing, there exists $p \in P$ such that $p \cdot ([\bar{\nabla}]_g) - (d_\Delta p)p^{-1} = (A_{\frac{dz}{z}} + H_T)\frac{dz}{z} + A_\Delta + f$ by Proposition 4.2. Since f is closed on Δ, it is exact. Choosing $\varphi \in \mathfrak{t}_\Delta^\flat$ such that $f = d_\Delta \varphi$, we obtain

$$\mathrm{Ad}(e^\varphi)\left(p \cdot ([\bar{\nabla}]_g) - d_\Delta(p)p^{-1}\right) - d_\Delta\varphi = (A_{\frac{dz}{z}} + H_T)\frac{dz}{z} + A_\Delta.$$

It follows that $e^\varphi g$ is a good compatible framing. $\qquad\square$

4.2. **Global Deformations.**

Definition 4.6. A *framed global deformation* is a triple $(\mathbf{g}, \bar{V}, \tilde{\nabla})$ consisting of:
   (1) a trivializable rank n vector bundle $\bar{V}$ on $\mathbb{P}^1 \times \Delta$;
   (2) an R-relative connection $\tilde{\nabla}$;
   (3) a collection of analytic framings $\mathbf{g} = (g_i)_{i \in I}$, $g_i : \Delta \to U_i \backslash \mathrm{GL}_n(\mathbb{C})$;
   (4) the restriction of $(\bar{V}, \tilde{\nabla})$ to $\mathbb{P}^1 \times \{y\}$ must lie in $\widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$ with compatible framing $\mathbf{g}(y)$.

We say that a framed deformation is *integrable* if there exists a flat $\mathbb{C}$-relative connection $\bar{\nabla}$ on $\mathbb{P}^1 \times \Delta$ with $\mathbb{P}^1$-part $\tilde{\nabla}$.

We note that $(\mathbf{g}, \bar{V}, \tilde{\nabla})$ determines a smooth map $\Delta \to \widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$. Specifically, there are maps $g_i(y)$ and $\alpha_i(y)$ such that the connection on the fiber above y corresponds to the point $(U_i g_i(y), \alpha_i(y))_{i \in I} \in \widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$.

Suppose that $(\mathbf{g}, \bar{V}, \tilde{\nabla})$ is an integrable framed global deformation. If we fix a trivialization for $\bar{V}$, we may write $[\bar{\nabla}] = \alpha\nu + \Upsilon$, where Υ is a section of $\Omega^1_{\Delta/\mathbb{C}}(\mathrm{End}(\bar{V}))$ with poles along $\{x_i\}$. The curvature of $\bar{\nabla}$ is given by

$$\Xi(\alpha, \Upsilon) = \bar{d}(\alpha\nu + \Upsilon) + \Upsilon \wedge \alpha\nu + \alpha\nu \wedge \Upsilon + \Upsilon \wedge \Upsilon \in \Omega^2_{\Delta \times \mathbb{P}^1}.$$

Thus, $\bar{\nabla}$ is flat if and only if the following hold:

(4.3) $$\tau_\nu \Upsilon = d_\Delta \alpha + [\Upsilon, \alpha] \quad \text{and} \quad 0 = d_\Delta \Upsilon + \Upsilon \wedge \Upsilon.$$

An integrable deformation $(\mathbf{g}, \bar{V}, \tilde{\nabla})$ determines a flat formal deformation $(\hat{V}_i, \bar{\nabla}_i)$ at each singular point. Therefore, if $\tilde{A}_i(y)$ is the normalized formal type of $(\hat{V}_i, \bar{\nabla}_i)$ at $y \in \Delta$, Proposition 4.5 implies that there exists $p_i \in P_i^1$ such that $p_i g_i \cdot [\bar{\nabla}] - \mathrm{Ad}(p_i)(d_\Delta g_i g_i^{-1}) - (d_\Delta p_i)p_i^{-1} = \tilde{A}_{i,\nu}\nu + A_{i,\Delta}$. Since $p_i^{-1}d_\Delta p_i \in \Omega^1_\Delta(\mathfrak{P}^1)$ by [3, Lemma 4.4], we deduce

(4.4) $$\Upsilon \in \mathrm{Ad}(g_i^{-1}p_i^{-1})A_{i,\Delta} + g_i^{-1}d_\Delta g_i + \Omega^1_\Delta((\mathfrak{P}_i^1)^{g_i}).$$

Set $\mathbf{r_e} = (r_i')_{i \in I} := (\lceil \frac{r_i}{e_i} \rceil)_{i \in I}$, where $\lceil \frac{r_i}{e_i} \rceil$ is the integer ceiling of $\frac{r_i}{e_i}$. Let $D_\mathbf{r}$ be the divisor $\sum_{i \in I} r_i'[x_i]$ on $\mathbb{P}^1$.

Definition 4.7. Let $\mathfrak{gl}_{n\mathbb{P}^1}$ denote the trivial $\mathfrak{gl}_n$-bundle on $\mathbb{P}^1$, and let $\mathfrak{gl}_{n\mathbb{P}^1}^{\mathbf{r_e}} = \mathfrak{gl}_{n\mathbb{P}^1}(D_\mathbf{r})$ be the sheaf corresponding to the divisor $D_\mathbf{r}$

Note that sections of $\mathfrak{gl}_{n\mathbb{P}^1}^{\mathbf{r_e}}$ have poles of order at most r_i' at x_i.

Fix a set of parameters $(z_i)_{i \in I}$ at each singular point x_i with the property that each z_i has a pole at a fixed point x_0. Define a $\mathbb{C}$-linear map $\phi_i : \mathfrak{gl}_n(F_i) \to \mathfrak{gl}_n(\mathbb{C})$ by

(4.5) $$\phi_i(X) = \mathrm{Res}_{x_i}(X\frac{dz_i}{z_i}).$$

Thus, ϕ_i extracts the constant term of X at x_i with respect to z_i. This induces a map $\bar{\phi}_i : \mathfrak{gl}_n(F_i)/\mathfrak{P}_i^1 \to \mathfrak{gl}_n(\mathbb{C})/\mathfrak{u}_i$.

Next, we define a map ς which assigns a global section of $\mathfrak{gl}_{n\mathbb{P}^1}^{\mathbf{r_e}}$ to a collection of principal parts at $\mathbf{x}$. Given $X_i \in \mathfrak{g}_i^{-r_i'}/\mathfrak{g}_i$, let $\tilde{X}_i \in H^0(\mathbb{P}^1; \mathfrak{gl}_{n\mathbb{P}^1}^{\mathbf{r_e}})$ be the section corresponding to the unique lift of X_i to $\mathfrak{gl}_n(z_i^{-1}\mathbb{C}[z_i^{-1}])$.

Definition 4.8. The map $\varsigma : \prod_{i \in I} \mathfrak{g}_i^{-r_i'}/\mathfrak{g}_i \to H^0(\mathbb{P}^1; \mathfrak{gl}_{n\mathbb{P}^1}^{\mathbf{r_e}})$ is given by

$$(4.6) \qquad \varsigma((X_i)_{i \in I}) = \sum_{i \in I} \tilde{X}_i.$$

We will usually write X^0 for $\varsigma((X_i)_{i \in I})$.

Remark 4.9. This map commutes with the adjoint action of $\mathrm{GL}_n(\mathbb{C})$, i.e., $\mathrm{Ad}(g)(X^0) = (\mathrm{Ad}(g)X)^0$ for any $g \in \mathrm{GL}_n(\mathbb{C})$. Indeed, $\mathrm{Ad}(g)(\tilde{X}_i) = \widetilde{\mathrm{Ad}(g)X_i}$ for each i since $\mathrm{Ad}(g)$ stabilizes $\mathfrak{gl}_n(z_i^{-1}\mathbb{C}[z_i^{-1}])$.

We are now ready to describe a system of differential equations satisfied by an integrable deformation. Let $(\bar{V}, \bar{\nabla})$ be a deformation of (V, ∇) as in Definition 4.6, corresponding to a map $(g_i(y), \alpha_i(y))_{i \in I}$ from Δ to $\widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$.

Fix a uniformizer z_0 at x_0.

Lemma 4.10. *If Υ' has principal part Υ_i at each x_i and is holomorphic elsewhere, then $\Upsilon' - \Upsilon^0 = \Upsilon'|_{z_0=0} \in \Omega_\Delta^1(\mathfrak{gl}_n(\mathbb{C}))$. Here, $\Upsilon^0 = \varsigma((\Upsilon_i)_{i \in I})$ as above.*

Proof. Since Υ' and Υ^0 have the same principal parts at each singular point, $\Upsilon' - \Upsilon^0 = X \in \Omega_\Delta^1(\mathfrak{gl}_n(\mathbb{C}))$. By construction, Υ^0 and Υ' are holomorphic at x_0, and $\Upsilon^0|_{z_0=0} = 0$. Therefore, $X = \Upsilon'|_{z_0=0}$. $\square$

Lemma 4.11. *Given $\Upsilon \in \Omega_\Delta^1(H^0(\mathbb{P}^1; \mathfrak{gl}_{n\mathbb{P}^1}^{\mathbf{r_e}}))$, define $\Upsilon_i = \Upsilon + \mathfrak{g}_i$. Any integrable framed deformation $\bar{\nabla}$ is $\mathrm{GL}_n(R)$-gauge equivalent to a deformation $\alpha\nu + \Upsilon$ satisfying $\Upsilon = \varsigma((\Upsilon_i)_{i \in I})$.*

Proof. Suppose that $[\bar{\nabla}] = \alpha'\nu + \Upsilon'$. Then, $\Upsilon' - (\Upsilon')^0 = X \in \Omega_\Delta^1(\mathfrak{gl}_n(\mathbb{C}))$ by Lemma 4.10. Since the image of ς is closed under conjugating by $\mathrm{GL}_n(R)$, it suffices to show that there exists $g \in \mathrm{GL}_n(R)$ such that $\Upsilon := \mathrm{Ad}(g)\Upsilon' - (d_\Delta g)g^{-1} = \mathrm{Ad}(g)(\Upsilon')^0$.

The system of differential equations $g^{-1}(d_\Delta g) = X$ has a solution if $d_\Delta X + X \wedge X = 0$. To see this, recall that $d_\Delta X = d_\Delta \Upsilon'|_{z_0=0}$. By flatness, $d_\Delta \Upsilon' + \Upsilon' \wedge \Upsilon' = 0$, so $d_\Delta X + X \wedge X = (d_\Delta \Upsilon' + \Upsilon' \wedge \Upsilon')|_{z_0=0} = 0$. $\square$

Now, suppose that $(\mathbf{g}, \bar{V}, \tilde{\nabla})$ is a deformation. Choose $p_i \in P_\Delta^1$ such that $p_i \cdot \alpha\nu = \tilde{A}_{i\nu}\nu$, and write $\hat{g}_i = p_i g_i$. Let $\Upsilon_i = \mathrm{Ad}(\hat{g}_i^{-1})A_{i,\Delta} + \Omega_\Delta^1(\mathfrak{g}_i)$, with corresponding global section $\Upsilon^0 \in \Omega_\Delta^1(H^0(\mathbb{P}^1; \mathfrak{gl}_{n\mathbb{P}^1}^{\mathbf{r_e}}))$.

Theorem 4.12. *A good framed deformation $(\mathbf{g}, \bar{V}, \tilde{\nabla})$ is integrable if and only if it is $\mathrm{GL}_n(R)$-gauge-equivalent to a deformation satisfying the following equations:*

(1) $\phi_i(\mathrm{Ad}(p_i^{-1})(A_{i,\Delta}) + (d_\Delta g_i)g_i^{-1}) \in \phi_i(\mathrm{Ad}(g_i)\Upsilon^0) + \Omega_\Delta^1(\mathfrak{u}_i)$;
(2) $d_\Delta \alpha \in \tau_\nu \Upsilon^0 + [\alpha, \Upsilon^0] + \Omega_\Delta^1(\mathfrak{g}_i \frac{dz_i}{z_i\nu})$; *and*
(3) $d\Upsilon^0 + \Upsilon^0 \wedge \Upsilon^0 = 0$.

Proof. Suppose that $(\mathbf{g}, \bar{V}, \tilde{\nabla})$ is integrable. By Lemma 4.11, there exists a gauge $g \in \mathrm{GL}_n(R)$ such that $[\bar{\nabla}]_g = \alpha\nu + \Upsilon$ with $\Upsilon = \Upsilon^0$. By Proposition 4.2,

$$(4.7) \qquad \Upsilon \in \mathrm{Ad}(\hat{g}_i^{-1})A_{i,\Delta} + \hat{g}_i^{-1}d_\Delta\hat{g}_i + (\mathfrak{P}_{i,\Delta}^1)^{g_i}.$$

Therefore, condition (1) is satisfied by applying $\mathrm{Ad}(g_i)$ and ϕ_i. Moreover, $\alpha\nu + \Upsilon$ is flat, so conditions (2) and (3) follow from (4.3).

To see the converse, note that condition (1) shows that Υ^0 satisfies (4.7). Condition (2) implies that the cross-term of the curvature of $\alpha\nu + \Upsilon^0$ is zero modulo $\mathfrak{g}_i^1 \frac{dz_i}{z_i\nu}$. Writing this term as $\sum f_j dy_j$, where the dy_j's are a basis for Ω_Δ^1, we see that $f_j \in H^0(\mathbb{P}^1; \Omega_{\mathbb{P}^1}^1) = \{0\}$. Since the term in Ω_Δ^2 vanishes by condition (3), we see that the curvature vanishes.

$\square$

Corollary 4.13. *A good framed deformation* $(\mathbf{g}, \bar{V}, \tilde{\nabla})$ *is integrable if and only if there exists* $g \in \mathrm{GL}_n(R)$ *such that*

(1) $\phi_i(\mathrm{Ad}(g_i)(g^{-1}d_\Delta g + \Upsilon^0)) \in \phi_i(\mathrm{Ad}(p_i^{-1})A_{\Delta,i} + (d_\Delta g_i)g_i^{-1}) + \Omega_\Delta^1(\mathfrak{u}_i);$

(2) $d_\Delta\alpha \in \tau_\nu\Upsilon^0 + [\alpha, \Upsilon^0 + g^{-1}d_\Delta g] + \Omega_\Delta^1(\mathfrak{g}_i^1\frac{dz_i}{z_i\nu});$ *and*

(3) $d_\Delta(\Upsilon^0 + g^{-1}d_\Delta g) + (\Upsilon^0 + g^{-1}d_\Delta g) \wedge (\Upsilon^0 + g^{-1}d_\Delta g) = 0.$

Proof. Set $[\bar{\nabla}] = \alpha\nu + \Upsilon$ and choose $g \in \mathrm{GL}_n(R)$ such that $[\bar{\nabla}]_g = \alpha'\nu + \Upsilon'$ with $\Upsilon' = (\Upsilon')^0$. The proof above shows that $(\mathbf{g}', \alpha', \Upsilon')$ satisfy the conditions in the theorem. One obtains the equations in the corollary by substituting $\Upsilon' = \mathrm{Ad}(g)\Upsilon^0$, $\alpha' = \mathrm{Ad}(g)(\alpha)$, and $g_i' = g_i g^{-1}$. The converse is proved similarly. $\square$

Remark 4.14. It will be shown later in Theorem 6.8 that the third condition in these two results is unnecessary.

Remark 4.15. In [1, Theorem 5.3], Bertola and Mo have proved an equivalent version of Theorem 4.12 in the case where $r_i \equiv -1 \pmod{e_i}$ for all $i \in I$ (as well as other cases not considered in our paper). Their approach is significantly different; for example, there is no equivalent notion of a compatible framing. Instead, their analysis uses results from the perturbation theory of matrices. One of the advantages of our approach is that we are able to find explicit generators for the corresponding differential ideal on the moduli space $\widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$; see Section 5 below.

5. The Differential Ideal

In this section and the next, we show that the equations from Corollary 4.13 determine a Frobenius integrable system on $\widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$. Throughout, we will fix a global meromorphic one-form ν on $\mathbb{P}^1(\mathbb{C})$. We will also use the convention that if B is a vector bundle on a manifold M and $N \subset M$ is open, then we will write $\Omega_N^1(B)$ for $\Omega_N^1(B|_N)$. Furthermore, if σ is a global section of $\Omega_M^1(B)$, we will abuse notation and let σ also denote $\sigma|_N$.

5.1. **Generators.** We will construct a differential ideal $\mathcal{I}$ on $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$ corresponding to the system of differential equations in Corollary 4.13. Throughout, we will simplify notation by suppressing $\mathbf{x}$, $\mathbf{P}$, and $\mathbf{r}$ in the notation and using d for the exterior differential on all spaces whenever there is no risk of ambiguity. We will also write $\widetilde{\mathcal{M}}_i = \widetilde{\mathcal{M}}(T_i, r_i)$ and $\mathcal{A}_i = \mathcal{A}(T_i, r_i)$.

Set $\mathfrak{t}_i^j = \mathfrak{t}_i \cap \mathfrak{P}_i^j$ and $T_i^j = T_i \cap P_i^j$. First, we define maps $\hat{\Gamma}_i : \widehat{\mathcal{M}} \to \mathcal{A}_i$ and $\hat{\Gamma}_{i\nu} : \widehat{\mathcal{M}} \to (\mathfrak{t}_i^{-r_i} + H_T)\frac{dz_i}{z_i\nu}$: $\hat{\Gamma}_i((U_jg_j, \alpha_j)_{i\in I}) = \Gamma_i(U_ig_i, \alpha_i)$ and $\hat{\Gamma}_{i\nu}((U_jg_j, \alpha_j))$

is the standard representative of $\Gamma_i(U_i g_i, \alpha_i)$ in $\mathfrak{P}_i^{-r} \frac{dz_i}{z_i \nu}$ with Laurent expansion $(\gamma_{-r} \varpi_i^{-r} + \ldots + \gamma_0 \varpi_i^0 + H_T) \frac{dz_i}{z_i \nu}$. We also write $\hat{\Gamma}_i^0$ for the coefficient of $\varpi_i^0 \frac{dz_i}{z_i \nu}$ in $\hat{\Gamma}_{i\nu}$.

We identify the tangent bundle of $\mathcal{A}_i \subset (\mathfrak{t}_i / \mathfrak{t}_i^{r+1})^\vee$ with $\mathcal{A}_i \times \mathfrak{t}_i^{-r_i} / \mathfrak{t}_i^1$ using the pairing $\langle , \rangle_{\frac{dz_i}{z_i}}$. The differential of the map $\hat{\Gamma}_i : \widehat{\mathcal{M}} \to \mathcal{A}_i$ determines a section $d\hat{\Gamma}_i$ of $\Omega_{\widehat{\mathcal{M}}}^1 (\mathfrak{t}_i^{-r_i} / \mathfrak{t}_i^1) \subset \Omega_{\widehat{\mathcal{M}}}^1 (\mathfrak{P}_i^{-r_i} / \mathfrak{P}_i^1)$.

Identify $\widetilde{\mathcal{M}}_i$ with $\mathfrak{V}_j \times_{U_j} \mathrm{GL}_n(\mathbb{C})$ as in Lemma 3.8. Throughout this section, we will use $m_j = (v_j, g_j) \in \mathfrak{V}_j \times_{U_j} \mathrm{GL}_n(\mathbb{C})$ to denote a point in $\widetilde{\mathcal{M}}_i$ and $m = (m_j)_{j \in I}$ to denote a point in $\widehat{\mathcal{M}} \subset \prod_{i \in I} \widetilde{\mathcal{M}}_i$. Let $\psi_i : \widehat{\mathcal{M}} \to U_i \backslash \mathrm{GL}_n(\mathbb{C})$ be the composition of the projections $\widehat{\mathcal{M}} \to \widetilde{\mathcal{M}}_i$ and $\tilde{\psi}_i : \widetilde{\mathcal{M}}_i \to U_i \backslash \mathrm{GL}_n(\mathbb{C})$.

We are now ready to define a collection of differential forms on $\widehat{\mathcal{M}}$.

(1) Define an endomorphism of $\mathfrak{t}_i$ by

$$\delta_{\mathrm{e}}'(t_{-j} \varpi_i^{-j}) = \begin{cases} \frac{-e}{j} t_{-j} \varpi_i^{-j} & \text{if } j \neq 0; \\ 0 & \text{if } j = 0. \end{cases}$$

Note that if $t = \sum_{i=-N}^\infty t_i \varpi_i$, then $\delta_e(\delta_{\mathrm{e}}'(t)) = t - t_0$. We obtain an induced map $\delta_{\mathrm{e}}' : \Omega_{\widehat{\mathcal{M}}}^*(\mathfrak{t}_i^{-r_i} / \mathfrak{t}_i^1) \to \Omega_{\widehat{\mathcal{M}}}^*(\mathfrak{t}_i^{-r_i} / \mathfrak{t}_i^1)$.

We shall clarify the dictionary between this notation and the notation in Section 4.1. Set $A_{\widetilde{\mathcal{M}}, i} = \delta_{\mathrm{e}}'(d\hat{\Gamma}_i) \in \Omega_{\widehat{\mathcal{M}}}^1(\mathfrak{t}_i^{-r_i} / \mathfrak{t}_i^1)$. If $\nu = \frac{dz_i}{z_i}$, the first part of Lemma 4.1 implies that

(5.1) $$\tau_\nu A_{\widetilde{\mathcal{M}}, i} - [A_{\widetilde{\mathcal{M}}, i}, \hat{\Gamma}_{i\nu}] = d\hat{\Gamma}_{i\nu} - d\hat{\Gamma}_i^0 + \mathfrak{t}_i^1.$$

Below, we will apply Proposition 4.2 several times with $A_{\widetilde{\mathcal{M}}, i}$ and $\hat{\Gamma}_{i\nu} - \hat{\Gamma}_i^0$ playing the roles of A_Δ and $\tilde{A}_\nu$ respectively.

(2) Recall, from (3.1), that $(\mathfrak{P}_i / \mathfrak{P}_i^{r_i+1})_{\mathrm{reg}}^\vee \cong \mathcal{A}_i \times_{T_i^1} P_i^1 / P_i^{r+1}$. Given $m_i \in \widetilde{\mathcal{M}}_i$, we may write $\pi_{\mathfrak{P}_i}(v_i) = (a_i, p_i) \in \mathcal{A}_i \times_{T^1} P_i^1 / P_i^{r_i+1}$. Thus, a_i is the image of v_i under the map Γ_i from Proposition 3.9, and p_i is characterized by $\mathrm{Ad}^*(p_i)(\pi_{\mathfrak{P}_i}(v_i)) = a_i$. Define a bundle $B_i = \mathfrak{P}_i^{-r} / \mathfrak{P}_i^1 \times_{U_i} \mathrm{GL}_n(\mathbb{C})$ over $U_i \backslash \mathrm{GL}_n(\mathbb{C})$, where U_i acts on $\mathrm{GL}_n(\mathbb{C})$ by left multiplication and on $\mathfrak{P}_i^{-r} / \mathfrak{P}_i^1$ by Ad. We define $\Phi_i \in \Omega_{\widehat{\mathcal{M}}}^1(\psi_i^*(B_i))$ by

(5.2) $$\Phi_i(m) = \mathrm{Ad}(p_i^{-1})(A_{\widetilde{\mathcal{M}}, i}).$$

Note that Φ_i is a well defined section of $\Omega_{\widehat{\mathcal{M}}}^1(\psi_i^*(B_i))$: if $u \in U_i$, then $u(a_i, p_i) = (a_i, p_i u^{-1})$; it follows that $\mathrm{Ad}(u)\Phi_i(um) = \Phi_i(m)$.

(3) We define a third form Υ^0 on $\widehat{\mathcal{M}}$ which has coefficients in $H^0(\mathbb{P}^1; \mathfrak{gl}_{n\mathbb{P}^1}^{\mathbf{r_e}})$. The form

$$\bar{\Phi}_i = \mathrm{Ad}(g_i^{-1})\Phi_i + \Omega_{\widehat{\mathcal{M}}}^1(\mathfrak{g}_i / \mathfrak{P}_i^1) \in \Omega_{\widehat{\mathcal{M}}}^1((\mathfrak{P}_i^{-r_i})^{g_i} / \mathfrak{g}_i)$$

is well defined on $\widehat{\mathcal{M}}$, so the product $\bar{\Phi} = (\bar{\Phi}_i)_{i \in I}$ determines a form in $\Omega_{\widehat{\mathcal{M}}}^1(\prod_{i \in I}((\mathfrak{P}_i^{-r_i})^{g_i} / \mathfrak{g}_i))$. Using Definition 4.8, we set $\Upsilon^0 = \varsigma(\bar{\Phi}) \in \Omega_{\widehat{\mathcal{M}}}^1(H^0(\mathbb{P}^1; \mathfrak{gl}_{n\mathbb{P}^1}^{\mathbf{r_e}}))$.

(4) The next form, Θ_i, is defined on $\widehat{\mathcal{M}}$ with coefficients in $\psi_i^*(T(U_i \backslash \mathrm{GL}_n(\mathbb{C})))$. Here, we identify $T(U_i \backslash \mathrm{GL}_n(\mathbb{C}))$ with $\mathfrak{u}_i \backslash \mathfrak{gl}_n(\mathbb{C}) \times_{U_i} \mathrm{GL}_n(\mathbb{C})$ in the usual

way. Define a section of $\Omega^1_{\widehat{\mathcal{M}}}(\psi_i^* T(U_i \backslash \mathrm{GL}_n(\mathbb{C})))$ by

$$\Theta_i(m) = (dg_i)g_i^{-1} + \bar{\phi}_i(\Phi_i) - \bar{\phi}_i(\mathrm{Ad}(g_i)(\Upsilon^0)) + \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{u}_i),$$

where $\bar{\phi}$ is the function defined in (4.5). The form Θ_i is a well defined. By construction, $\bar{\phi}_i(\Phi_i)$ is a well defined section of $\Omega^1_{\widehat{\mathcal{M}}}(\psi_i^* T(U_i \backslash \mathrm{GL}_n(\mathbb{C})))$. Furthermore, $\mathrm{Ad}(u_i)\bar{\phi}_i(\mathrm{Ad}(g_i)\Upsilon^0)(u_i m) = \bar{\phi}_i(\mathrm{Ad}(g_i)\Upsilon^0)(m)$ and

$$d(u_i g_i)g_i^{-1}u_i^{-1} \in \mathrm{Ad}(u_i)((dg_i)g_i^{-1}) + \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{u}_i).$$

(5) By Theorem 3.16, each point of $\widehat{\mathcal{M}}$ determines a unique form $\alpha\nu \in H^0(\mathbb{P}^1\backslash\mathbf{x}; \Omega^1_{\mathbb{P}^1}(\mathfrak{gl}_{n\mathbb{P}^1}))$. In particular, $d\alpha$ is a one-form with coefficients in $H^0(\mathbb{P}^1\backslash\mathbf{x}; \mathfrak{gl}_{n\mathbb{P}^1})$. Note that $d\alpha$ depends on the choice of global form ν.

(6) We define the last collection of forms on $\widehat{\mathcal{M}}$ to have coefficients in $(\mathfrak{g}_i^{-r'_i}\frac{dz_i}{z_i\nu})/(\mathfrak{g}_i^1\frac{dz_i}{z_i\nu})$, where $\mathbf{r_e} = (r'_i)_{i\in I}$ as in Definition 4.7. Let Ξ_i be defined on $\widehat{\mathcal{M}}$ by

$$\Xi_i((v_j, g_j)_{j\in I}) = \tau_\nu \Upsilon^0 - d\alpha - [\Upsilon^0, \alpha] + \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{g}_i^1 \frac{dz_i}{z_i\nu}).$$

Intuitively, we have a framed global deformation on any neighborhood of $\widehat{\mathcal{M}}$ isomorphic to an analytic polydisk determined by $\mathbf{g} = (g_i)_{i\in I}$ and $\tilde{\nabla} = d + \alpha\nu$ (see Definition 4.6). Here, Υ^0 represents the dy term of the connection (up to a section of $\Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{gl}_n(\mathbb{C}))$). The form Ξ_i corresponds to the expansion, in non-positive degrees of z_i, of the $dz \wedge dy$ term of the curvature. Moreover, the vanishing of Θ_i gives a relation on the framing g_i analogous to part (1) of Theorem 4.12.

5.2. **The Differential Ideal.** We are now ready to construct a differential ideal $\mathcal{I}$ on $\widehat{\mathcal{M}}$ corresponding to the isomonodromy equations in Section 4. Throughout, we fix sections $\hat{g}_i$ of the trivial $\mathrm{GL}_n(\mathfrak{o}_i)$-bundle over $\widehat{\mathcal{M}}$ such that $\hat{g}_i g_i^{-1} \in P_i^1$ and $\hat{g}_i \cdot \alpha = \hat{\Gamma}_{i\nu}$. We write $\hat{p}_i = \hat{g}_i g_i^{-1}$. Observe that $\mathrm{Ad}(g_i)(\hat{g}_i^{-1}d\hat{g}_i) \in (dg_i)g_i^{-1} + \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{P}_i^1)$ and $p_i = \hat{p}_i P_i^{r_i+1}$.

For convenience, we will take $A'_{\widetilde{\mathcal{M}},i} \in \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{t}_i^{-r})$ to be the lift of $A_{\widetilde{\mathcal{M}},i}$ which has zero components in positive degrees. By Lemma 4.1, this lift has the property

$$(5.3) \qquad\qquad \tau_\nu A'_{\widetilde{\mathcal{M}},i} + [\hat{\Gamma}_{i\nu}, A'_{\widetilde{\mathcal{M}},i}] = d\hat{\Gamma}_{i\nu} - d\hat{\Gamma}_i^0.$$

Moreover, $dA'_{\widetilde{\mathcal{M}},i} = 0$, since $A'_{\widetilde{\mathcal{M}},i} = \delta_{\mathrm{e}}'(d\hat{\Gamma}_{i\nu})$. The same holds for $A_{\widetilde{\mathcal{M}},i}$.

If B is a vector bundle over $\widehat{\mathcal{M}}$, $\sigma \in \Omega^*_{\widehat{\mathcal{M}}}(B)$ and $\mathcal{J}$ is a differential ideal on $\widehat{\mathcal{M}}$, we will use the shorthand "$\sigma \in \mathcal{J}$" to mean $\sigma \in \mathcal{I} \otimes_{\mathcal{O}_{\widehat{\mathcal{M}}}} B$. If B' is another bundle and $f : B \to B'$ is a bundle map, then $\sigma \in \mathcal{J}$ implies that $f(\sigma) \in \mathcal{J}$. In particular, this is the case if the bundles are trivial with f induced by a $\mathbb{C}$-linear map on their fibers. We also say that $\mathcal{J}$ is generated by σ if it is locally generated by the coefficients of σ in a trivialization of B.

Throughout, we may assume that $\nu = \frac{dz_i}{z_i}$ when working with Ξ_i and Θ_i. Let $\chi_i = \phi_i(\mathrm{Ad}(\hat{g}_i^{-1})A'_{\widetilde{\mathcal{M}},i} + \hat{g}_i^{-1}d\hat{g}_i - \Upsilon^0) \in \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{gl}_n(\mathbb{C}))$. Note that $\mathrm{Ad}(g_i)(\chi_i) + \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{u}_i)$ is a well-defined section of the bundle B_i from the previous section. Choose some $j \in I$. Let $\mathcal{I}$ be the differential ideal on $\widehat{\mathcal{M}}$ is generated by $\Xi_i - [\chi_j, \alpha]$, $\Theta_i - \mathrm{Ad}(g_i)(\chi_j)$, and $d\hat{\Gamma}_i^0$ for all $i \in I$. We will show late that this definition is independent of the choice of j.

Theorem 5.1.

(1) *The differential ideal $\mathcal{I}$ is a Pfaffian ideal which satisfies the Frobenius integrability condition $d\mathcal{I} \subset \mathcal{I}$ and therefore determines a foliation of $\widehat{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$. Moreover, any map $f : \Delta \to \widehat{\mathcal{M}}$ corresponds to a framed integrable deformation if and only if $f^*\mathcal{I} = 0$.*

(2) *The ideal $\mathcal{I}$ is the pullback of an integrable Pfaffian differential ideal $\overline{\mathcal{I}}$ on $\widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$, and any map $f : \Delta \to \widetilde{\mathcal{M}}$ corresponds to a framed integrable deformation if and only if $f^*\overline{\mathcal{I}} = 0$.*

We will prove this theorem in the subsequent two sections. The remainder of this section is devoted to a local analysis of the ideal $\mathcal{I}$ which will be useful later on.

Over a sufficiently small neighborhood $N_i \subset U_i \backslash \operatorname{GL}_n(\mathbb{C})$, we may choose a smooth algebraic slice of coset representatives $\sigma_i : N_i \to \operatorname{GL}_n(\mathbb{C})$. Let $\widetilde{N}_i \subset \operatorname{GL}_n(\mathbb{C})$ be the inverse image of N_i, and let $N = \bigcap_{i \in I} \psi_i^{-1}(N_i) \subset \widehat{\mathcal{M}}$.

For brevity, we will write $g_i^\sigma = \sigma_i(U_i g_i)$. When $\chi \in \mathfrak{gl}_n(\mathbb{C}) = T_{g_i^\sigma} \operatorname{GL}_n(\mathbb{C})$, we write $\chi^\sigma \in \mathfrak{gl}_n(\mathbb{C})$ for the image of χ under the tangent map corresponding to the composition $\widetilde{N}_i \to N_i \xrightarrow{\sigma_i} \widetilde{N}_i$. We also define $\Theta_i^\sigma \in \Omega_N^1(\mathfrak{gl}_n(\mathbb{C}))$ by

$$(5.4) \qquad \Theta_i^\sigma = (g_i^\sigma)^{-1} dg_i^\sigma + T\sigma_i(\bar{\phi}_i(\Phi_i) - \bar{\phi}_i(\operatorname{Ad}(g_i)(\Upsilon^0))).$$

Observe that $\operatorname{Ad}(g_i^\sigma)(\Theta_i^\sigma) + \Omega_N^1(\mathfrak{u}_i) = \Theta_i|_N$; indeed, $\Theta_i^\sigma = T\sigma_i(\Theta_i)$.

We define a map $\xi_i : \mathfrak{g}_i \to \Omega_N^1$ using the residue-trace pairing in (2.1):

$$(5.5) \qquad \xi_i(X) = \langle \operatorname{Ad}(g_i^\sigma)(\Xi_i - [\Theta_i^\sigma, \alpha]), X \rangle_\nu.$$

Lemma 5.2. *Let $\mathcal{J}$ be a differential ideal on $N \subset \widehat{\mathcal{M}}$, and let Ξ be a section of $\Omega_N^1(\mathfrak{P}^{-R}/\mathfrak{g}^1)$ for some integer $R > 0$. Define $\xi \in \Omega_N^1(\mathfrak{g}^\vee)$ by $\xi(X) = \langle \Xi, X \rangle_\nu$. Then $\xi(\mathfrak{g}) \subset \mathcal{J}$ if and only if $\Xi \in \mathcal{J}$. Furthermore, $\xi(\mathfrak{P}^s) \in \mathcal{J}$ if and only if $\Xi \in \mathcal{J} + \Omega_N^1(\mathfrak{P}^{-s+1})$*

Proof. By assumption, $\mathfrak{P}^{R+1} \subset \ker(\xi)$, so ξ induces an element $\bar{\xi} \in \operatorname{Hom}_{\mathbb{C}}(\mathfrak{g}/\mathfrak{P}^{R+1}, \Omega_N^1) \cong \Omega_N^1((\mathfrak{g}/\mathfrak{P}^{R+1})^\vee)$. Note that $\xi(\mathfrak{g}) \subset \mathcal{J}$ if and only $\bar{\xi} \in \mathcal{J}$. Moreover, $\bar{\xi}$ and Ξ correspond under the isomorphism $(\mathfrak{g}/\mathfrak{P}^{R+1})^\vee \cong \mathfrak{P}^{-R}/\mathfrak{g}^1$ given by the perfect pairing $\langle, \rangle_\nu$, so $\bar{\xi} \in \mathcal{J}$ if and only if $\Xi \in \mathcal{J}$. The first result now follows. A similar argument, replacing $\mathfrak{g}$ with $\mathfrak{P}^s$, proves the second statement. $\qquad \square$

We will write $\hat{\rho}$ for the action of $\operatorname{GL}_n(\mathbb{C})$ on $\prod_{i \in I} \widetilde{\mathcal{M}}_i$. (This is the product over the action of $\operatorname{GL}_n(\mathbb{C})$ on each factor $\widetilde{\mathcal{M}}(T, r)$ from Lemma 3.13.) Define $\operatorname{T}_{\mathfrak{gl}_n} \subset \prod_{i \in I} \operatorname{T}(\widetilde{\mathcal{M}}_i)$ to be the subbundle of $\operatorname{T}(\prod_{i \in I} \widetilde{\mathcal{M}}_i)$ consisting of vectors tangent to the $\operatorname{GL}_n(\mathbb{C})$-orbits. Since the action of $\operatorname{GL}_n(\mathbb{C})$ is free, this is an integrable subbundle. Moreover, the action map of $\hat{\rho}$ gives an isomorphism between the trivial $\mathfrak{gl}_n(\mathbb{C})$-bundle on $\prod_{i \in I} \widetilde{\mathcal{M}}_i$ and $\operatorname{T}_{\mathfrak{gl}_n}$.

We may identify the preimage $\tilde{\psi}_i^{-1}(N_i)$ of N_i in $\widetilde{\mathcal{M}}_i$ with $\mathfrak{V}_i \times N_i$ via

$$(v, g_i) \mapsto (\operatorname{Ad}(g_i^\sigma g_i^{-1})v, U_i g_i).$$

Thus, there is a local isomorphism

$$(5.6) \qquad \operatorname{T}(\tilde{\psi}_i^{-1}(N_i)) \cong \operatorname{T}(\mathfrak{V}_i) \times \operatorname{T}(N_i).$$

This equation induces a local bundle map $\operatorname{p}_i^{\mathrm{U}} : \operatorname{T}_{\mathfrak{gl}_n}|_N \to \operatorname{T}(\mathfrak{V}_i) \subset T(\prod_{i \in I} \tilde{\psi}^{-1}(N_i))|_N$. We also have a global bundle map $\operatorname{p}_i^{\mathrm{L}} : \operatorname{T}_{\mathfrak{gl}_n}|_N \to \psi_i^{-1} T(U_i \backslash \operatorname{GL}_n(\mathbb{C}))$.

Recall from Lemma 3.8 that $\mathfrak{V}_i$ is open dense in $Z_i = (\mathfrak{g}_i/\widetilde{W}_{r_i})^\vee$. Therefore, $T(\mathfrak{V}_i) \cong \mathfrak{V}_i \times Z_i$.

Lemma 5.3. *Let χ be a section of $\Omega^1_N(\mathrm{T}_{\mathfrak{gl}_n})$. Then, $\mathrm{p}^{\mathrm{L}}_{\mathrm{i}}(\chi) = -\operatorname{Ad}(g_i)(\chi) + \mathfrak{u}_i \in \Omega^1_N(B_i)$. Moreover, $\mathrm{p}^{\mathrm{U}}_{\mathrm{i}}(\chi) \in \Omega^1_N(Z_i^\vee)$ is the functional defined by $\operatorname{Ad}(g_i^\sigma)\left([\chi - \chi^\sigma, \alpha]\right)$.*

Proof. The action of $\mathrm{GL}_n(\mathbb{C})$ on $U_i \backslash \mathrm{GL}_n(\mathbb{C})$ is the usual (left) action, $h(U_i g) = U_i g h^{-1}$. The first statement follows. The map $\widetilde{N}_i \to U_i$ defined by $g \mapsto g(g^\sigma)^{-1}$ has tangent map $X \mapsto \operatorname{Ad}(g^\sigma)(X - X^\sigma)$. Since the projection from the $\tilde{\psi}_i(N_i) \subset \widetilde{\mathcal{M}}_i$ onto $\mathfrak{V}_i$ has the form $(v_i, g_i) \mapsto \operatorname{Ad}(g_i^\sigma g_i^{-1})(v_i)$, it follows that $\mathrm{p}^{\mathrm{U}}_{\mathrm{i}}(\chi) = [\operatorname{Ad}(g_i^\sigma)(\chi - \chi^\sigma), \operatorname{Ad}(g_i^\sigma)(\alpha)])$. $\qquad\square$

Choose a vector space lift $\widehat{Z}_i \subset \mathfrak{g}_i$ of $\mathfrak{g}_i/\widetilde{W}_{r_i}$. Define $\Xi^{\mathfrak{V}}_i \in (\widehat{Z}_i)^\vee \cong Z_i$ by $\Xi^{\mathfrak{V}}_i = \xi_i|_{\widehat{Z}_i}$, using the notation in (5.5). In other words, $\Xi^{\mathfrak{V}}_i$ is the functional on $\widehat{Z}_i$ determined by $\operatorname{Ad}(g_i^\sigma)(\Xi_i - [\Theta_i^\sigma, \alpha])$. The I-tuple $\tilde{\kappa} = (-\Xi^{\mathfrak{V}}_i, \Theta_i)_{i \in I}$ determines a section of $\Omega^1_N(\prod_{i \in I} \mathrm{T}(\widetilde{\mathcal{M}}_i))$. We define κ to be the differential form on $\Omega^1_N((\prod_{i \in I} \mathrm{T}(\widetilde{\mathcal{M}}_i))/\mathrm{T}_{\mathfrak{gl}_n})$ determined by the image of $\tilde{\kappa}$.

Definition 5.4. We define $\mathcal{I}^\sigma$ to be the differential ideal on N determined by the coefficients of κ and by $d\hat{\Gamma}^0_i$ for each $i \in I$.

We wish to show that $\mathcal{I}^\sigma$ is the restriction of $\mathcal{I}$ to N.

Lemma 5.5. *Suppose that $\mathcal{J}$ is a differential ideal on N containing $d\hat{\Gamma}^0_i$ and χ is a section of $\Omega^1_N(\mathrm{T}_{\mathfrak{gl}_n})$. If*

$$\operatorname{Ad}(g_i)(\Upsilon^0) + \operatorname{Ad}(g_i)(\chi) - (dg_i)g_i^{-1} \in \Phi_i + \mathcal{J} + \Omega^1_N(\mathfrak{P}^1_i),$$

then $\operatorname{Ad}(g_i)(\Xi_i - [\chi, \alpha]) \in \Omega^1_N([\mathfrak{P}^1_i, \operatorname{Ad}(g_i)(\alpha)] + \mathfrak{P}^1_i \frac{dz_i}{z_i \nu}) + \mathcal{J}$.

Proof. Without loss of generality, let $\nu = \frac{dz_i}{z_i}$. By construction, $\tau_\nu(A'_{\widetilde{\mathcal{M}},i}) - d\hat{\Gamma}_{i\nu} + d\hat{\Gamma}^0_i + [\hat{\Gamma}_{i\nu}, A'_{\widetilde{\mathcal{M}},i}] = 0$. Write $\alpha' = \hat{g}_i^{-1} \cdot (\hat{\Gamma}_{i\nu} - \hat{\Gamma}^0_i)$, so $\alpha - \alpha' = \operatorname{Ad}(\hat{g}_i^{-1})(\hat{\Gamma}^0_i)$. Set $\Upsilon = \operatorname{Ad}(\hat{g}_i^{-1})A'_{\widetilde{\mathcal{M}},i} + \hat{g}_i^{-1} d\hat{g}_i$.

By Proposition 4.2, $\hat{g}_i^{-1} \cdot \left((\hat{\Gamma}_{i\nu} - \hat{\Gamma}^0_i)\nu + A'_{\widetilde{\mathcal{M}},i}\right) + \hat{g}_i^{-1} d\hat{g}_i = \alpha'\nu + \Upsilon$ determines a flat connection on $\mathrm{Spec}(F_i) \times N$. Therefore, $\tau_\nu(\Upsilon) - d\alpha' + [\alpha', \Upsilon] = 0$. It follows that

$$\tau_\nu(\Upsilon) - d\alpha + [\alpha, \Upsilon] = -\operatorname{Ad}(\hat{g}_i^{-1})d\hat{\Gamma}^0_i \in \mathcal{J}.$$

By definition, $\operatorname{Ad}(\hat{g}_i^{-1})A_{\widetilde{\mathcal{M}},i} + \Omega^1_N((\mathfrak{P}^1_i)^{g_i}) = \operatorname{Ad}(g_i^{-1})\Phi_i$ and $\hat{g}_i^{-1} d\hat{g}_i \in g_i^{-1} dg_i + \Omega^1_N((\mathfrak{P}^1_i)^{g_i})$. Therefore, subtracting the above expression from $\Xi_i - [\chi, \alpha]$ and using the hypothesis yields

$$\Xi_i - [\chi, \alpha] \in \tau_\nu(\Upsilon^0 + \chi - \operatorname{Ad}(\hat{g}_i^{-1})A_{\widetilde{\mathcal{M}},i} - \hat{g}_i^{-1} d\hat{g}_i) -$$
$$[\Upsilon^0 + \chi - \operatorname{Ad}(\hat{g}_i^{-1})A_{\widetilde{\mathcal{M}},i} - \hat{g}_i^{-1} d\hat{g}_i, \alpha] + \Omega^1_N((\mathfrak{P}^1_i)^{g_i}) + \mathcal{J}.$$

The lemma follows after applying $\operatorname{Ad}(g_i)$ to both sides of the equation above. $\quad\square$

Lemma 5.6. *Let $\mathcal{J}$ be a differential ideal on N satisfying the following property: there exists a section χ of $\Omega^1_N(\mathrm{T}_{\mathfrak{gl}_n})$ such that $\Theta_i + \mathrm{p}^{\mathrm{L}}_{\mathrm{i}}(\chi) \in \mathcal{J}$ for all i. Then,*

$$\operatorname{Ad}(g_i)(\Upsilon^0 + \chi) \in \operatorname{Ad}(\hat{p}_i^{-1})A'_{\widetilde{\mathcal{M}},i} + (dg_i)g_i^{-1} + \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{P}^1_i) + \mathcal{J}.$$

Proof. We note that $\Upsilon^0 - \mathrm{Ad}(\hat{g}_i^{-1})(A'_{\widetilde{\mathcal{M}},i}) \in \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{g}_i)$ by construction. Since the map $\bar{\phi}_i$ induces an isomorphism $\mathfrak{P}^1_i \backslash \mathfrak{g}_i \cong \mathfrak{u}_i \backslash \mathfrak{gl}_n(\mathbb{C})$, it suffices to show

$$\bar{\phi}_i(\mathrm{Ad}(g_i)(\Upsilon^0 + \chi) - \mathrm{Ad}(\hat{p}_i^{-1})(A'_{\widetilde{\mathcal{M}},i}) - (dg_i)g_i^{-1}) \in \mathcal{J}.$$

This is equal to $-\Theta_i + \mathrm{Ad}(g_i)(\chi) + \Omega^1_N(\mathfrak{u}_i)$ as a section of $\Omega^1_N(\psi_i^* T(U_i \backslash \mathrm{GL}_n(\mathbb{C})))$. By Lemma 5.3,

$$-\Theta_i + \mathrm{Ad}(g_i)(\chi) + \Omega^1_N(\mathfrak{u}_i) = -\Theta_i - \mathrm{p}_i^{\mathrm{L}}(\chi) \in \mathcal{J}.$$

It follows that $\mathrm{Ad}(g_i)(\Upsilon^0 + \chi) \in \mathrm{Ad}(\hat{p}_i^{-1})(A'_{\widetilde{\mathcal{M}},i}) + (dg_i)g_i^{-1} + \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{P}^1_i) + \mathcal{J}.$ $\square$

Proposition 5.7. *On a sufficiently small neighborhood N, there exists a section χ of $\Omega^1_N(\mathrm{T}_{\mathfrak{gl}_n})$ such that $\tilde{\kappa} + \chi \in \mathcal{I}^\sigma$. Any such χ will satisfy the conditions of Lemma 5.6 for $\mathcal{J} = \mathcal{I}^\sigma$; furthermore, $\mathcal{I}^\sigma$ is generated by $\Xi_i - [\chi, \alpha]$, $\Theta_i + \mathrm{p}_i^{\mathrm{L}}(\chi)$, and $d\hat{\Gamma}^0_i$. In particular, $\mathcal{I}^\sigma$ is independent of the choice of lift $\widehat{Z}_i$.*

For the next few results, we will assume that N is taken to be sufficiently small as in the statement of the lemma. However, we shall see below in Corollary 5.10 that we may choose χ to be the restriction of the globally defined χ_j.

Proof. Shrinking N if necessary, we may choose a direct sum decomposition $\prod_{i \in I} \mathrm{T}(\widetilde{\mathcal{M}}_i))|_N \cong \mathrm{T}_{\mathfrak{gl}_n}|_N \oplus \mathrm{T}^\perp_{\mathfrak{gl}_n}$. Here, $\mathrm{T}^\perp_{\mathfrak{gl}_n} \cong ((\prod_{i \in I} \mathrm{T}(\widetilde{\mathcal{M}}_i))/\mathrm{T}_{\mathfrak{gl}_n})|_N$. Thus, we may choose a section χ of $\Omega^1_N(\mathrm{T}_{\mathfrak{gl}_n})$ such that $\tilde{\kappa} + \chi \in \mathcal{I}^\sigma$. In particular, $\Theta_i + \mathrm{p}_i^{\mathrm{L}}(\chi) \in \mathcal{I}^\sigma$ and $-\Xi_i^{\mathfrak{Y}} + \mathrm{p}_i^{\mathrm{U}}(\chi) \in \mathcal{I}^\sigma$. By Lemma 5.6,

$$\mathrm{Ad}(g_i)(\Upsilon^0) + \mathrm{Ad}(g_i)(\chi) - (dg_i)g_i^{-1} \in \mathrm{Ad}(\hat{p}_i^{-1})(A'_{\widetilde{\mathcal{M}},i}) + \mathcal{I}^\sigma + \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{P}^1_i).$$

The right hand side is equal to $\Phi_i + \mathcal{I}^\sigma + \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{P}^1_i)$. We may conclude, using Lemma 5.5, that $\mathrm{Ad}(g_i)(\Xi_i - [\chi, \alpha]) \in \Omega^1_N(\mathfrak{P}^{-r_i+1}_i) + \mathcal{I}^\sigma$.

Consider the identity
(5.7)
$$\mathrm{Ad}(g_i^\sigma)(-\Xi_i + [\Theta_i^\sigma, \alpha] + [\chi - \chi^\sigma, \alpha]) = \mathrm{Ad}(g_i^\sigma)(-\Xi_i + [\chi, \alpha]) + \mathrm{Ad}(g_i^\sigma)([\Theta_i^\sigma - \chi^\sigma, \alpha]).$$

Note that $\Theta_i^\sigma - \chi^\sigma \in \Omega^1_N(\mathfrak{gl}_n(\mathbb{C}))$ lies in $\mathcal{I}^\sigma$, since it is the image of $\Theta_i + \mathrm{p}_i^{\mathrm{L}}(\chi)$ under the tangent map of σ_i. We obtain that the right hand side of (5.7) lies in $\Omega^1_N(\mathfrak{P}^{-r_i+1}_i) + \mathcal{I}^\sigma$ since $\Theta_i^\sigma - \chi^\sigma \in \Omega^1_N(\mathfrak{gl}_n(\mathbb{C}))$ is the image of $\Theta_i + \mathrm{p}_i^{\mathrm{L}}(\chi) \in \mathcal{I}^\sigma$ under the tangent map of σ_i. The left hand side determines a functional $\xi'_i \in \Omega^1_N(\mathfrak{g}_i^\vee)$ using the residue-trace pairing. Equation (5.2) and the second part of Lemma 5.3 show that ξ'_i is equal to $-\xi_i + \mathrm{p}_i^{\mathrm{U}}(\chi)$. It follows that $(-\xi_i + \mathrm{p}_i^{\mathrm{U}}(\chi))(\mathfrak{P}^{r_i}_i) \subset \mathcal{I}^\sigma$. Since $\widetilde{W}_{r_i} \subset \mathfrak{P}^{r_i}_i$, we see that $\widehat{Z}_i + \mathfrak{P}^{r_i}_i = \mathfrak{g}_i$. Thus, $(-\xi_i + \mathrm{p}_i^{\mathrm{U}}(\chi))(\mathfrak{g}_i) \subset \mathcal{I}^\sigma$. Lemma 5.2 implies that both sides of (5.7) lie in $\mathcal{I}^\sigma$. In particular, $\Xi_i - [\chi, \alpha] \in \mathcal{I}^\sigma$.

Finally, let $\mathcal{I}'$ be the ideal generated by $\Xi_i - [\chi, \alpha]$, $\Theta_i + \mathrm{p}_i^{\mathrm{L}}(\chi)$, and $d\hat{\Gamma}^0_i$. Since $-\Xi_i^{\mathfrak{Y}} + \mathrm{p}_i^{\mathrm{U}}(\chi)$ is the restriction of the difference of the first two generators, it lies in the ideal $\mathcal{I}'$. Therefore, $\kappa' = \tilde{\kappa} + \chi$ lies in $\mathcal{I}'$. The image of κ' in $\Omega^1_N((\prod_i(\mathrm{T}(\widetilde{\mathcal{M}}_i)))/\mathrm{T}_{\mathfrak{gl}_n})$ is κ, so $\kappa \in \mathcal{I}'$. It follows that $\mathcal{I}^\sigma = \mathcal{I}'$. $\square$

We will now show that $\mathcal{I}^\sigma$ is the restriction of $\mathcal{I}$ to N.

Lemma 5.8. *Let $\mathcal{J}$ be a differential ideal on N. Suppose that $X \in \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{P}^1_i)$ satisfies*

$$(5.8) \qquad \qquad \tau_\nu(X) - [X, \hat{\Gamma}_{i\nu}] \in \mathcal{J}.$$

Then, $X \in \mathcal{J}$.

Proof. Without loss of generality, assume that $\nu = \frac{dz_i}{z_i}$ and $\tau_\nu = z_i \partial_{z_i}$. To simplify notation, we will suppress the subscript i in the proof. Let $\psi : \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{P}^1) \to \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{P}^{-r+1})$ be defined by $\psi(Y) = \tau_\nu(Y) - [Y, \hat{\Gamma}_\nu]$.

First, we prove the case $r = 0$, so $e = 1$ and $P = \mathrm{GL}_n(\mathfrak{o})$. Note that $\hat{\Gamma}_\nu \in (\mathfrak{t}^\flat)'$ by Definition 3.3 and $\tau_\nu(Y) = \ell Y$ for $Y \in \Omega^1_{\widehat{\mathcal{M}}}(z^\ell \, \mathfrak{gl}_n(\mathbb{C}))$. It follows that $\psi : z^\ell \, \mathfrak{gl}_n(\mathbb{C}) \to z^\ell \, \mathfrak{gl}_n(\mathbb{C})$ is an isomorphism, since $-\ell$ is never an eigenvalue of $\mathrm{ad}(\hat{\Gamma}_\nu)$. Now, suppose that $X \in \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{g}^\ell) + \mathcal{J}$ for $\ell \geq 1$, say $X \in X_\ell + \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{g}^{\ell+1}) + \mathcal{J}$ for some $X_\ell \in \Omega^1_{\widehat{\mathcal{M}}}(z^\ell \, \mathfrak{gl}_n(\mathbb{C}))$. (This holds trivially for $\ell = 1$.) We see that the image of $\psi(X_\ell) \in \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{g}^{\ell+1}) + \mathcal{J}$. However, we have shown that ψ induces an automorphism of $\Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{g}^\ell/\mathfrak{g}^{\ell+1})$, so $X_\ell \in \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{g}^{\ell+1}) + \mathcal{J}$. By induction, we conclude that $X \in \mathcal{J}$.

Let $W_\ell = \ker(\bar{\pi}_\mathfrak{t}) \subset \mathfrak{P}^\ell/\mathfrak{P}^{\ell+1}$. We define the projection $\mathrm{p}_\mathfrak{t} : \mathfrak{gl}_n(F) \to \ker(\pi_\mathfrak{t})$ by $\mathrm{p}_\mathfrak{t}(X) = X - \pi_\mathfrak{t}(X)$, and write $\bar{\mathrm{p}}_\mathfrak{t} : \mathfrak{P}^\ell/\mathfrak{P}^{\ell+1} \to W_\ell$ for the induced map on cosets. Let $\psi : \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{P}^1) \to \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{P}^{-r+1})$ be defined by $\psi(Y) = \tau_\nu(Y) - [Y, \hat{\Gamma}_\nu]$. Since Lemma 4.1 states that $\psi(t) = \delta_e(t) \in \mathfrak{t}$ for all $t \in \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{t}^1)$, $\mathrm{p}_\mathfrak{t} \circ \psi(t) = 0$. This implies that $\mathrm{p}_\mathfrak{t}(\psi(Y)) = \mathrm{p}_\mathfrak{t}(\psi(\mathrm{p}_\mathfrak{t}(Y)))$ for all $Y \in \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{P}^\ell)$.

First, we show inductively that $X' = \mathrm{p}_\mathfrak{t}(X) \in \mathcal{J}$. Suppose that $X' \in \mathcal{J} + \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{P}^\ell)$, say $X' \in X'_\ell + \mathcal{J}$ with $X'_\ell \in \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{P}^\ell)$. Since $\psi(X) \in \mathcal{J}$, we obtain $\mathrm{p}_\mathfrak{t}(\psi(X'_\ell)) = \mathrm{p}_\mathfrak{t}(\psi(X)) + \mathcal{J} \subset \mathcal{J}$. By Corollary 2.5, the map $\bar{\psi} : W_\ell \to W_{\ell-r}$ induced by $\mathrm{p}_\mathfrak{t} \circ \psi$ is an isomorphism, so $X'_\ell \in \mathcal{J} + \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{P}^{\ell+1})$. It follows that $X' \in \mathcal{J} + \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{P}^{\ell+1})$. Since the inductive hypothesis holds trivially for $\ell = 1$, we conclude that $X' \in \mathcal{J}$.

We now have $X \in \pi_\mathfrak{t}(X) + \mathcal{J}$ and also $\delta_e(\pi_\mathfrak{t}(X)) = \psi(\pi_\mathfrak{t}(X)) = \psi(X) - \psi(X') \in \mathcal{J}$. Since δ_e is an automorphism of $\mathfrak{t}^1$, this gives $\pi_\mathfrak{t}(X) \in \mathcal{J}$, and hence $X \in \mathcal{J}$. $\qquad \square$

In the following, let χ be the section of $\mathrm{T}_{\mathfrak{gl}_n}|_N$ constructed in Lemma 5.3.

Proposition 5.9. *The following identity holds modulo $\mathcal{I}^\sigma$:*

$$\Upsilon^0 + \chi \in \mathrm{Ad}(\hat{g}_i^{-1}) A'_{\widetilde{\mathcal{M}},i} + \hat{g}_i^{-1} d\hat{g}_i + \mathcal{I}^\sigma.$$

Proof. By Proposition 5.7,

$$\tau_\nu \Upsilon^0 - d\alpha - [\Upsilon^0 + \chi, \alpha] + \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{g}_j^1 \frac{dz_j}{z_j \nu}) = \Xi_j - [\chi, \alpha] \in \mathcal{I}^\sigma.$$

Thus, if ν is a global meromorphic one-form on $\mathbb{P}^1$, we obtain $\nu \wedge \big(\tau_\nu \Upsilon^0 - d\alpha - [\Upsilon^0 + \chi, \alpha]\big) \in \mathcal{I}^\sigma$. This follows from the observation that the principal part at each $x_j \in \mathbb{P}^1$ lies in $\mathcal{I}^\sigma$. Applying the interior derivative ι_{τ_ν} implies that

$$(5.9) \qquad \qquad \tau_\nu \Upsilon^0 - d\alpha - [\Upsilon^0 + \chi, \alpha] \in \mathcal{I}^\sigma.$$

Now, we set $\nu = \frac{dz_i}{z_i}$ and $\tau_\nu = z_i \partial_{z_i}$. Note that $\Upsilon^0 - \mathrm{Ad}(\hat{g}_i^{-1})(A'_{\widetilde{\mathcal{M}},i}) \in \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{g}_i)$ by construction. Therefore, setting $\hat{p}_i^\sigma = \hat{g}_i(g_i^\sigma)^{-1}$,

$$\mathrm{Ad}(g_i^\sigma)(\Upsilon^0) + \mathrm{Ad}(g_i^\sigma)(\chi) - (\mathrm{Ad}((\hat{p}_i^\sigma)^{-1})(A'_{\widetilde{\mathcal{M}},i}) + (dg_i^\sigma)(g_i^\sigma)^{-1}) \in$$
$$- \phi_i((dg_i^\sigma)(g_i^\sigma)^{-1} + \mathrm{Ad}((\hat{p}_i^\sigma)^{-1})(A'_{\widetilde{\mathcal{M}},i}) - \mathrm{Ad}(g_i^\sigma)(\Upsilon^0)) + \mathrm{Ad}(g_i^\sigma)(\chi) + \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{P}_i^1) =$$
$$- \mathrm{Ad}(g_i^\sigma)(\Theta_i^\sigma - \chi) \in \mathcal{I}^\sigma.$$

It follows that $\Upsilon^0 + \chi \in \mathrm{Ad}(\hat{g}_i^{-1})A'_{\widetilde{\mathcal{M}},i} + \hat{g}_i^{-1}d\hat{g}_i + \mathcal{I}^\sigma + \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{P}_i^1)$.

Let $\Upsilon' = \mathrm{Ad}(\hat{g}_i)(\Upsilon^0 + \chi) - (d\hat{g}_i)\hat{g}_i^{-1}$. From the argument above, we see that $\Upsilon' - A'_{\widetilde{\mathcal{M}},i} \in \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{P}_i^1) + \mathcal{I}^\sigma$. Substituting $\mathrm{Ad}(\hat{g}_i^{-1})(\Upsilon') + \hat{g}_i^{-1}d\hat{g}_i = \Upsilon^0 + \chi$ and $\mathrm{Ad}(\hat{g}_i^{-1})\hat{\Gamma}_{i\nu} + \hat{g}_i^{-1}\tau_\nu(\hat{g}_i) = \alpha$ into (5.9), we obtain

$$\tau_\nu(\Upsilon') - d\hat{\Gamma}_{i\nu} - [\Upsilon', \hat{\Gamma}_{i\nu}] \in \mathcal{I}^\sigma.$$

Since $d\hat{\Gamma}_{i\nu} - d\hat{\Gamma}_i^0 = \tau_\nu(A'_{\widetilde{\mathcal{M}},i}) - [A'_{\widetilde{\mathcal{M}},i}, \hat{\Gamma}_{i\nu}]$ by (5.3), we deduce that

$$\tau_\nu(\Upsilon' - A'_{\widetilde{\mathcal{M}},i}) - [\Upsilon' - A'_{\widetilde{\mathcal{M}},i}, \hat{\Gamma}_{i\nu}] \in d\hat{\Gamma}_i^0 + \mathcal{I}^\sigma.$$

The left hand side lies in $\mathcal{I}^\sigma$, since $d\hat{\Gamma}_i^0 \in \mathcal{I}^\sigma$. The statement of the Lemma now follows, since by Lemma 5.8, $\Upsilon' - A'_{\widetilde{\mathcal{M}},i} \in \mathcal{I}^\sigma$.

$\square$

Corollary 5.10. *The sections χ_i of $\mathrm{T}_{\mathfrak{gl}_n}$ satisfy $\tilde{\kappa} + \chi_i \in \mathcal{I}^\sigma$. Moreover, $\chi_i \in \chi_j + \mathcal{I}^\sigma$ and $d\chi_i + \chi_i \wedge \chi_i \in \mathcal{I}^\sigma$.*

Proof. Let χ be the section of $\mathrm{T}_{\mathfrak{gl}_n}$ constructed in Proposition 5.7. To prove the first statement, it suffices to show that $\chi_i - \chi \in \mathcal{I}^\sigma$. However, for any $i \in I$,

$$\chi_i - \chi = \phi_i(\mathrm{Ad}(\hat{g}_i^{-1})A'_{\widetilde{\mathcal{M}},i} + \hat{g}_i^{-1}d\hat{g}_i - \Upsilon^0 - \chi) \in \mathcal{I}^\sigma$$

by Proposition 5.9.

In order to prove the second part, we recall that Υ^0 is defined as $\varsigma((\bar{\Phi}_j)_{j \in I})$. It follows that $d\Upsilon^0 = \varsigma((d\bar{\Phi}_j)_{j \in I})$. By construction, $\bar{\Phi}_j = \mathrm{Ad}(\hat{g}_j^{-1})(A_{\widetilde{\mathcal{M}},j})$. Therefore, $d\bar{\Phi}_j \in -[\mathrm{Ad}(\hat{g}_j^{-1})A_{\widetilde{\mathcal{M}},j}, \hat{g}_j^{-1}d\hat{g}_j] + \Omega^1_N(\mathfrak{g}_j)$ since $dA_{\widetilde{\mathcal{M}},i} = 0$.

By Proposition 5.9, $\mathrm{Ad}(\hat{g}_j^{-1})A_{\widetilde{\mathcal{M}},j} + \hat{g}_j^{-1}d\hat{g}_j \in \Upsilon^0 + \chi_i + \mathcal{I}^\sigma$. We conclude that

$$[\mathrm{Ad}(\hat{g}_j^{-1})A_{\widetilde{\mathcal{M}},j}, \hat{g}_j^{-1}d\hat{g}_j] \in -(\Upsilon^0 + \chi_i) \wedge (\Upsilon^0 + \chi_i) + \mathcal{I}^\sigma + \Omega^2_N(\mathfrak{g}_j),$$

and moreover $\varsigma((d\bar{\Phi}_j)_{j \in I}) \in -\varsigma(((\Upsilon^0 + \chi_i) \wedge (\Upsilon^0 + \chi_i))_{j \in I}) + \mathcal{I}^\sigma$. Thus, by Lemma 4.10, $(\Upsilon^0 + \chi_i) \wedge (\Upsilon^0 + \chi_i) + \varsigma((d\bar{\Phi}_j)_{j \in I}) \in \chi_i \wedge \chi_I + \mathcal{I}^\sigma$.

Finally,

$$d(\mathrm{Ad}(\hat{g}_i^{-1})A_{\widetilde{\mathcal{M}},i} + \hat{g}_i^{-1}d\hat{g}_i) =$$
$$- (\mathrm{Ad}(\hat{g}_i^{-1})A_{\widetilde{\mathcal{M}},i} + \hat{g}_i^{-1}d\hat{g}_i) \wedge (\mathrm{Ad}(\hat{g}_i^{-1})A_{\widetilde{\mathcal{M}},i} + \hat{g}_i^{-1}d\hat{g}_i)$$
$$\in -(\Upsilon^0 + \chi_i) \wedge (\Upsilon^0 + \chi_i) + \mathcal{I}^\sigma.$$

It follows that

$$d\chi_i + \chi_i \wedge \chi_i = \phi_i\left(d\left(\mathrm{Ad}(\hat{g}_i^{-1})A'_{\widetilde{\mathcal{M}},i} + \hat{g}_i^{-1}d\hat{g}_i - \Upsilon^0\right)\right) + \chi_i \wedge \chi_i$$
$$\in \phi_i(-(\Upsilon^0 + \chi_i) \wedge (\Upsilon^0 + \chi_i) + (\Upsilon^0 + \chi_i) \wedge (\Upsilon^0 + \chi_i) - \chi_i \wedge \chi_i) + \chi_i \wedge \chi_i + \mathcal{I}^\sigma$$
$$= \mathcal{I}^\sigma.$$

$\square$

Corollary 5.11. *Fix $j \in I$. The ideal $\mathcal{I}^\sigma$ is generated by $\Xi_i - [\chi_j, \alpha]$, $\Theta_i + \mathrm{p}_i^{\mathrm{L}}(\chi_j)$, and $d\hat{\Gamma}_i^0$ for all $i \in I$. In particular, $\mathcal{I}^\sigma$ is the restriction of $\mathcal{I}$ to N, and $\mathcal{I}$ does not depend on the choice of j.*

Proof. By Corollary 5.10, χ_j satisfies the the conditions of Proposition 5.7. The same proposition implies that $\mathcal{I}^\sigma$ has the given generators.

$\square$

5.3. Invariance.

In this section, we will show that $\mathcal{I}$ is the pullback of a differential ideal $\overline{\mathcal{I}}$ on $\widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$.

Proposition 5.12. *Let X be a section of $\mathrm{T}_{\mathfrak{gl}_n}$, and take $\omega \in \mathcal{I}$. Then, $\iota_X \omega \in \mathcal{I}$. In particular, if $\omega \in \mathcal{I} \cap \Omega^1_{\widehat{\mathcal{M}}}$, then $\iota_X \omega = 0$.*

Proof. It suffices to show that the generators of $\mathcal{I}$ are in the kernel of ι_X. Since $\hat{\Gamma}_i$ is invariant under the action of $\mathrm{GL}_n(\mathbb{C})$, we see that $\iota_X d\hat{\Gamma}_i^0 = 0$ and $\iota_X(A_{\widetilde{\mathcal{M}},i}) = 0$. It follows that any term involving $A_{\widetilde{\mathcal{M}},i}$ lies in the kernel of ι_X. We now calculate $\iota_X(\Xi_i - [\chi_j, \alpha])$. By the argument above,

$$\iota_X(\Xi_i) = \iota_X(-d\alpha) = -[X, \alpha].$$

On the other hand,

$$(5.10) \qquad \iota_X(\chi_j) = \iota_X \phi_j(\hat{g}_j^{-1} d\hat{g}_j) = -X.$$

It follows that $\iota_X(\Xi_i - [\chi_j, \alpha]) = -[X, \alpha] + [X, \alpha] = 0$. On the other hand,

$$\iota_X(\Theta_i) = \iota_X((dg_i)g_i^{-1} + \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{u}_i)) = \mathrm{Ad}(g_i)(X) + \mathfrak{u}_i.$$

Using (5.10), we see that $\iota_X(\Theta_i + \mathrm{p}_i^{\mathrm{L}}(X)) = \mathrm{Ad}(g_i)(X) - \mathrm{Ad}(g_i)(X) + \mathfrak{u}_i = \mathfrak{u}_i$. $\square$

We now show that the ideal $\mathcal{I}$ is $\mathrm{GL}_n(\mathbb{C})$-invariant.

Proposition 5.13. *Let $g : V \to \mathrm{GL}_n(\mathbb{C})$ be a function defined on a neighborhood $V \subset \widehat{\mathcal{M}}$, and let $\hat{\rho}_g : V \to \widehat{\mathcal{M}}$ be the map determined by the action of g. Then, $\hat{\rho}_g^* \mathcal{I} = \mathcal{I}$. Moreover, $\hat{\rho}_g^*(\Xi_i - [\chi_j, \alpha]) = \mathrm{Ad}(g)(\Xi_i - [\chi_j, \alpha])$ and $\hat{\rho}_g^*(\Theta_i + \mathrm{p}_i^{\mathrm{L}}(\chi_j)) = \Theta_i + \mathrm{p}_i^{\mathrm{L}}(\chi_j)$.*

Proof. First, we calculate $\hat{\rho}_g^*(\chi_j)$. One easily checks that

$$(5.11) \qquad \hat{\rho}_g^*(\chi_j) = \mathrm{Ad}(g)\chi_j - (dg)g^{-1}.$$

We also see that $\hat{\rho}_g^*(\Xi_i) = \mathrm{Ad}(g)(\Xi_i) - [(dg)g^{-1}, \alpha]$ and $\hat{\rho}_g^*(\Theta_i) = -\mathrm{Ad}(g_i g^{-1})((dg)g^{-1}) + \Theta_i$. We see that $\hat{\rho}_g^*(\Xi_i - [\chi_j, \alpha]) = \mathrm{Ad}(g)(\Xi_i - [\chi_j, \alpha]) \in \mathcal{I}$ and $\hat{\rho}_g^*(\Theta_i + \mathrm{p}_i^{\mathrm{L}}(\chi_j)) = \Theta_i + \mathrm{p}_i^{\mathrm{L}}(\chi_j) \in \mathcal{I}$. Finally, $\hat{\rho}_g^* d\hat{\Gamma}_i^0 = d\hat{\Gamma}_i^0$. It follows that $\hat{\rho}_g^*(\mathcal{I}) \subset \mathcal{I}$. $\square$

In the following, $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$ and $q : \widehat{\mathcal{M}} \to \widetilde{\mathcal{M}}$ is the natural projection.

Corollary 5.14. *There exists a differential ideal $\overline{\mathcal{I}}$ on $\widetilde{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$ such that $q^* \overline{\mathcal{I}} = \mathcal{I}$.*

Proof. Since $\widehat{\mathcal{M}}$ is a principal $\mathrm{GL}_n(\mathbb{C})$-bundle over $\widetilde{\mathcal{M}}$, we may choose a cover of $\widetilde{\mathcal{M}}$ by neighborhoods W with the property that $q^{-1}(W) \cong \mathrm{GL}_n(\mathbb{C}) \times W$. Define Θ_i^G and Ξ_i^G on $q^{-1}(W)$ by $\Theta_i^G(g, w) = \Theta_i + \mathrm{p}_i^{\mathrm{L}}(\chi_j) + \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{u}_i)$ and $\Xi_i^G(g, w) =$

$\mathrm{Ad}(g^{-1})(\Xi_i - [\chi_j, \alpha])$. It is clear that the collection Θ_i^G, Ξ_i^G, and $d\hat{\Gamma}_i^0$ over all $i \in I$ generates $\mathcal{I}|_{q^{-1}(W)}$. By Proposition 5.13, $\hat{\rho}_h^*(\Theta_i^G) = \Theta_i^G$ and $\hat{\rho}_h^*(\Xi_i^G) = \Xi_i^G$.

Proposition 5.12 shows that each of the generators determines a well-defined form on $q^*T(W) \cong T(q^{-1}(W))/\mathfrak{gl}_n(\mathbb{C})$. By invariance, Θ_i^G, Ξ_i^G, and $d\hat{\Gamma}_i^0$ descend to differential forms on $T(\widetilde{\mathcal{M}})$. These generate the ideal $\overline{\mathcal{I}}$.

$\square$

6. INTEGRABILITY

We will now show that $\mathcal{I}$ is an integrable Pfaffian system in the sense of [8, II.2.4]. In general, if $\mathcal{J}$ is a differential ideal on a smooth variety X, we write $\mathcal{J}^1 = \mathcal{J} \cap \Omega_X^1$. It suffices to show that the coherent sheaf $\mathcal{I}^1$ is locally free and that $d\mathcal{I} \subset \mathcal{I}$.

Let $\mathcal{J}$ be the ideal on $\widehat{\mathcal{M}}$ generated by Ξ_i, $d\hat{\Gamma}_i^0$, and Θ_i for all $i \in I$. Note that $\mathcal{I} \subset \mathcal{J}$. We will first show that $\mathcal{J}$ is a Pfaffian system and that $\mathcal{I}^1 \subset \mathcal{J}^1$ has constant corank.

We need to describe a (local) minimal basis for $\mathcal{J}$. Since $\mathcal{J}$ is independent of the choice of one-form ν, we may assume that $\nu = \frac{dz_i}{z_i}$ when working locally at x_i. As in Section 5.2, we will restrict to sufficiently small neighborhoods $N_i \subset U_i \backslash \mathrm{GL}_n(\mathbb{C})$ and the corresponding neighborhood $N \subset \widehat{\mathcal{M}}$.

By part (2) of Proposition 2.4, the map $\pi_{\mathfrak{t}_i}$ induces a map $\pi_{\mathfrak{t}_i}' : \mathfrak{P}_i/\mathfrak{P}_i^{r_i} \to \mathfrak{t}_i/\mathfrak{t}_i^{r_i}$. Choose a vector space lift $Z_i^{od} \subset \mathfrak{P}_i$ of $\ker(\pi_{\mathfrak{t}_i}')$ and a lift $Z_i^{\mathfrak{u}} \subset \mathfrak{g}_i$ of $\mathfrak{g}_i/\mathfrak{P}_i$. Note that these vector spaces are trivial when $r = 0$. We define locally the 'off-diagonal' and $\mathfrak{u}$ components of Ξ_i to be $\Xi_i^{od} = \xi_i|_{Z^{od}} \in \Omega_N^1((Z^{od})^\vee)$ and $\Xi_i^{\mathfrak{u}} = \xi_i|_{Z_i^{\mathfrak{u}}} \in \Omega_N^1((Z_i^{\mathfrak{u}})^\vee)$ respectively.

Proposition 6.1. *The ideal $\mathcal{J}|_N$ is generated by Θ_i, Ξ_i^{od}, $\Xi_i^{\mathfrak{u}}$, and $d\hat{\Gamma}_i^0$ for $i \in I$.*

Proof. As above, we assume that $\nu = \frac{dz_i}{z_i}$. By Lemma 5.2, it suffices to show that $\xi_i(\mathfrak{g}_i)$ lies in the ideal $\mathcal{J}'$ generated by $d\hat{\Gamma}_i^0$, Θ_i, Ξ_i^{od}, and $\Xi_i^{\mathfrak{u}}$, since $\mathcal{J}' \subset \mathcal{J}$.

Let $\mathcal{J}_1$ be the ideal generated by Θ_i and $d\hat{\Gamma}_i^0$ for each $i \in I$. By Lemma 5.6 (applied with $\chi = 0$), $\Upsilon^0 \in \mathrm{Ad}(\hat{g}_i^{-1})A'_{\widetilde{\mathcal{M}},i} + \hat{g}_i^{-1}d\hat{g}_i + \Omega_W^1((\mathfrak{P}_i^1)^{g_i}) + \mathcal{J}_1$. Lemma 5.5 states that

$$(6.1) \qquad \mathrm{Ad}(g_i^\sigma)\Xi_i \in \Omega_W^1([\mathfrak{P}_i^1, \mathrm{Ad}(g_i^\sigma)(\alpha)] + \mathfrak{P}_i^1) + \mathcal{J}_1 \subset \Omega_W^1(\mathfrak{P}^{-r+1}) + \mathcal{J}_1.$$

Therefore, $\xi_i(\mathfrak{P}_i^r) \subset \mathcal{J}_1$.

We will now show that $\mathrm{Ad}(g_i^\sigma)(\Xi_i) \in \mathcal{J}' + \Omega_N^1(\mathfrak{P}_i^1)$. Take $1 \leq j \leq r$, and assume inductively that there exists $X \in \Omega_N^1(\mathfrak{P}_i^j)$ such that

$$(6.2) \qquad \mathrm{Ad}(g_i^\sigma)\Xi_i \in [X, \mathrm{Ad}(g_i^\sigma)(\alpha)] + \mathcal{J}' + \Omega_N^1(\mathfrak{P}_i^1).$$

Part (3) of Proposition 2.4 shows that $[X, \mathrm{Ad}(g_i^\sigma)(\alpha)] + \Omega_N^1(\mathfrak{P}_i^{j-r+1}) \in \ker(\bar{\pi}_{\mathfrak{t}_i})$. Thus, by part (4) of the same proposition, $\langle [X, \mathrm{Ad}(g_i^\sigma)(\alpha)], \mathfrak{t}_i^{r-j}\rangle_\nu = \{0\}$. Since $\Xi_i^{od} \in \mathcal{J}'$, $\langle [X, \mathrm{Ad}(g_i^\sigma)(\alpha)], Y\rangle_\nu \in \mathcal{J}'$ for $Y \in \ker(\pi_{\mathfrak{t}_i}) \cap \mathfrak{P}^{r-j}$. Combining these two facts gives $\langle [X, \mathrm{Ad}(g_i^\sigma)(\alpha)], \mathfrak{P}^{r-j}\rangle_\nu \in \mathcal{J}'$, and we conclude that $[X, \mathrm{Ad}(g_i^\sigma)(\alpha)] \in \Omega_N^1(\mathfrak{P}_i^{j-r+1}) + \mathcal{J}'$. Corollary 2.5 now implies that $X \in \pi_{\mathfrak{t}}(X) + \Omega_N^1(\mathfrak{P}_i^{j+1}) + \mathcal{J}'$. Finally, since there exists $p \in P_i^1$ such that $\mathrm{Ad}(p)(\mathrm{Ad}(g_i^\sigma)(\alpha)) \in \mathfrak{t}_i^{-r} + \mathfrak{P}_i^1$, we see that $X - \mathrm{Ad}(p^{-1})(\pi_{\mathfrak{t}}(X)) \in \Omega_N^1(\mathfrak{P}_i^{j+1}) + \mathcal{J}'$ satisfies (6.2) for $j+1$. By induction, we obtain (6.2) for $r+1$. This gives $\mathrm{Ad}(g_i^\sigma)(\Xi_i) \in \mathcal{J}' + \Omega_N^1(\mathfrak{P}_i^1)$ and hence $\xi_i(\mathfrak{P}_i) \subset \mathcal{J}'$. Finally, since $\mathfrak{g}_i = Z_i^{\mathfrak{u}} + \mathfrak{P}_i$, we see that $\xi_i(\mathfrak{g}_i) \subset \mathcal{I}'$ as desired.

$\square$

In the following, let $\iota : \widehat{\mathcal{M}} \to \prod_{i \in I} \widetilde{\mathcal{M}}_i$ be the inclusion and $T^*\iota : \iota^*(\Omega^1_{\prod_{i \in I} \widetilde{\mathcal{M}}_i}) \to \Omega^1_{\widehat{\mathcal{M}}}$ the induced bundle map. The generators of $\mathcal{I}$ (resp. $\mathcal{J}$) lift to a set of generators for a differential ideal $\widetilde{\mathcal{I}} \subset \iota^*\Omega^*_{\prod_{i \in I} \widetilde{\mathcal{M}}_i}$ (resp. $\widetilde{\mathcal{J}}$) on $\widehat{\mathcal{M}}$. Only the lift $\tilde{\Xi}_i$ of Ξ_i requires explanation. Suppose that $m = ((U_j g_j, \alpha_j)_{j \in I}) \in \widehat{\mathcal{M}}$. We write $\alpha_{i\nu}$ for a representative of α in $(\mathfrak{g}_i^{-r'_i} \frac{dz_i}{z_i\nu})/(\mathfrak{g}_i^1 \frac{dz_i}{z_i\nu})$. Therefore, if $m \in \widehat{\mathcal{M}}$, $\alpha_{i\nu} = \alpha + \mathfrak{g}_i^1 \frac{dz_i}{z_i\nu}$. We set

$$\tilde{\Xi}_i = \tau_\nu \Upsilon^0 - d\alpha_{i\nu} - [\Upsilon^0, \alpha_{i\nu}].$$

It is clear that $T^*\iota(\tilde{\Xi}_i) = \Xi_i$. All other forms involved in the definitions of $\mathcal{I}$ and $\mathcal{J}$ are restrictions of forms on $\prod_{i \in I} \widetilde{\mathcal{M}}_i$, and we will not make any notational distinction between such a form and its restriction to $\widehat{\mathcal{M}}$.

Lemma 6.2. *Let μ be the moment map for the action of $\mathrm{GL}_n(\mathbb{C})$ on $\prod_{i \in I} \widetilde{\mathcal{M}}(P_i, r_i)$. Then, $d\mu \subset \widetilde{\mathcal{I}}$.*

Proof. Let ν be the global choice of one form. By construction, $\mu((U_i g_i, \alpha_i)_{i \in I}) = \sum_{i \in I} \mathrm{res}(\alpha_i)$. Let $\chi = \chi_j$, so $\tilde{\Xi}_i - [\chi, \alpha]$ is one of the generators of $\widetilde{\mathcal{I}}$. We have

$$\sum_{i \in I} \mathrm{Res}_i(\nu \wedge (\tilde{\Xi}_i - [\chi, \alpha])) = \sum_{i \in I} \mathrm{Res}_i(\nu \wedge \tau_\nu(\Upsilon^0)) - \mathrm{Res}_i(\nu \wedge d\alpha_{i\nu}) - \mathrm{Res}_i(\nu \wedge [\Upsilon^0 + \chi, \alpha])$$

$$= \sum_{i \in I} \mathrm{Res}_i(\nu \wedge d\alpha_{i\nu}).$$

Here, $\sum_{i \in I} \mathrm{Res}_i(\nu \wedge \tau_\nu(\Upsilon^0))$ and $\sum_{i \in I} \mathrm{Res}_i(\nu \wedge [\Upsilon^0, \alpha])$ vanish by the residue theorem. Since $\mathrm{Res}_i(\nu \wedge \alpha_{i\nu}) = \mathrm{res}_i(\alpha_i)$, we see that $d\mu = \sum_{i \in I} \mathrm{Res}_i(\nu \wedge d\alpha_{i\nu})$ lies in $\mathcal{I}$. $\square$

Let $\widehat{\mathcal{M}}(\mathbf{A}) = q^{-1}(\widehat{\mathcal{M}}(\mathbf{A}))$ and $N(\mathbf{A}) = N \cap \widehat{\mathcal{M}}(\mathbf{A})$. We also set $\overline{N} = \prod_{i \in I} (\tilde{\psi}_i)^{-1}(N_i) \subset \prod_{i \in I} \widetilde{\mathcal{M}}_i$ and $\overline{N}(\mathbf{A}) = \overline{N} \cap \left(\prod_{i \in I} \widetilde{\mathcal{M}}(A_i) \right)$.

Proposition 6.3. *The ideal $\mathcal{J}$ is a Pfaffian system. Moreover, the restriction map taking $\mathcal{J}^1 \subset \Omega^1_{\widehat{\mathcal{M}}}$ to $\Omega^1_{\widehat{\mathcal{M}}(\mathbf{A})}$ is surjective.*

Proof. This is a local statement, so we will work with the restriction of $\mathcal{J}$ to N. Fix a point $m = (U_i g_i, \alpha_i)_{i \in I} \in N(\mathbf{A})$. We will first show that the restriction of $\widetilde{\mathcal{J}}^1$ to $\overline{N}(\mathbf{A})$ spans $\Omega^1_{\overline{N}(\mathbf{A})}$.

There are isomorphisms $\overline{N} \cong \prod_i (\mathfrak{P}_i/\mathfrak{P}_{r+1})^{\vee}_{reg} \times N_i$ and $\overline{N}(\mathbf{A}) \cong \prod_{i \in I} (\pi_{\mathfrak{P}^1})^{-1}(\mathcal{O}_i^1) \times N_i$. Restricting $\widetilde{\mathcal{J}}$ to $\overline{N}(\mathbf{A})$, we see that all terms in Θ_i and $\tilde{\Xi}_i$ involving $A_{\widetilde{\mathcal{M}},i}$ vanish. In particular, Υ^0 and Φ_i vanish, so Θ_i becomes $(dg_i)g_i^{-1} + \Omega^1_{\overline{N}(\mathbf{A})}(\mathfrak{u}_i)$, and $\tilde{\Xi}_i$ is simply $d\alpha + \Omega^1_{\overline{N}(\mathbf{A})}(\mathfrak{g}_i^1 \frac{dz_i}{z_i\nu})$. Write $\mathrm{Ad}(g_i)(\alpha) = v \in (\pi_{\mathfrak{P}^1})^{-1}(\mathcal{O}_i^1)$. Since $d\alpha = \mathrm{Ad}(g_i^{-1})(dv - [(dg_i)g_i^{-1}, v])$, it is easily checked that the coefficients of $(dg_i)g_i^{-1}$ and $d\alpha$ span $T^*_m(\overline{N}(\mathbf{A}))$.

On the other hand, a calculation using [3, Lemma 3.21] shows that $\dim T^*_m(\overline{N}(\mathbf{A})) = \dim \mathfrak{gl}_n(\mathbb{C})/\mathfrak{u} + \dim \mathfrak{g}/\mathfrak{P}^r - \dim \mathfrak{t}^1/\mathfrak{t}^r$, and so the coefficients of the set of generators for $\widetilde{\mathcal{J}}$ in Proposition 6.1 give a basis for $T^*_m(\overline{N}(\mathbf{A}))$. Thus, $\widetilde{\mathcal{J}}$ is a rank $\sum_{i \in I} \dim(\widetilde{\mathcal{M}}(A_i))$ Pfaffian system.

Lemma 6.2 shows that $d\mu \in \widetilde{\mathcal{J}}$. Since the coefficients of $d\mu$ span the conormal bundle of $\widehat{\mathcal{M}}$ in $\prod_{i \in I} \widetilde{\mathcal{M}}(T_i, r_i)$, we see that the pullback of $\widetilde{\mathcal{J}}$ in $\Omega^*_{\widehat{\mathcal{M}}}$, i.e., $\mathcal{J}$, is a rank $\left(\sum_{i \in I} \dim(\widetilde{\mathcal{M}}(A_i)) \right) - n^2$ Pfaffian system. Moreover, the restriction map taking $\mathcal{J}^1$ to $\Omega^1_{N(\mathbf{A})}$ induced by the map $\widetilde{\mathcal{J}}^1 \to \Omega^1_{\overline{N}(\mathbf{A})}$ is still surjective. $\square$

Lemma 6.4. *Fix $j \in I$. The ideal $\mathcal{J}$ is generated by $\mathcal{I}$ and the coefficients of χ_j.*

Proof. We know that $\mathcal{I} \subset \mathcal{J}$. By Corollary 5.10, $\tilde{\kappa} + \chi_j \in \mathcal{I}$. However, by construction, $\tilde{\kappa} \in \mathcal{J}$; it follows that $\chi_j \in \mathcal{J}$.

In order to prove the reverse inclusion, Proposition 5.7 shows that $\Xi_i - [\chi_j, \alpha] \in \mathcal{I}$. By assumption, $\Theta_i + \mathrm{p}^{\mathrm{L}}_i(\chi_j) \in \mathcal{I}$ as well. It follows that Ξ_i and Θ_i lie in the ideal generated by $\mathcal{I}$ and χ_j. $\square$

There is a natural restriction map $\Omega^1_{\widehat{\mathcal{M}}} \to (\mathrm{T}_{\mathfrak{gl}_n})^\vee$. We denote by F the induced map $F : \mathcal{J}^1 \to (\mathrm{T}_{\mathfrak{gl}_n})^\vee$.

Proposition 6.5. *The map F is surjective, and $\mathcal{I}^1$ is the kernel of F. Moreover, $\mathcal{I}$ is a Pfaffian system.*

Proof. First, we prove surjectivity. For a point $m \in \widehat{\mathcal{M}}(\mathbf{A})$, the map F factors through restriction to $T((\widehat{\mathcal{M}}(\mathbf{A}))$, since $\widehat{\mathcal{M}}(\mathbf{A})$ is $\mathrm{GL}_n(\mathbb{C})$-invariant. By Proposition 6.3, the restriction map $\mathcal{J}^1 \to \Omega^1_{\widehat{\mathcal{M}}(\mathbf{A})}$ is a surjective bundle map. Since the map $\Omega^1_{\widehat{\mathcal{M}}(\mathbf{A})} \to \mathrm{T}^\vee_{\mathfrak{gl}_n}$ is surjective, it follows that F is surjective.

By Proposition 5.12, $\mathcal{I}^1 \subset \mathrm{T}^\perp_{\mathfrak{gl}_n}$. We conclude that $\mathcal{I} \subset \ker(F)$. Finally, since $\mathcal{I}^1$ and χ_j generate $\mathcal{J}^1$, the rank of $\mathcal{I}^1$ must be at least $\mathrm{rank}(\mathcal{J}^1) - n^2$. The rank of the kernel of F is $\mathrm{rank}(\mathcal{J}^1) - n^2$, so $\mathcal{I}^1 = \ker(F)$. Therefore, $\mathcal{I}$ has constant rank and is thus a Pfaffian system. $\square$

We will now prove that $\mathcal{I}$ satisfies the Frobenius integrability condition.

Lemma 6.6. *Set $\chi = \chi_j$ as above. Then,*

$$d(\Upsilon^0 + \chi) + (\Upsilon^0 + \chi) \wedge (\Upsilon^0 + \chi) \in \mathcal{I}.$$

Proof. First, we will show that

$$(6.3) \qquad d(\Upsilon^0) \in -(\Upsilon^0 + \chi) \wedge (\Upsilon^0 + \chi) + \Omega^2_{\widehat{\mathcal{M}}}(\mathfrak{g}_i) + \mathcal{I}.$$

By construction, $\Upsilon^0 \in \mathrm{Ad}(\hat{g}_i^{-1}(A'_{\widetilde{\mathcal{M}},i})) + \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{g}_i)$. Therefore,

$$d(\Upsilon^0) \in -[\mathrm{Ad}(\hat{g}_i^{-1})(A'_{\widetilde{\mathcal{M}},i}), \hat{g}_i^{-1} d\hat{g}_i] + \Omega^2_{\widehat{\mathcal{M}}}(\mathfrak{g}_i).$$

On the other hand, by Proposition 5.9,

$$(6.4) \quad (\Upsilon^0 + \chi) \wedge (\Upsilon^0 + \chi) \in$$
$$(\mathrm{Ad}(\hat{g}_i^{-1}) A'_{\widetilde{\mathcal{M}},i} + \hat{g}_i^{-1} d\hat{g}_i) \wedge (\mathrm{Ad}(\hat{g}_i^{-1}) A'_{\widetilde{\mathcal{M}},i} + \hat{g}_i^{-1} d\hat{g}_i) + \Omega^2_{\widehat{\mathcal{M}}}(\mathfrak{g}_i) + \mathcal{I}.$$

The right hand side is equivalent to $[\mathrm{Ad}(\hat{g}_i^{-1}) A'_{\widetilde{\mathcal{M}},i}, \hat{g}_i^{-1} d\hat{g}_i] + \Omega^2_{\widehat{\mathcal{M}}}(\mathfrak{g}_i) + \mathcal{I}$. This proves (6.3).

It suffices to show that

$$(6.5) \qquad (\Upsilon^0 + \chi) \wedge (\Upsilon^0 + \chi) = \varsigma(((\Upsilon^0 + \chi) \wedge (\Upsilon^0 + \chi) + \Omega^2_{\widehat{\mathcal{M}}}(\mathfrak{g}_i))_{i \in I}) + \chi \wedge \chi,$$

since the right hand side is equal to $-d(\Upsilon^0 + \chi) \pmod{\mathcal{I}}$ by the arguments above and Corollary 5.10. By Lemma 4.10, $(\Upsilon^0 + \chi)|_{z_0=0} = \chi$. Moreover,

$$\left((\Upsilon^0 + \chi) \wedge (\Upsilon^0 + \chi)\right)|_{z_0=0} = \chi \wedge \chi.$$

Another application of Lemma 4.10 gives (6.5).

$\square$

Lemma 6.7.
$$d(\Theta_i - \mathrm{Ad}(g_i)(\chi)) \in \Omega^2_{\widehat{\mathcal{M}}}(\mathfrak{u}_i) + \mathcal{I}.$$

Proof. Throughout, we will write $\Upsilon = \Upsilon^0 + \chi$. As in the beginning of Section 5.2, we write $\hat{p}_i = \hat{g}_i g_i^{-1}$. By Proposition 5.9, we see that

$$\mathrm{Ad}(g_i)(\Upsilon) \in \mathrm{Ad}(\hat{p}_i^{-1})(A'_{\widetilde{\mathcal{M}},i}) + \hat{p}_i^{-1} d\hat{p}_i + (dg_i)g_i^{-1} + \mathcal{I}.$$

A sequence of calculations using the expression above and Lemma 6.6 produces

$$\begin{aligned}
d(\mathrm{Ad}(g_i)\Upsilon) &= [(dg_i)g_i^{-1}, \mathrm{Ad}(g_i)(\Upsilon)] + \mathrm{Ad}(g_i)d(\Upsilon) \\
&\in [(dg_i)g_i^{-1}, \mathrm{Ad}(g_i)(\Upsilon)] - \mathrm{Ad}(g_i)(\Upsilon \wedge \Upsilon) + \mathcal{I} \\
&= [(dg_i)g_i^{-1}, \mathrm{Ad}(\hat{p}_i^{-1})(A'_{\widetilde{\mathcal{M}},i}) + \hat{p}_i^{-1} d\hat{p}_i + (dg_i)g_i^{-1}] \\
&\quad - (\mathrm{Ad}(\hat{p}_i^{-1})(A'_{\widetilde{\mathcal{M}},i}) + \hat{p}_i^{-1} d\hat{p}_i + (dg_i)g_i^{-1}) \wedge (\mathrm{Ad}(\hat{p}_i^{-1})(A'_{\widetilde{\mathcal{M}},i}) + \hat{p}_i^{-1} d\hat{p}_i + (dg_i)g_i^{-1}) + \mathcal{I} \\
&= -dg_i dg_i^{-1} + d\hat{p}_i^{-1} d\hat{p}_i - [\hat{p}_i^{-1} d\hat{p}_i, \mathrm{Ad}(\hat{p}_i^{-1})A'_{\widetilde{\mathcal{M}},i}] + \mathcal{I} \\
&= -dg_i dg_i^{-1} + d\hat{p}_i^{-1} d\hat{p}_i + d(\mathrm{Ad}(\hat{p}_i^{-1})A'_{\widetilde{\mathcal{M}},i}) + \mathcal{I}.
\end{aligned}$$

The last line is equivalent to $d\Phi_i - dg_i dg_i^{-1} + \mathcal{I} \pmod{\Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{P}_i^1)}$. Therefore,

$$d(\Phi_i + (dg_i)g_i^{-1} - \mathrm{Ad}(g_i)(\Upsilon^0 + \chi) \in \Omega^2_{\widehat{\mathcal{M}}}(\mathfrak{P}_i^1) + \mathcal{I}.$$

Applying $\bar{\phi}_i$ to the equation above gives $d(\Theta_i - \mathrm{Ad}(g_i)(\chi)) \in \Omega^2_{\widehat{\mathcal{M}}}(\mathfrak{u}_i) + \mathcal{I}$.

$\square$

Proof of Theorem 5.1, part (1). As above, write $\Upsilon = \Upsilon^0 + \chi$. We have already shown that $d(\Theta_i - \mathrm{Ad}(g_i)(\chi)) \in \Omega^2_{\widehat{\mathcal{M}}}(\mathfrak{u}_i) + \mathcal{I}$. A standard, but tedious, calculation shows that

$$d(\Xi_i - [\chi, \alpha]) = \tau_\nu(d\Upsilon + \Upsilon \wedge \Upsilon) - [d\Upsilon + \Upsilon \wedge \Upsilon, \alpha] + [\Xi_i, \Upsilon] + \Omega^1_{\widehat{\mathcal{M}}}(\mathfrak{g}_i^1 \frac{dz_i}{z_i \nu}),$$

so by Lemma 6.6, $d(\Xi_i - [\chi, \alpha]) \in \mathcal{I}$.

Finally, by Theorem 3.16, we may identify trivialized framed global deformations $(\mathbf{g}, \bar{V}, \tilde{\nabla})$ on $\mathbb{P}^1 \times \Delta$ with analytic maps $\sigma : \Delta \to \widehat{\mathcal{M}}(\mathbf{x}, \mathbf{T}, \mathbf{r})$. We will show that the conditions of Corollary 4.13 are equivalent to the vanishing of $\sigma^* \mathcal{I}$. Suppose that $\sigma^* \mathcal{I} = 0$. By Corollary 5.10, $d\chi + \chi \wedge \chi = 0$. It follows that there exists an analytic solution $g \in \mathrm{GL}_n(R)$ on Δ to the integrable differential equation $dg = g.\chi$. In particular, $\chi = g^{-1} dg$.

We observe that $\sigma^* \mathcal{I} = \{0\}$ if and only if $\sigma^*(\Theta_i - \mathrm{Ad}(g_i)(\chi)) = 0$ and $\sigma^* \Xi_i - [\chi, \alpha] = 0$ for all i, and the vanishing of these forms is equivalent to conditions (1) and (2) of Corollary 4.13. Also, by Lemma 6.6, $\sigma^* \mathcal{I} = \{0\}$ implies $\sigma^*(d\Upsilon + \Upsilon \wedge \Upsilon) = 0$, which is equivalent to condition (3).

$\square$

Theorem 5.1 and Lemma 6.6 immediately show that the third conditions in Theorem 4.12 and Corollary 4.13 are redundant.

Theorem 6.8. *Let* $(\mathbf{g}, \bar{V}, \tilde{\nabla})$ *be a good framed deformation. The following statements are equivalent.*

   (1) $(\mathbf{g}, \bar{V}, \tilde{\nabla})$ *is integrable.*
   (2) $(\mathbf{g}, \bar{V}, \tilde{\nabla})$ *is* $\mathrm{GL}_n(R)$*-gauge-equivalent to a deformation satisfying the first two conditions of Theorem 4.12.*
   (3) *There exists* $g \in \mathrm{GL}_n(R)$ *such that the first two conditions of Corollary 4.13 are satisfied.*

Proof of Theorem 5.1, part (2). Since $q^*\bar{\mathcal{I}} = \mathcal{I}$, and $\mathcal{I}$ is a Pfaffian system by Theorem 5.1, one immediately sees that $\bar{\mathcal{I}}$ is Pfaffian. Specifically, the rank of $\bar{\mathcal{I}}^1$ at any point $m \in \widetilde{\mathcal{M}}$ is equal to the rank of $\mathcal{I}^1$ at a point in $q^{-1}(m)$.

In order to prove integrability, suppose that $\omega \in \bar{\mathcal{I}}^1$. Then, $d(q^*\omega) = q^*(d\omega) \in \mathcal{I}$. It now suffices to work on a neighborhood $W \subset \widetilde{\mathcal{M}}$. Choosing a local section $\sigma : W \to q^{-1}(W)$ of the principal $\mathrm{GL}_n(\mathbb{C})$-bundle, we see that $\sigma^*\mathcal{I}|_{q^{-1}(W)} = \bar{\mathcal{I}}|_W$. We deduce that $d\omega = \sigma^*q^*(d\omega) \in \bar{\mathcal{I}}|_W$.

Finally, suppose that $f : \Delta \to \widetilde{\mathcal{M}}$. Again, we choose a local section $\sigma : W \to q^{-1}(W)$ of a neighborhood W containing the image of Δ. By the first part of Theorem 5.1, f corresponds to a framed integrable deformation if and only if $f^*\sigma^*(\mathcal{I}) = 0$. However, this is equivalent to the statement that $f^*(\bar{\mathcal{I}}) = 0$ since $\sigma^*(\mathcal{I}) = \bar{\mathcal{I}}|_W$.

$\square$

7. EXAMPLE

In this section, we give an explicit example of the system of equations constructed above. We will consider a space of rank n meromorphic connections on $\mathbb{P}^1$ with m singularities of slope $\frac{1}{n}$. Let $\mathbf{x}$ be a set of m finite points. Let T_i be the standard embedding of $F((z^{1/n}))^\times$ in $\mathrm{GL}_n(F_i)$ as in (2.3), thus giving $P_i = I_i$, and let $r_i = 1$ for all i. If z is the usual coordinate on $\mathbb{P}^1$, we write $z_i = (z - \xi_i)$. Accordingly, $\varpi_i = \varpi_{T_i} = N + z_i E \in \mathfrak{P}_i^1$, where N is the regular nilpotent matrix in Jordan form and E is the elementary matrix E_{n1}. Note that $\varpi_i^{-1} = N' + \frac{1}{z_i}E'$, where N' and E' are the transposes of N and E respectively.

We choose our one form to be $\nu = dz$, so $\tau_\nu = \partial_z = \frac{d}{dz}$. Choose a point $(U_i g_i, \alpha_i)_{i \in I} \in \widehat{\mathcal{M}}$ corresponding to a connection ∇, and write $[\nabla] = \alpha\nu$. Thus, $\alpha \in \alpha_{i\nu} + \mathfrak{I}_i^1$ at each i and $\sum_{i \in I} \mathrm{Res}_i(\alpha\nu) = 0$. To simplify calculations, we assume that the normalized formal type of ∇ at ξ_i has the representative $\frac{1}{z_i}(-\frac{a_i}{n}\varpi_i^{-1} + H_T)$ under the pairing $\langle, \rangle_\nu$.

We write $A_{\widetilde{\mathcal{M}},i} = \varpi_i^{-1} da_i$ and $\mathrm{Ad}(g_i)(\alpha) = \frac{1}{z_i}(-\frac{a_i}{n}\varpi_i^{-1} - \frac{1}{n}(D_i + X_i) + H_T) + \mathfrak{g}_i$ for some traceless diagonal matrix D_i and $X_i \in \mathfrak{u}_i$. We let ϱ_i denote the residue term of $\alpha\nu$ at x_i. Explicitly, $\varrho_i = \phi_i(z_i\alpha) = \mathrm{Ad}(g^{-1})(-\frac{1}{n}(N' + D_i + X_i) + H_T)$.

Proposition 7.1. *The isomonodromy equations for the system above are*
(7.1a)

$$((dg_i)g_i^{-1}, g_i) = \left(\left[\sum_{j \in I \setminus i} \frac{1}{\xi_i - \xi_j} \mathrm{Ad}(g_i g_j^{-1})(E' da_j) \right] - \left[N' da_i + D_i \frac{da_i}{a_i} \right] + \mathfrak{u}_i, g_i \right)$$

$$\in (\mathfrak{u}_i \setminus \mathfrak{gl}_n(\mathbb{C})) \times_{U_i} \mathrm{GL}_n(\mathbb{C}) \ and$$

(7.1b)
$$d\varrho_i = \sum_{j \in I \setminus \{i\}} \left(\frac{1}{\xi_j - \xi_i} \left([\mathrm{Ad}(g_j^{-1})E', \varrho_i] da_j + [\mathrm{Ad}(g_i^{-1})E', \varrho_j] da_i \right) \right.$$

$$\left. + \frac{1}{n} \frac{1}{(\xi_j - \xi_i)^2} [\mathrm{Ad}(g_i^{-1})E', \mathrm{Ad}(g_j^{-1})E'] d(a_i a_j) \right).$$

Remark 7.2. In this system of partial differential equations, the independent variables are the a_i's, which are essentially the formal types, while the dependent variables are the framings g_i and the residues ϱ_i of $\alpha\nu$ (which can be written explicitly in terms of g_i, D_i, and X_i).

Proof. First, observe that by definition,

$$\Phi_i = (\varpi_i^{-1} da_i + D_i \frac{da_i}{a_i}) + \Omega^1_\Delta(\mathfrak{I}_i^1).$$

Therefore,

$$\Upsilon^0 = \sum_{i \in I} \mathrm{Ad}(g_i^{-1}) \left(z_i^{-1} E' da_i \right),$$

and

$$\phi_i(\Upsilon^0) = \sum_{j \in I \setminus \{i\}} \frac{1}{\xi_i - \xi_j} \mathrm{Ad}(g_j^{-1})(E' da_j).$$

We conclude that equation (1) from Theorem 4.12 is equivalent to (7.1a).

Now, we consider equation (2) of Theorem 4.12. At ξ_i, applying $\mathrm{Ad}(g_i)$ to the principal part of the curvature gives us

(7.2) $- z_i^{-2} E' da_i \in$

$$- \frac{1}{n z_i}(\varpi_i^{-1} da_i + dD_i + dX_i) + [\mathrm{Ad}(g_i)(\Upsilon^0) - (dg_i)g_i^{-1}, \mathrm{Ad}(g_i)(\alpha)] + \Omega^1_\Delta(\mathfrak{g}_i).$$

First, we calculate $[z_i^{-1} E', \mathrm{Ad}(g_i)(\alpha)] + \mathfrak{g}_i$. Observe that $[z_i^{-1} E', \mathrm{Ad}(g_i)(\alpha)dz]$ is a one form on $\mathbb{P}^1_\Delta$ with poles along $\mathbf{x}$. Thus,

(7.3) $[z_i^{-1} E', \mathrm{Ad}(g_i)(\alpha)dz] \in (-\frac{1}{n z_i}[z_i^{-1} E', N' a_i + D_i + X_i - n H_T] + \frac{1}{z_i} Y) dz + \mathfrak{g}_i dz,$

where Y is the residue term. By the residue theorem, $Y + \sum_{j \in I} \mathrm{Res}_j([z_i^{-1} E', \mathrm{Ad}(g_i)(\alpha)dz]) = 0$. We conclude that

$$Y = \frac{1}{n} \sum_{j \in I \setminus \{i\}} \mathrm{Res}_j([z_i^{-1} E', \mathrm{Ad}(g_i g_j^{-1})(\varpi_i^{-1} a_j + D_j + X_j - n H_T)] \frac{dz}{z_j})$$

$$= \frac{1}{n} \sum_{j \in I \setminus \{i\}} \frac{1}{\xi_j - \xi_i} [E', \mathrm{Ad}(g_i g_j^{-1})(N' a_j + D_j + X_j - n H_T)] - \frac{1}{(\xi_j - \xi_i)^2} [E', \mathrm{Ad}(g_i g_j^{-1}) E' a_j].$$

By Proposition 4.2 we see that

(7.4) $$- z_i^{-2} E' da_i = - \frac{1}{n z_i}(\varpi_i^{-1} da_i) + \frac{1}{z_i}[\varpi_i^{-1} da_i, H_T].$$

Comparing (7.2), (7.3), and (7.4), we obtain the condition

$$(7.5) \quad -\frac{1}{nz_i}\left(d(D_i + X_i) + [z_i^{-1}E'da_i, N'a_i + D_i + X_i - nH_T]\right) +$$

$$\frac{1}{z_i}Yda_i + [\mathrm{Ad}(g_i)(\Upsilon^0) - z_i^{-1}E'da_i - (dg_i)g_i^{-1}, \mathrm{Ad}(g_i)(\alpha)] - \frac{1}{z_i}[\varpi_i^{-1}da_i, H_T] \in \Omega^1_\Delta(\mathfrak{g}_i).$$

Now, by (7.1a),

$$(7.6) \quad [\mathrm{Ad}(g_i)\Upsilon^0 - z_i^{-1}E'da_i - (dg_i)g_i^{-1}, z_i^{-1}E'a_i] \in$$

$$[\phi_i(\mathrm{Ad}(g_i)\Upsilon^0) - (dg_i)g_i^{-1}, z_i^{-1}E'a_i] - \sum_{j \in I\setminus\{i\}} \frac{1}{(\xi_j - \xi_i)^2}[\mathrm{Ad}(g_ig_j^{-1})(E'), E']a_ida_j + \Omega^1_\Delta(\mathfrak{g}_i^1)$$

$$= [N'da_i + D_i\frac{da_i}{a_i}, z_i^{-1}E'a_i] - \sum_{j \in I\setminus\{i\}} \frac{1}{(\xi_j - \xi_i)^2}[\mathrm{Ad}(g_ig_j^{-1})(E'), E']a_ida_j + \Omega^1_\Delta(\mathfrak{g}_i^1).$$

On the other hand,

$$(7.7) \quad [\mathrm{Ad}(g_i)\Upsilon^0 - z_i^{-1}E'da_i, N'a_i + D_i + X_i - nH_T] \in$$

$$[\phi_i(\mathrm{Ad}(g_i)\Upsilon^0), N'a_i + D_i + X_i - nH_T] + \Omega^1_\Delta(\mathfrak{g}_i^1)$$

$$= [\sum_{j \in I\setminus\{i\}} \frac{1}{\xi_i - \xi_j}\mathrm{Ad}(g_ig_j^{-1})(E'da_j), N'a_i + D_i + X_i - nH_T] + \Omega^1_\Delta(\mathfrak{g}_i^1).$$

Finally,

$$(7.8) \qquad\qquad [z_i^{-1}E'da_i, H_T] - [\varpi_i^{-1}da_i, H_T] = [-N'da_i, H_T].$$

We substitute (7.6), (7.7), and (7.8) into (7.5):

$$d(D_i + X_i) - [(dg_i)g_i^{-1}, N'a_i + D_i + X_i - nH_T] \equiv [-N'da_i, nH_T] +$$

$$\sum_{j \in I\setminus\{i\}} \frac{1}{\xi_j - \xi_i}[E'da_i, \mathrm{Ad}(g_ig_j^{-1})(N'a_j + D_j + X_j - nH_T)] - \frac{1}{(\xi_j - \xi_i)^2}[E', \mathrm{Ad}(g_ig_j^{-1})E']a_jda_i$$

$$+ \sum_{j \in I\setminus\{i\}} \frac{1}{(\xi_j - \xi_i)^2}[\mathrm{Ad}(g_ig_j^{-1})(E'), E']a_ida_j$$

$$- \sum_{j \in I\setminus\{i\}} \frac{1}{\xi_i - \xi_j}[\mathrm{Ad}(g_ig_j^{-1})(E'da_j), N'a_i + D_i + X_i - nH_T] \quad (\mathrm{mod}\ \Omega^1_\Delta(\mathfrak{g}_i^1)).$$

Since $[N'da_i, H_T] = \frac{1}{n}N'da_i$, applying $\mathrm{Ad}(g_i^{-1})$ and dividing both sides by $-n$ shows that this equation is equivalent to (7.1b). $\qquad\square$

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