LIE GROUP ACTIONS ON SIMPLE ALGEBRAS

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Abstract. Let G be a connected Lie group acting by algebra automorphisms on a finite-dimensional complex central simple algebra A. The algebra A is isomorphic to the endomorphism algebra of a projective representation V of G. We study the invariant subalgebras of A. In particular, we show that if V is irreducible, then the invariant subalgebras appear in dual pairs arising from factorizations of V. We apply this result to find a very simple description of the invariant subalgebras when G is compact. For example, if G is simple, the only stable subalgebras are A, $\mathbf{C}$, and $\{0\}$. We also determine the invariant ideals of A.

1. Introduction

Let G be a group and V a finite-dimensional complex representation of G. A fundamental problem in representation theory is to classify the subrepresentations of V, or in other words, to determine those subspaces of V which are stabilized by the group action. For the case when G is a compact group, the solution goes back to Hermann Weyl. The representation can be decomposed canonically into a direct sum of subrepresentations $V = U_1 \oplus \cdots \oplus U_m$, where each U_i is the direct sum of n_i copies of an irreducible representation V_i and the V_i's are pairwise nonisomorphic. The G-invariant subspaces of U_i are parametrized by subspaces of $\mathbf{C}^{n_i}$ while the subrepresentations of V are direct sums of subrepresentations of the U_i's which may be chosen independently. As long as a decomposition of V into irreducible components is given explicitly (which may be very difficult in practice), this classification is also entirely explicit.

In this paper, we will consider an analogue of this problem, motivated by application to physics, in which G is a connected Lie (or topological) group and the representation V is replaced by a continuous G-algebra A, i.e. a unital associative algebra on which G acts by continuous algebra automorphisms. In this context, it is natural to study the subrepresentations of A with multiplicative significance. In particular, we would like to understand the G-invariant subalgebras and ideals of A. These problems are much more difficult than the classification of subrepresentations, and it is unreasonable to expect to find a way of determining G-invariant subalgebras and ideals that works for all G-algebras. Indeed, if we let G act trivially on A, then this result would give a uniform way of classifying ideals and subalgebras. It is thus necessary to limit the class of algebras under consideration.

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In this paper, we restrict attention to simple algebras over $\mathbf{C}$, that is, complex matrix algebras. When the unique simple (left) module V of A is an irreducible projective representation, we show that all nonzero invariant subalgebras are themselves simple and appear in dual pairs. In particular, we show that a nonzero representation which is closed under ring multiplication must contain the identity. Moreover, we give an explicit description of these dual pairs in terms of V. We then apply these results when G is a compact connected Lie group to obtain a particularly elegant solution of the classification problem due to Etingof. For example, when G is simple, the only invariant subalgebras are $\{0\}$, $\mathbf{C}$, and V. In the final section of the paper, we classify the invariant ideals of A without any restrictions on V.

Our initial motivation for looking at these questions came from a problem in solid state physics. The study of G-actions on endomorphism algebras is important in understanding how physical properties such as conductivity, elasticity, and piezoelectricity of a composite material depend on the properties of its constituents. These physical characteristics are described by elements of a symmetric tensor space $\mathrm{Sym}^2(T)$, where T is a certain real representation of the rotation group $SO(n)$. In general, a property of a composite depends heavily on the microstructure, i.e. the arrangement of the component materials. Let $M \subset \mathrm{Sym}^2(T)$ be the set of all possible values of a fixed property for composites made with their constituents taken in prescribed volume fractions. Typically, M is the closure of an open set in $\mathrm{Sym}^2(T)$ and may be described by a system of inequalities, so that away from the boundary of M, it is possible to make any desired small change in the property by varying the microgeometry. However, in certain unusual situations, some of the inequalities become equations, determining a proper closed submanifold E in which M is locally closed. The submanifold E and also the equations defining E are called exact relations for the property. The variability of the property with microstructure is thus drastically reduced when an exact relation is present. Recent work of Grabovsky, Milton, and Sage has shown how to classify exact relations in terms of the multiplication of $SO(n)$-subrepresentations in the endomorphism algebra $\mathrm{End}(T)$; in particular, invariant algebras and ideals of this simple algebra have great physical significance [GS, GMS].

It is a great pleasure to thank Yury Grabovsky for first bringing these problems to my attention and for many illuminating discussion on their physical significance. I would also like to thank Pavel Etingof for letting me use an unpublished result on invariant subalgebras of compact connected Lie groups.

2. Invariant subalgebras

Let A be a finite-dimensional (central) simple algebra over $\mathbf{C}$, and let V be a simple (left) A-module. The module V is unique up to isomorphism and is a finite-dimensional complex vector space. It is well-known that A is isomorphic to the matrix algebra $\mathrm{End}(V) \stackrel{\mathrm{def}}{=} \mathrm{End}_{\mathbf{C}}(V)$, and from now on, we assume without loss of generality that $A = \mathrm{End}(V)$.

It is easy to construct examples of simple algebras on which a group G acts by algebra automorphisms. Recall that a mapping $\rho : G \to GL(V)$ is called a projective representation of G if $\rho(1) = 1_V$ and if there exists $\alpha : G \times G \to \mathbf{C}^*$ such that $\rho(xy) = \alpha(x, y)\rho(x)\rho(y)$ for all $x, y \in G$. We will sometimes denote the linear map $\rho(g)$ by $\bar{g}$. (Equivalently, we can view a projective representation as a homomorphism $G \to PGL(V)$.) The homomorphism $\pi : G \to GL(A)$ then makes A into a (linear) representation of G with $(\pi(g)f)(v) = \rho(g)(f(\rho(g)^{-1}(v))$ for all $g \in G$, $f \in A$, and $v \in V$. Moreover, the linear map $\pi(g)$ is in fact an algebra automorphism. It turns out that all complex simple G-algebras are of this type.

Proposition 2.1. *Suppose that G acts on $A = End(V)$ by algebra automorphisms, i.e. A is a representation of G via a homomorphism $G \xrightarrow{\pi} Aut(A)$. Then V is a projective representation of G determined up to projective equivalence, and the G-action on A is the natural action induced by the projective G-action on V.*

Proof. Any automorphism of A is inner by the Skolem-Noether theorem. Hence, we obtain a function $\hat{\rho} : G \to A^{\times} \subseteq GL(V)$ such that $\pi(g)(a) = \hat{\rho}(g)a\hat{\rho}(g)^{-1}$ for all $g \in G$ and $a \in A$. Since $\pi(1) = 1_A$, we have $\hat{\rho}(1) \in Z(A)^{\times} = \mathbf{C}^*$. Setting $\rho(g) = \hat{\rho}(g)/\hat{\rho}(1)$ gives $\rho(1) = 1_V$. Also, the equation $\pi(gh) = \pi(g)\pi(h)$ implies that $\rho(gh)\rho(h)^{-1}\rho(g)^{-1}$ is central and therefore a nonzero multiple of the identity. It follows that (V, ρ) is a projective representation of G giving rise to π. □

From now on, let G be a connected Lie group and A a continuous G-algebra. The identification of $\mathrm{Aut}(A)$ with $PGL(V)$ shows that a continuous homomorphism $G \to \mathrm{Aut}(A) \subset GL(A)$ gives rise to a continuous projective representation $G \to PGL(V)$ and vice versa. In particular, the projective representation V associated to A is continuous.

In general, invariant subalgebras of a simple G-algebra can be very badly behaved. For example, if we let G act trivially on $\mathrm{End}(V)$, then every subalgebra is invariant. This means that if V has dimension n, then $\mathrm{End}(V)$ contains every n-dimensional $\mathbf{C}$-algebra as an invariant subalgebra. We will therefore need to place additional restrictions on the G-algebra A.

For the remainder of this section, we assume that the projective representation V associated to the G-algebra A is irreducible. We will call such an algebra *G-simple.*

First, we show that all unital invariant subalgebras of A are simple.

Proposition 2.2. *Let B be an invariant subalgebra of A containing 1_A. Then B is also simple.*

Proof. The inclusion of B in A makes the A-module V into a B-module. Let U be a simple B-submodule of V; for example, take U to be a B-submodule of minimal dimension. Consider the translate $\bar{g}U$ for $g \in G$. Note that the G-invariance of B implies that

$$b\bar{g}(u) = \bar{g}\bar{g}^{-1}b\bar{g}(u) = \bar{g}(g^{-1} \cdot b)(u) \in \bar{g}U \tag{1}$$

for all $b \in B$ and $u \in U$. Here, we have used the fact that $\overline{g^{-1}} = \alpha(g, g^{-1})\bar{g}^{-1}$, where α is the cocycle defined by (V, ρ). Thus, $\bar{g}U$ is a B-submodule of V. Moreover, $\bar{g}U$ is simple, since the same argument shows that if W is a submodule of $\bar{g}U$, then $\overline{g^{-1}}W$ is a submodule of U. The sum $\sum_{g \in G} \bar{g}U$ is evidently a nonzero G-invariant subspace of V, and by irreducibility, $V = \sum_{g \in G} \bar{g}U$. Thus, V is a semisimple B-module, and we can choose $g_1, \dots, g_l \in G$ such that $V = \oplus_{i=1}^{l} \bar{g}_i U$. Without loss of generality, set $g_1 = 1_V$.

Let $u_1, \dots, u_k$ be an F-basis for U. The map $B \to \oplus_{i=1}^{l} k(\overline{g_i}U)$ given by $b \mapsto (b\overline{g_1}u_1, \dots, b\overline{g_1}u_k, \dots, b\overline{g_l}u_1, \dots, b\overline{g_l}u_k)$ is a B-homomorphism. If b is in the kernel, then b kills an F-basis of V, and since $b \in A \subseteq \mathrm{End}_F(V)$, we have $b = 0$; hence, the map is injective. This shows that B is a semisimple F-algebra, and any simple B-module is isomorphic to $\bar{g}U$ for some $g \in G$.

Write $B = B_1 \oplus \cdots \oplus B_t$, where the B_i are simple. The restriction of the G-action π to B gives rise to a permutation representation of G on the set of B_i's because the algebra automorphism $\pi(g)$ must permute the minimal two-sided ideals of B. More explicitly, let $X = \{e_1, \dots, e_t\}$ with $e_i = 1_{B_i}$ be the set of central primitive idempotents of B. Since e_i is the unique nonzero idempotent in the center of B_i, it is clear that if $\pi(g)(B_i) = B_j$, then $\pi(g)(e_i) = e_j$. We thus obtain a homomorphism $\bar{\pi}_B : G \to S_t$, where we have identified $S(X)$ with S_t in the obvious way.

This permutation representation is transitive. Indeed, choose $g \in G$ such that $\bar{g}U$ is a simple B_i module. By definition, e_1 acts on U by the identity map, so for all $u \in U$, we have $(g \cdot e_1)(\bar{g}u) = \bar{g}(e_1(\bar{g}^{-1}(\bar{g}u) = \bar{g}(e_1(u)) = \bar{g}(u)$. Since e_i is the unique central primitive idempotent acting as the identity on $\bar{g}U$, this implies that $g \cdot e_1 = e_i$.

Thus, we see that G acts transitively and continuously on the finite set of Wedderburn components of B. But G is connected, implying that B has only one component, i.e. B is simple. □

Remark. If G is not connected, the conclusion of the theorem is not true. For example, let $G = S_3$, the symmetric group on three elements, and V its two-dimensional irreducible representation. Then $\mathrm{End}(V) \cong \mathbf{C} \oplus W \oplus V$, where W is the alternating representation. The submodule $\mathbf{C} \oplus W$ is an invariant subalgebras isomorphic to the algebra $\mathbf{C} \oplus \mathbf{C}$.

In fact, we can use this result to show that all nonzero invariant subalgebras are unital. First, we need a lemma.

Lemma 2.3. *For $t \geq 2$, the matrix algebra $M_t(\mathbf{C})$ has no nonunital subalgebras of codimension one.*

Proof. Suppose that Q is a codimension one nonunital subalgebra. First note that any element of Q must be singular. To see this, take $q \in Q$ invertible. Since $\det q \neq 0$, the Cayley-Hamilton theorem implies that q^{-1} can be expressed as a polynomial in q and so lies in Q. But then $1_A = qq^{-1} \in Q$, a contradiction. Thus, $Q \subseteq V(\det)$, the hypersurface of $M_t(F)$ cut out by the determinant. However, Q is a codimension one linear subvariety, so $Q = V(f)$ for some homogeneous degree one polynomial f. As a result, f divides det, and this cannot be true, since the determinant is an irreducible polynomial of degree t. □

Now, let Q be an nonunital invariant subalgebra. Then $Q' = Q + F1_A$ is a unital invariant subalgebra. By the previous proposition, Q' is simple, hence isomorphic to $M_t(F)$ for some $t \geq 1$. Since Q is a codimension one nonunital subalgebra, the lemma forces $t = 1$, and so $Q = \{0\}$. Applying the lemma finishes the proof. We have thus proved

Proposition 2.4. *Any nonzero subrepresentation of the continuous G-algebra A closed under multiplication must contain the identity. Equivalently, $\{0\}$ is the only nonunital invariant subalgebra of A.*

We now show how to construct unital invariant subalgebras. First observe that if B is invariant, then so is $Z_A(B)$, the centralizer of B. This follows from the fact that $(g \cdot z)b = g \cdot (z(g^{-1} \cdot b)) = g \cdot ((g^{-1} \cdot b)z) = b(g \cdot z)$ for all $g \in G$, $b \in B$, and $z \in Z_A(B)$. By the double centralizer theorem, $B = Z_A(Z_A(B))$, so invariant subalgebras appear in dual pairs $(B, Z_A(B))$. Since B is simple, $Z_A(B)$ is as well; in fact $Z_A(B)$ is isomorphic to $M_m(\mathbf{C})$, where $V \cong mU$ as B-modules. Moreover, the natural map $B \otimes_{\mathbf{C}} Z_A(B) \to A$ is a G-algebra isomorphism.

Let W be the simple left module for $Z_A(B)$; it is also a projective representation of G. The tensor product $U \otimes W$ is a simple module for $B \otimes_{\mathbf{C}} Z_A(B)$. Thus, we obtain a G-isomorphism $U \otimes W \to V$ inducing the G-algebra isomorphism $B \otimes_{\mathbf{C}} Z_A(B) = \mathrm{End}(U) \otimes \mathrm{End}(W) \to \mathrm{End}(V) = A$. It is obvious that U and W are irreducible projective representations, so we see that each dual pair of invariant (nonzero) subalgebras of A is induced by a factorization of V into the tensor product of irreducible projective representations.

Conversely, any such factorization $V \cong U \otimes W$ gives rise to a dual pair of invariant subalgebras given by the images of $\mathrm{End}(U) \otimes \mathbf{C}$ and $\mathbf{C} \otimes \mathrm{End}(W)$ under the isomorphism $\mathrm{End}(U) \otimes \mathrm{End}(W) \to \mathrm{End}(V)$. Finally, suppose another factorization $V \cong U' \otimes W'$ induces the same dual pair (taken in the same order). The the composite G-algebra isomorphism $\mathrm{End}(U) \otimes \mathrm{End}(W) \to \mathrm{End}(U') \otimes \mathrm{End}(W')$ gives G-algebra isomorphisms $\mathrm{End}(U) \to \mathrm{End}(U')$ and $\mathrm{End}(W) \to \mathrm{End}(W')$. This implies that U' and W' are projectively equivalent to U and W respectively. Summing up, we have the following theorem.

Theorem 2.5. *Let $A = End(V)$ be G-simple. Suppose that $V \cong U \otimes W$, where U and W are irreducible projective representations of G. Then $A \cong End(U) \otimes End(W)$, and the images of the two factors in A are a dual pair of invariant subalgebras. Conversely, all nonzero invariant subalgebras arise in this way. Indeed, there is a one-to-one correspondence between unordered pairs of irreducible projective representations $\{U, W\}$ modulo projective equivalence such that $V \cong U \otimes W$ and dual pairs of nonzero invariant subalgebras.*

Remark. The analogous result holds with the same proof when G is a connected topological group.

In general, finding all (or even some) factorizations for a given of V into a tensor product of projective representations is a difficult problem. (For a discussion, see [St].) As a first example, suppose G is a simply connected complex Lie group. Note that projective representations of G are in fact linear. Every continuous representation can be expressed uniquely as the tensor product of one that is holomorphic and one that is anti-holomorphic [St]. Thus, every continuous G-algebra comes equipped with a natural dual pair of invariant subalgebras arising from the holomorphic and anti-holomorphic factors.

In the case that G is a compact connected Lie group, the theorem can be applied to obtain a complete classification of invariant subalgebras of a continuous G-algebra. We start with a lemma.

Lemma 2.6. *Suppose that G is a simple compact connected Lie group, and let $V(\lambda)$ and $V(\mu)$ be irreducible representations with highest weights λ and μ. Then $V(\lambda) \otimes V(\mu)$ is irreducible if and only if λ or μ is 0.*

Proof. Since $V(\lambda + \mu)$ is a component of $V(\lambda) \otimes V(\mu)$, it suffices to compare the dimension of these representations. The Weyl dimension formula states that

$$\dim V(\lambda) = \prod_{\alpha \in R^+} \frac{\langle \alpha, \lambda + \rho \rangle}{\langle \alpha, \rho \rangle},$$

where R^+ is the set of positive roots, ρ is half the sum of the positive roots, and $\langle\ ,\ \rangle$ is a Weyl group invariant inner product on the $\mathbf{R}$-span of the roots. The equation $\langle\alpha, \lambda+\mu+\rho\rangle\langle\alpha,\rho\rangle + \langle\alpha,\lambda\rangle\langle\alpha,\mu\rangle = \langle\alpha,\lambda+\rho\rangle\langle\alpha,\mu+\rho\rangle$ shows that

$$\frac{\langle\alpha,\lambda+\mu+\rho\rangle}{\langle\alpha,\rho\rangle} \leq \frac{\langle\alpha,\lambda+\rho\rangle}{\langle\alpha,\rho\rangle}\frac{\langle\alpha,\mu+\rho\rangle}{\langle\alpha,\rho\rangle},$$

with equality if and only if $\langle\alpha,\lambda\rangle\langle\alpha,\mu\rangle = 0$. Here we have used the fact that $\langle\alpha,\lambda\rangle \geq 0$ and $\langle\alpha,\rho\rangle > 0$ for every positive root α and dominant weight λ. If β is the highest root, then $\langle\beta,\nu\rangle > 0$ for any nonzero dominant weight ν. Multiplying over all positive roots, it follows easily that $\dim V(\lambda+\mu) < \dim V(\lambda)\dim V(\mu)$ if and only if both λ and μ are nonzero. □

Remarks. 1. This result is known, but we have included a proof for lack of a satisfactory reference.

2. The analogue of this lemma does not hold for simple algebraic groups in positive characteristic. In the notation of the Atlas of Finite Groups, $U_4(2) = PSp_4(\mathbf{F}_3)$ has irreducible representations χ_3 and χ_4 of dimensions five and six respectively such that $\chi_3 \otimes \chi_4 \cong \chi_{12}$ is also irreducible [C].

Let G be a compact connected Lie group. It is well-known that the universal covering group of G is of the form $\tilde{G} = G_1 \times \cdots \times G_s \times \mathbf{R}^n$, where each G_i is a simple, simply connected, compact Lie group. Let V be an irreducible projective representation of G. Then V can be lifted to an irreducible representation of $\tilde{G}$, which can be expressed as $V_1 \otimes \ldots \otimes V_s \otimes L$, where V_i is a complex irreducible representation of G_i and L is a character of $\mathbf{R}^n$. This means that V is projectively equivalent to $\tilde{V} = V_1 \otimes \ldots \otimes V_s$. Moreover, simple Lie groups have no nontrivial characters, so projective and linear equivalence are the same for representations of $G_1 \times \cdots \times G_s$. The lemma shows that any factorization of $\tilde{V} = W \otimes W'$ into the tensor product of two representations of $\tilde{G}$ must have W and W' as complementary partial products of $V_1 \otimes \ldots \otimes V_s$. More precisely, let $I = \{i \mid V_i \neq \mathbf{C}\}$ and take $J \subset I$. Set $W_J = \bigotimes_{i=1}^s W_{J_i}$ and $W'_J = \bigotimes_{i=1}^s W'_{J_i}$, where W_{J_i} is V_i if $i \in J$ and $\mathbf{C}$ otherwise and W'_{J_i} is V_i if $i \notin J$ and $\mathbf{C}$ otherwise. We get a factorization $\tilde{V} = W_J \otimes W'_J$, and $J \mapsto W_J$ gives a one-to-one correspondence between the subsets of I and the factors of $\tilde{V}$. This observation combined with Theorem 2.5 proves the following theorem due to Etingof:

Theorem 2.7. *Let G be a compact connected Lie group, and let $A = End_{\mathbf{C}}(V)$ where V is an irreducible projective representation of G projectively equivalent to $V_1 \otimes \ldots \otimes V_s$. Then there is a bijective correspondence between $\mathcal{P}(I)$, the power set of $I = \{i \mid V_i \neq \mathbf{C}\}$, and the set of nonzero invariant subalgebras of A, given by $J \mapsto End(W_J)$. Moreover, the duality operator corresponds to taking complements in $\mathcal{P}(I)$, i.e. it is given by $End(W_J) \mapsto End(W_{I-J})$.*

Corollary 2.8. *There are exactly $2^{|I|}+1$ subrepresentations of $End(V)$ which are closed under matrix multiplication: $2^{|I|}$ unital subalgebras and $\{0\}$.*

In particular, if G is a simple compact connected Lie group, then no G-simple algebra has any nontrivial invariant subalgebras.

3. Invariant ideals

In this section, we briefly describe the G-invariant ideals of a simple G-algebra. We no longer assume that A is G-simple, so $A \cong \mathrm{End}(V)$ where V is an arbitrary finite-dimensional projective representation of G.

We now recall the ideal structure of A. Let $\mathcal{S}(V)$ denote the set of subspaces of V partially ordered by inclusion. This poset is in fact a complete lattice, with the greatest lower bound and least upper bound of a collection of subspaces given by their intersection and sum respectively. Similarly, the sets $\mathcal{L}(A)$ and $\mathcal{R}(A)$ of left and right ideals of A are complete lattices. It will be convenient to work with the dual lattice $\mathcal{L}(A)^*$ of left ideals under reverse inclusion (and with the supremum and infimum reversed). If L is a subspace of V, we define the annihilator and coannihilator of L by $\mathrm{Ann}(L) = \{f \in A \mid f(L) = 0\}$ and $\mathrm{Coann}(L) = \{f \in A \mid f(V) \subset L\}$; these are respectively left and right ideals of A. We then have the well-known fact that all left and right ideals of A are of this form.

Proposition 3.1. *The maps $\mathcal{S}(V) \overset{Ann}{\to} \mathcal{L}(A)^*$ and $\mathcal{S}(V) \overset{Coann}{\longrightarrow} \mathcal{R}(A)$ are isomorphisms of complete lattices. The inverses are given by $I \mapsto \bigcap_{f\in I} Ker(f)$ and $J \mapsto \sum_{f\in J} f(V)$, where $I \in \mathcal{L}(A)$ and $J \in \mathcal{R}(A)$.*

Remark. In matrix language, this simply says that a left ideal consists of all matrices (with respect to some basis depending on the ideal) with zeroes in given columns while a right ideal consists of all matrices with zeros in given rows.

Let $\mathcal{S}_G(V) \subset \mathcal{S}(V)$ be the complete sublattice of all subspaces of V preserved by the G-action on V. Similarly, we define the complete sublattices $\mathcal{L}_G(A) \subset \mathcal{L}(A)$ and $\mathcal{R}_G(A) \subset \mathcal{R}(A)$ of G-invariant left and right ideals of A. It is natural to conjecture that the sublattices $\mathcal{L}_G(A)$ and $\mathcal{R}_G(A)$ are just the images of $\mathcal{S}_G(V)$ under the above isomorphisms, i.e. invariant left and right ideals are annihilators and coannihilators respectively of subrepresentations of V. This is indeed the case.

Theorem 3.2. *The restrictions of the maps Ann and Coann define isomorphisms of complete lattices $\mathcal{S}_G(V) \overset{Ann}{\to} \mathcal{L}_G(A)^*$ and $\mathcal{S}_G(V) \overset{Coann}{\longrightarrow} \mathcal{R}_G(A)$.*

Proof. In order to prove the first isomorphism, it suffices to show that $\mathrm{Ann}(\mathcal{S}_G(V)) \subset \mathcal{L}_G(A)^*$ and $\mathrm{Ann}^{-1}(\mathcal{L}_G(A)^*) \subset \mathcal{S}_G(V)$. If L is a subrepresentation of V and $f \in \mathrm{Ann}(L)$, then $(g \cdot f)(v) = \bar{g}(f(\bar{g}^{-1}(v))) = \bar{g}(0) = 0$ for all $g \in G$ and $v \in L$. Thus, $\mathrm{Ann}(L)$ is G-invariant. Conversely, if I is an invariant left ideal and $v \in \mathrm{Ann}^{-1}(I) = \bigcap_{f\in I} \mathrm{Ker}(f)$, then we also have $v \in \bigcap_{f\in I} \mathrm{Ker}(g \cdot f)$. Since $\rho(g)$ is bijective, this gives $f(\bar{g}^{-1}v) = 0$ for all $g \in G$ and $f \in I$. It follows that $\mathrm{Ann}^{-1}(I)$ is G-invariant.

The proof for invariant right ideals is similar. □

Remarks. 1. Since A is simple, the only two-sided ideals are $\{0\}$ and A which are of course G-invariant. However, it is a general fact that if B is an arbitrary G-algebra on which G acts by inner automorphisms, then all two-sided ideals are G-invariant. Indeed, if I is a two-sided ideal and the action of g on B is given by conjugation by $b_g \in B^\times$, then $gI = b_g I b_g^{-1} \subset I$.

2. Suppose that V is a completely reducible linear representation of G, say $V \cong n_1V_1 \oplus \cdots \oplus n_mV_m$ where the V_i's are pairwise nonisomorphic irreducible representations. Then the G-invariant left (and right) ideals of $\mathrm{End}(V)$ are parametrized by $\prod_{i=1}^m \{\text{subspaces of } \mathbf{C}^{n_i}\}$.

3. An analogous result holds for the space of linear maps between two linear representations of G. If V and W are two representations of G, then $\mathrm{Hom}(V, W)$ is a representation whose G-action is compatible with the $(\mathrm{End}(W), \mathrm{End}(V))$-bimodule structure. A similar proof shows that the lattice of subrepresentations of V is isomorphic to the lattice of invariant left $\mathrm{End}(W)$-submodules of $\mathrm{Hom}(V, W)$ while the lattice of subrepresentations of W is isomorphic to the lattice of invariant right $\mathrm{End}(V)$-submodules of $\mathrm{Hom}(V, W)$ under reverse inclusion.

This theorem allows us to characterize certain properties of representations in terms of the associated endomorphism algebras.

Corollary 3.3. (1) *The projective representation V is irreducible if and only if End(V) has no proper invariant one-sided ideals.*
(2) *Suppose that V is completely reducible. Then V is multiplicity free if and only if End(V) has a finite number of invariant one-sided ideals.*

Proof. The first statement is clear from the theorem. The other follows from the second remark and the fact that a complex vector space has an infinite number of subspaces if and only if it has dimension larger than one. □

Note that in spite of the strong connection between subrepresentations and invariant ideals, the group action on a subrepresentation does not determine the action on the corresponding left and right invariant ideals or vice versa.

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