Moy-Prasad filtrations and flat G-bundles on curves

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Geometric and categorical representation theory
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Overview

New approach (joint with C. Bremer) to the local theory of flat G-bundles over curves, i.e. formal flat G-bundles, using methods from representation theory:
Systematic study of the "leading terms" of the flat structures with respect to Moy-Prasad filtrations

Two main motivations:

- Moduli spaces and the isomonodromy problem for meromorphic flat G-bundles with nondiagonalizable irregular singularities (nonabelian Hodge theory)
- The wild ramification case of the geometric Langlands program

Minimal K-types

G reductive group over p-adic field k, V admissible irrep

Classical idea (Bushnell, Fröhlich, Kutzko, Moy-Prasad): study in terms of "generalized congruence subgroups"

x filtration on parahoric $P_x \subset G(k)$, r level, ϕ character of $P_{x,r}/P_{x,r+}$

Definition: A K-type of V is a triple (x, r, ϕ) with $V^{P_{x,r+}} \neq 0$ and containing ϕ. It is called a minimal K-type if ϕ satisfies a certain nondegeneracy condition.
Level of minimal K-types in V always the same, the depth of V.

Might hope that K-types have a role on the Galois side of the Langlands correspondence, but no such is known.

Geometric Langlands

Replace k by $F = \mathbb{C}((z))$. $\mathrm{Spec}(F) = \Delta^\times$, formal punctured disk

Local geometric Langlands according to Frenkel-Gaitsgory:

$$\left\{ \begin{matrix} \text{flat } {}^L G\text{-bundles} \\ \text{on } \Delta^\times \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{appropriate categorical representations} \\ \text{of loop group } G(F) \end{matrix} \right\}$$

The categorical reps should be certain categories of affine Kac-Moody algebra reps at the critical level.

One might expect that an analogous theory of minimal K-types should hold on the rep-theoretic side. (In progress)

More surprisingly, minimal K-types (or rather, closely related objects called fundamental strata) show up on the Galois side).

Expect payoff–Induced Langlands duality on fundamental strata.

Flat G-bundles

X curve, $\mathcal{O}$ structure sheaf of $\mathbb{P}^1(\mathbb{C})$, K function field, $\Omega^1_{K/\mathbb{C}}$ meromorphic 1-forms

Recall: A flat GL_n-bundle on $\mathbb{P}^1(\mathbb{C})$ is a rank n trivializable vector bundle with a meromorphic connection, i.e., a $\mathbb{C}$-derivation $\nabla : V \to V \otimes_{\mathcal{O}} \Omega^1_{K/\mathbb{C}}$.

If one fixes a trivialization $\phi : V \to V^{\mathrm{triv}}$, then

$$\nabla = d + [\nabla]_\phi, \text{ where } [\nabla]_\phi \in M_n(\Omega^1_{K/\mathbb{C}}).$$

Definition: A flat G-bundle on X is a trivializable principal G-bundle $E \to X$ with an abstract meromorphic connection ∇; equivalently, a compatible family of flat vector bundles $(E \times_G W, \nabla_W)$, $W \in \mathrm{Rep}(G)$, with structure group G.

Here, $\nabla = d + [\nabla]_\phi$ with $[\nabla]_\phi \in \Omega^1_{K/\mathbb{C}}(\mathfrak{g})$.

Change of trivialization by a section g gives rise to gauge change on the connection matrix:

$$g \cdot [\hat{\nabla}] = \mathrm{Ad}(g)([\hat{\nabla}]) - (dg)g^{-1}, \text{ where } g \text{ is a section.}$$

Localization

(E, ∇) flat G-bundle induces formal flat structures at each $y \in \mathbb{P}^1$
Let y be a singular point of ∇, z a parameter at y
$\mathfrak{o} = \mathbb{C}[[z]]$ completion of local ring at y, $F = \mathbb{C}((z))$ fraction field,
$\Delta^\times = \mathrm{Spec}(F)$ the formal punctured disk at y
One obtains an induced formal connection $(\hat{E}, \hat{\nabla})$ on $\Delta^\times$. Note that $[\hat{\nabla}] \in \mathfrak{g}(F)\frac{dz}{z}$.
If $[\hat{\nabla}]_\phi$ has a simple pole for some trivialization ϕ, then y is a regular singular point. Otherwise, it is irregular singular.

Write $[\hat{\nabla}] = (M_{-r}z^{-r} + M_{1-r}z^{1-r} + \dots)\frac{dz}{z}$, $M_i \in \mathfrak{g}$, $M_{-r} \neq 0$.

Classical approach to studying local behavior uses the naive leading term $M_{-r}z^{-r}\frac{dz}{z}$.

For example, M_{-r} nonnilpotent $\implies$ the slope is r.

Much more can be said when M_{-r} is regular semisimple.

Nonresonant case for GL_n (reg semisimple leading term)

Assume that M_{-r} is regular semisimple, $r > 0$. (Such an irregular singularity is called *nonresonant*.)

- $[\hat{\nabla}]$ is gauge equivalent to an element of $\mathcal{A}(r)\frac{dz}{z} = \{D_{-r}z^{-r} + \cdots + D_0 \mid D_i \text{ diag}, D_{-r} \text{ reg}\}\frac{dz}{z}$. ($\mathcal{A}(r)$ is the set of "formal types").
- If all irregular singularities are nonresonant, $G = \mathrm{GL}_n$ and $X = \mathbb{P}^1$, Boalch (2001) (building on Jimbo-Miwa-Ueno (1981)) constructed symplectic moduli spaces of connections with given nonresonant formal types at each singular point and realized the isomonodromy equations as an integrable system.

Nonexample: (Generalized Airy connections):

$$d + \begin{pmatrix} 0 & z^{-(s+1)} \\ z^{-s} & 0 \end{pmatrix}\frac{dz}{z} = d + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z^{-(s+1)}\frac{dz}{z} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} z^{-s}\frac{dz}{z}$$

Here, the naive leading term is nilpotent (indeed, wrt any trivialization), and the slope is $s + \frac{1}{2}$ (not $s + 1$)

Classical techniques don't work. How to proceed?

The Frenkel-Gross rigid connection

The nonexample with $s = 0$ is the GL_2 version of a flat G-bundle of great importance in geometric Langlands.

Take G simple (for convenience), $G \supset B \supset T$. Let N be a principal nilpotent element involving only the simple root spaces of the opposite Borel, E a nonzero root vector for the highest root.

Definition: The Frenkel-Gross flat G-bundle on $\mathbb{P}^1$ is $\nabla = d + (Ez^{-1} + N)\frac{dz}{z}$.

- Regular singular with principal unipotent monodromy at ∞.
- Irregular singular with slope $1/h$ (h Coxeter number) at 0.
- Rigid–uniquely determined by its local behavior.

It is the de Rham analogue of the Kloosterman sheaf of Katz, Heinloth-Ngô-Yun. This is an l-adic sheaf on $\mathbb{P}^1_{\mathbb{F}_q}$ which corresponds under Langlands to the unique cuspidal automorphic representation ramified only at 0 and ∞, with local factors:

- a "simple supercuspidal" (in sense of Gross-Reeder) at 0; this has depth $1/h$
- the Steinberg representation at ∞.

Bruhat-Tits building

$T \subset B \subset G$, T maximal torus, B Borel subgroup
$\pi : G(\mathfrak{o}) \to G$, $z \mapsto 0$

Definition

An Iwahori subgroup is a $\hat{G} := G(F)$-conjugate of $I = \pi^{-1}(B)$. A parahoric subgroup (G semisimple) is a subgroup containing an Iwahori subgroup. Iwahori (parahoric) subalgebras defined similarly.

The (enlarged) Bruhat-Tits building $\mathfrak{B}$ of $\hat{G}$ is a $\hat{G}$-complex:

- The cells are parameterized by parahorics; $x \in \mathfrak{B} \rightsquigarrow \hat{G}_x, \hat{\mathfrak{g}}_x$.
- Adjacency is given by containment; in particular alcoves (maximal cells) correspond to Iwahoris.
- It is constructed from "apartments" corresponding to split maximal tori. The standard apartment (associated to $\hat{T} = T(F)$) is $\mathcal{A} = \mathcal{A}(T) = X_*(T) \otimes \mathbb{R}$.

Moy-Prasad filtrations

Definition (Moy-Prasad filtrations)

For $x \in \mathfrak{B}$, $V \in \mathrm{Rep}(G)$:

- $\exists$ a decreasing $\mathbb{R}$-filtration by $\mathfrak{o}$-lattices on $\hat{V} = V \otimes F$: $(\hat{V}_{x,r})_{r \in \mathbb{R}}$;
- $z\hat{V}_{x,r} = \hat{V}_{x,r+1}$ (periodicity);
- $V_{gx,r} = gV_{x,r}$ (equivariance);
- The parahoric subgroup $\hat{G}_x$ stabilizes $\hat{V}_{x,r}$;
- The set of critical numbers r with $\hat{V}_{x,r+} = \bigcup_{s>r} \hat{V}_{x,s} \subsetneq \hat{V}_{x,r}$ is discrete;
- For the adjoint rep, $\hat{\mathfrak{g}}_x = \hat{\mathfrak{g}}_{x,0}$; there is also a $\mathbb{R}_{\geq 0}$-filtration $(\hat{G}_{x,r})$ of $\hat{G}_x$ with $\hat{G}_x = \hat{G}_{x,0}$.

Enough to define for $x \in \mathcal{A}$. In this case, let $\hat{V}_x(s)$ be the s-eigenspace of $z\frac{\partial}{\partial z} + x$. Then $V_{x,r}$ is generated by $\hat{V}_x(s)$ for $s \geq r$.

Each weight space of V is assigned a degree determined by x; one completes to a filtration via periodicity.

Examples

Here, one assigns each weight space of V a degree determined by x and completes to a filtration via periodicity.

1. The "origin" $o \in \mathcal{A}$ gives the naive filtration of $\hat{V}$ by powers of z.
2. Consider the complete lattice chain in F^2:

   $$\supset L_{-1} = z^{-1}L_1 \supset L_0 = \mathfrak{o}^2 \supset L_1 = \mathfrak{o}e_1 \oplus z\mathfrak{o}e_2 \supset L_2 = zL_0 \supset$$

   Its stabilizer in $GL_2(F)$ is the standard Iwahori I. If x_I is the barycenter of the corresponding edge in the reduced building, the critical numbers are $\frac{1}{2}\mathbb{Z}$ and the filtration of $\hat{\mathfrak{g}}$ is by congruence subalgebras:

   $$\hat{\mathfrak{g}}_{x_I,k/2} = \{x \in \mathfrak{gl}_2(F) \mid x(L_i) \subset L_{i+k} \forall i\} \text{ for } k \in \mathbb{Z}.$$

   In particular, $\begin{pmatrix} 0 & z^{-(s+1)} \\ z^{-s} & 0 \end{pmatrix} \in \hat{\mathfrak{g}}_{x_I,-s-1/2} \setminus \hat{\mathfrak{g}}_{x_I,-s}$.
3. More generally,for GL_n, a parahoric P is the stabilizer of a lattice chain and the MP filtration of $\hat{\mathfrak{g}}$ at the barycenter x_P comes from the lattice chain filtration.

Fundamental strata

In p-adic representation theory, fundamental strata (or minimal K-types) were introduced by Bushnell and Kutzko (GL_n) and Moy and Prasad.

Definitions

- A stratum (x, r, β) consists of $x \in \mathfrak{B}$, a real number $r \geq 0$, and a functional $\beta \in (\hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+})^\vee \cong \hat{\mathfrak{g}}_{x,-r}/\hat{\mathfrak{g}}_{-x,-r+}$.
- (x, r, β) is fundamental if β is a semistable point in the $\hat{G}_x/\hat{G}_{x+}$ representation $(\hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+})^\vee$; equivalently if every representative $\tilde{\beta} \in \hat{\mathfrak{g}}_{x,-r}$ of β is non-nilpotent. If $x \in \mathcal{A}$, enough to check for unique graded representative.
- Two fundamental strata (x, r, β), (x', r', β') are associate if $r = r'$ and (for $r > 0$) the $G(F)$-orbits of $\tilde{\beta} + \hat{\mathfrak{g}}_{x,-r+}$ and $\tilde{\beta}' + \hat{\mathfrak{g}}_{x',-r'+}$ intersect.

Moy-Prasad: Every irreducible admissible representation W of a p-adic group contains a minimal K-type. Any such has the same depth, allowing one to define the depth of W.

We have a geometric analogue of their result.

Fundamental strata give the correct notion of the leading term of a formal flat G-bundle.
Fix a G-invariant nondegenerate symm bilinear form $(,)$ on $\mathfrak{g}$
eg for GL_n, $(X, Y) = \mathrm{Tr}(XY)$
$[\hat{\nabla}]$ may be viewed as an element of $\mathfrak{g}(F)^\vee$ via

$$X \mapsto \mathrm{Res}(X, [\hat{\nabla}]), \text{ where } X \in \hat{\mathfrak{g}} := \mathfrak{g}(F).$$

Definition

The formal flat G-bundle $\hat{\nabla}$ contains the stratum (x, r, β) ($x \in \mathcal{A}$, $r > 0$) if $\mathrm{Res}([\hat{\nabla}], \mathfrak{g}_{x,r+}) = 0$ and $[\hat{\nabla}]$ induces the same functional as β on $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$.

- $[\hat{\nabla}] = \left(z^{-r}M_{-r} + z^{-r+1}M_{1-r} + \mathrm{h.o.t.}\right)\frac{dz}{z}$ with $M_i \in \mathfrak{g}$.
  $\hat{\nabla}$ contains the G-stratum (o, r, β), $\beta \in (z^r\mathfrak{g}(\mathfrak{o})/z^{r+1}\mathfrak{g}(\mathfrak{o}))^\vee$ induced by $z^{-r}M_{-r}\frac{dz}{z}$, fundamental if M_{-r} is non-nilpotent.
- $\hat{V} = F^2$, $[\hat{\nabla}] = \begin{pmatrix} 0 & z^{-(s+1)} \\ z^{-s} & 0 \end{pmatrix}\frac{dz}{z}$.
  Here, $(\hat{V}, \hat{\nabla})$ contains the fundamental GL_2-stratum $(X_I, s + \frac{1}{2}, \beta)$, $\beta \in (\hat{\mathfrak{g}}_{x_I,s+1/2}/\hat{\mathfrak{g}}_{x_I,s+1})^\vee$.

Theorem (Bremer-S.)

Every formal flat G-bundle $\hat{\nabla}$ contains a fundamental stratum (x, r, β) with x an optimal point (so $r \in \mathbb{Q}$); the depth r is positive iff $\hat{\nabla}$ is irregular singular. Moreover,

- *If $\hat{\nabla}$ contains a stratum (x', r', β'), then $r' \geq r$.*
- *If $r > 0$, (x', r', β') is fundamental if and only if $r' = r$.*
- *Any two fundamental strata contained in $\hat{\nabla}$ are associate.*

We can now define the slope of $\hat{\nabla}$ as this minimal depth.

Theorem (Bremer-S)

The slope of the formal flat G-bundle $(\hat{E}, \hat{\nabla})$ is a nonnegative rational number. It is positive if and only if $(\hat{E}, \hat{\nabla})$ is irregular singular. The slope may also be characterized as

1. *the maximum slope of the associated flat connections; or*
2. *the maximum slope of the flat connections associated to the adjoint representations and the characters.*

Other equiv defs of slope by Frenkel-Gross and Chen-Kamgarpour.

Regular strata

Want a condition on strata that is analogous to the nonresonance condition for diagonalizable connections.

Let $S \subset G(F)$ be a (possibly non-split) maximal torus. There is a unique Moy-Prasad filtration $\{\mathfrak{s}_r\}$ on $\mathfrak{s} = \mathrm{Lie}(S)$.

Definition

A point $x \in \mathfrak{B}$ is compatible with $\mathfrak{s}$ if $\mathfrak{s}_r = \hat{\mathfrak{g}}_{x,r} \cap \mathfrak{s}$ for all r.

Definition

A fundamental stratum (x, r, β) is S-regular if x is compatible with $\mathfrak{s}$ and $Z^0(\tilde{\beta})$ is conjugate to S for any representative $\tilde{\beta}$ (for $r > 0$).

$$\left\{\begin{matrix}\text{conj classes maximal} \\ \text{tori in } G(F)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{conj classes} \\ \text{in } W\end{matrix}\right\}$$

Proposition

A torus centralizes a regular stratum (x, r, β) if and only if its class corresponds to a regular conjugacy class in W. In this case, $e^{2\pi i r}$ is a regular eigenvalue of this class.

Regular strata (cont.)

For $G = \mathrm{GL}_n$, S is regular if it is uniform i.e., $S = (E^\times)^k$ for some field extension E/F, or if it is of the form $S' \times \mathbb{C}^*$ where S' is uniform for GL_{n-1}.

Examples

- If M_{-r} is reg. semisimple, then $(o, r, z^{-r} M_{-r} \frac{dz}{z})$ is $S = Z_{\hat{G}}(M_{-r})$-regular (split torus).
- The Frenkel-Gross rigid flat G-bundle is S-regular of slope $1/h$ at 0, where h is the Coxeter number and S corresponds to the Coxeter element in W.
  Explicitly for $G = \mathrm{GL}_2$: Let $\omega = \left(\begin{smallmatrix} 0 & 1 \\ z & 0 \end{smallmatrix}\right)$, so $S = \mathbb{C}((\omega))^*$ is a non-split maximal torus in $\mathrm{GL}_2(F)$. Then, $(x_I, \frac{1}{2}, \left(\begin{smallmatrix} 0 & z^{-1} \\ 1 & 0 \end{smallmatrix}\right) \frac{dz}{z})$ is S-regular (as are the other generalized Airy connections).

Toral connections and formal types

Definition: $\hat{\nabla}$ is *S-toral* if it contains an S-regular stratum.

Theorem

Any fundamental stratum contained in an S-toral $\hat{\nabla}$ is S-regular.

An S-toral $\hat{\nabla}$ can be "diagonalized" into S. More precisely, one can define a quasiaffine variety $\mathcal{A}(S, r) \subset \mathfrak{s}_{-r}\frac{dz}{z}$ of S-formal types of depth r.

- T diagonal,
  $\mathcal{A}(T, r) = \{D_{-r}z^{-r} + \cdots + D_0 \mid D_i \text{ diag}, D_{-r} \text{ reg}\}\frac{dz}{z}$.
- $S = \mathbb{C}((\omega))^\times$, $\mathcal{A}(S, s + 1/2) = \{\text{deg } 2s + 1 \text{ polys in } \omega^{-1}\}\frac{dz}{z}$.

Theorem (Bremer-S.)

If $\hat{\nabla}$ contains the S-regular stratum (x, r, β), then $[\hat{\nabla}]$ is $\hat{G}_{x+}$-gauge equivalent to a unique elt of $\mathcal{A}(S, r)$ with "leading term" β.

Formal types vs formal isomorphism classes

$W_S = N(S)/S$, $W_S^{\mathrm{aff}} = N(S)/S_0 \cong W_S \ltimes S/S_0$ relative Weyl and affine Weyl groups
Gauge action of $N(S)$ induces natural action of W_S^{aff} on $\mathcal{A}(S, r)$.

$\mathcal{A}(S, r)$ is a W_S^{aff}-torsor over the set of formal isomorphism classes.

Let $\mathcal{C}(S, r)$ be the full subcategory of formal flat G-bundles $\hat{\nabla}$ containing an S-regular stratum with formal type in $\mathcal{A}(S, r)$.

One can construct a "framed" version $\mathcal{C}^{\mathrm{fr}}(S, r)$ of this category, together with a forgetful "deframing" functor $\mathcal{C}^{\mathrm{fr}}(S, r) \to \mathcal{C}(S, r)$.

Theorem (Bremer-S)

This functor induces the quotient map $\mathcal{A}(S, r) \to \mathcal{A}(S, r)/W_S^{\mathrm{aff}}$ on moduli spaces.

Generalized Kloosterman sheaves

Back to arithmetic setting:

Reeder and Yu classified (x, r) for which there exists (x, r, β) with β a stable point. They then defined epipelagic reps–families of supercuspidals containing a stable stratum (x, r, β) with $r > 0$ minimal (for that x).

Example

The simple supercuspidals of Gross-Reeder are epipelagic; they contain $(x_I, 1/h, \beta)$.

Yun's generalized Kloosterman sheaves are l-adic local systems on $\mathbb{G}_m$, tamely ramified at ∞ and wildly ramified at 0; the singularity at 0 corresponds to an epipelagic rep. They are conjecturally rigid.

Construction of de Rham analogues

Yun has posed the question of constructing de Rham analogues of generalized Kloosterman sheaves.

This can be done by the theory of regular strata by means of the following result.

Proposition (S.)

In the geometric setting, (x, r, β) is stable $\iff$ (x, r, β) is S-regular for an elliptic regular maximal torus.

For such an $x \in \mathcal{A}$ with r minimal, choose a regular semisimple element of $\gamma \in \hat{\mathfrak{g}}_x(r)$. One can then define a connection $\gamma \frac{dz}{z}$ on $\mathbb{P}^1$. It has the desired irregular singularity at 0, and the only other singular point is a regular singularity at ∞. Moreover, one can use the information contained in the regular stratum to show rigidity.

One can do a similar construction for non-elliptic S.

Sketch of proof of rigidity for the Frenkel-Gross connection

For connections whose irregular singularities are S-toral, one can construct symplectic moduli spaces of connections with the specified local data.

Consider a flat connection ($G = \mathrm{GL}_n$) with the Frenkel-Gross formal type A at 0 and a regular singular point at ∞ with formal iso class given by a (nonresonant) coadjoint orbit $\mathcal{O}$.

Proposition (Bremer-S)

The moduli space of such connections $\mathcal{M}(A, \mathcal{O})$ is a singleton when $\mathcal{O}$ is regular and empty otherwise. Thus, one obtains a family of rigid connections including the Frenkel-Gross example.

Idea of proof when $\mathcal{O}$ irregular ($n = 3$)

- Let $X = \{\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & y & 0 \end{pmatrix} + b \mid x, y \in \mathbb{C}^*, b \in \mathfrak{b} \cap \mathfrak{sl}_3(\mathbb{C})\}$.
- One can check that $\mathcal{M}(A, \mathcal{O})$ is the set of B orbits in the set $X \cap \mathcal{O}$.
- All elements of X are regular, so if $\mathcal{O}$ is not regular, the moduli space is empty.