$(\mathrm{SL}(N), q)$-OPERS, THE q-LANGLANDS CORRESPONDENCE, AND QUANTUM/CLASSICAL DUALITY

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ABSTRACT. A special case of the geometric Langlands correspondence is given by the relationship between solutions of the Bethe ansatz equations for the Gaudin model and opers–connections on the projective line with extra structure. In this paper, we describe a deformation of this correspondence for $\mathrm{SL}(N)$. We introduce a difference equation version of opers called q-opers and prove a q-Langlands correspondence between nondegenerate solutions of the Bethe ansatz equations for the XXZ model and nondegenerate twisted q-opers with regular singularities on the projective line. We show that the quantum/classical duality between the XXZ spin chain and the trigonometric Ruijsenaars-Schneider model may be viewed as a special case of the q-Langlands correspondence. We also describe an application of q-opers to the equivariant quantum K-theory of partial flag varieties.

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1. Introduction

1.1. **Opers and the Gaudin model.** One formulation of the geometric Langlands correspondence is the existence of an isomorphism between spaces of conformal blocks for the classical W-algebra associated to a simple complex Lie algebra $\mathfrak{g}$ and the dual affine Kac-Moody algebra ${}^L\hat{\mathfrak{g}}$ at the critical level. Since both these algebras admit deformations, it is natural to conjecture the existence of deformed versions of the Langlands correspondence, and indeed, this has been the subject of considerable recent interest [AFO, GF1805, Pes1707]. In this paper, we describe a q-Langlands correspondence which is a deformation of an important example of geometric Langlands, the classical correspondence between the spectra of the Gaudin model and opers on the projective line with regular singularities and trivial monodromy.

Let G be a simple complex algebraic group of adjoint type, and let ${}^L\mathfrak{g}$ be the Lie algebra of the Langlands dual group ${}^LG$. Fix a collection of distinct points $z_1, \dots, z_n$ in $\mathbb{C}$. The Gaudin Hamiltonians are certain mutually commuting elements of the algebra $U({}^L\mathfrak{g})^{\otimes n}$. They are contained in a commutative subalgebra $\mathcal{Z}_{(z_i)}({}^L\mathfrak{g})$ called the Gaudin algebra. The simultaneous eigenvalues of the actions of the Gaudin Hamiltonians on N-fold tensor products of ${}^L\mathfrak{g}$-modules is given by the (maximal) spectrum of this algebra, namely, the set of algebra homomorphisms $\mathcal{Z}_{(z_i)}({}^L\mathfrak{g}) \longrightarrow \mathbb{C}$.

Feigin, Frenkel, and Reshetikhin found a geometric interpretation of this spectrum in terms of flat G-bundles on $\mathbb{P}^1$ with extra structure [FFR94, Frea, Freb]. Let B be a Borel subgroup of G. A G-oper on a smooth curve X is a triple $(\mathcal{F}, \nabla, \mathcal{F}_B)$, where $(\mathcal{F}, \nabla)$ is a flat G-bundle on X and $\mathcal{F}_B$ is a reduction of $\mathcal{F}$ satisfying a certain transversality condition with respect to ∇. As an example, for PGL(2)-opers, this condition is that $\mathcal{F}_B$ is nowhere preserved by ∇. The space of G-opers can be realized more concretely as a certain space of differential operators. For example, a PGL(2)-oper can be identified with projective connections: second-order operators $\partial_z^2 - f(z)$ mapping sections of $K^{-1/2}$ to sections of $K^{3/2}$, where K is the canonical bundle. It turns out that the spectrum of $\mathcal{Z}_{(z_i)}({}^L\mathfrak{g})$ may be identified with the set of G-opers on $\mathbb{P}^1$ with regular singularities at $z_1, \dots, z_n$ and ∞.

We now consider the action of the Gaudin algebra on the tensor product of irreducible finite-dimensional modules $V_{\boldsymbol{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$, where $\boldsymbol{\lambda}$ is an n-tuple of dominant integral weights. The Bethe ansatz is a method of constructing such simultaneous eigenvectors. One starts with the unique (up to scalar) vector $|0\rangle \in V_{\boldsymbol{\lambda}}$ of highest weight $\sum \lambda_i$; it is a simultaneous eigenvector. Given a set of distinct complex numbers $w_1, \dots, w_m$ labeled by simple roots α_{k_j}, one applies a certain order m lowering operator with poles at the w_j's to $|0\rangle$. If this vector is nonzero and $\sum \lambda_i - \sum \alpha_{k_j}$ is dominant, it is an eigenvector of the Gaudin Hamiltonians if and only if certain equations called the Bethe ansatz equations are satisfied (see (2.13)). Frenkel has shown that the corresponding point in the spectrum of the Gaudin algebra is a G-oper with regular singularities at the z_i's and ∞ and and with trivial monodromy [Freb].

In fact, it is possible to give a geometric description of all solutions of the Bethe equations (i.e., without assuming $\sum \lambda_i - \sum \alpha_{k_j}$ is dominant) in terms of an enhanced version of opers. A *Miura G-oper* on $\mathbb{P}^1$ is a G-oper together with an additional reduction $\mathcal{F}'_B$ which is preserved by ∇. The set of Miura opers with the same underlying oper is parametrized by the flag manifold G/B. Frenkel has shown that there is a one-to-one correspondence between the set of solutions to the ${}^L\mathfrak{g}$ Bethe ansatz and "nondegenerate" Miura G-opers with regular singularities and trivial monodromy [Frea]. To see how this works, let $H \subset B$ be a maximal torus. The initial data of the Bethe ansatz gives rise to the explicit flat H-bundle (a *Cartan connection*)

$$\partial_z + \sum_{i=1}^{n} \frac{\lambda_i}{z - z_i} - \sum_{j=1}^{m} \frac{\alpha_{k_j}}{z - w_j}.$$

There is a map from Cartan connections to Miura opers given by the Miura transformation; this is just a generalization of the standard Miura transformation in the theory of KdV integrable models. It turns out that the Bethe equations are precisely the conditions necessary for the corresponding Miura oper to be regular at the w_j's.

In the global geometric Langlands correspondence for $\mathbb{P}^1$, the objects on the Galois side are flat G-bundles (with singularities) on $\mathbb{P}^1$ while on the automorphic side, one considers D-modules on enhanced versions of the moduli space of ${}^LG$-bundles over $\mathbb{P}^1$. The correspondence between opers and spectra of the Gaudin model provides an example of geometric Langlands. Indeed, the eigenvector equations for the Gaudin Hamiltonians for fixed eigenvalues determines a D-module on the moduli space of ${}^LG$-bundles with parabolic structures at $z_1, \dots, z_n$ and ∞ while the oper gives the flat G-bundle.

1.2. **q-opers and the q-Langlands correspondence.** Recall that the geometric Langlands correspondence may be viewed as an identification of conformal blocks for the classical $\mathcal{W}$-algebra associated to $\mathfrak{g}$ and conformal blocks for the affine Kac-Moody algebra ${}^L\hat{\mathfrak{g}}$ at the critical level. Both these algebras admit deformations. For example, one may pass from ${}^L\hat{\mathfrak{g}}$ to the associated quantum affine algebra while at the same time moving away from the critical level. This led Aganagic, Frenkel, and Okounkov to formulate a two-parameter deformation of geometric Langlands called the quantum q-Langlands correspondence [AFO]. This is an identification of certain conformal blocks of a quantum affine algebra with those of a deformed $\mathcal{W}$-algebra, working over the infinite cylinder. They prove this correspondence in the simply-laced case; their proof is based on a study of the equivariant K-theory of Nakajima quiver varieties whose quiver is the Dynkin diagram of $\mathfrak{g}$.

In this paper, we take another more geometric approach, involving *q-connections*, a difference equation version of flat G-bundles. Our goal is to establish a q-Langlands correspondence between *q-opers* with regular singularities and the spectra of the XXZ spin chain model. Here, we only consider this correspondence in type A.

Fix a nonzero complex number q which is not a root of unity. We are interested in (multiplicative) difference equations of the form $s(qz) = A(z)s(z)$; here $A(z)$ is an $N \times N$ invertible matrix whose entries are rational functions. To express this more geometrically, we start with a trivializable rank n vector bundle E on $\mathbb{P}^1$, and let E^q denote the pullback of E via the map $z \mapsto qz$. A $(\mathrm{GL}(N), q)$-connection on $\mathbb{P}^1$ is an invertible operator A taking sections of E to sections of E^q. If the matrices $A(z)$ have determinant one in some trivialization, (E, A) is called an $(\mathrm{SL}(N), q)$-connection. Just as in the classical setting, an $(\mathrm{SL}(N), q)$-oper is a triple (E, A, E_B), where E_B is a reduction to a Borel subgroup satisfying a certain transversality condition with respect to A. We also define a Miura q-oper to be a q-oper with an additional reduction E'_B preserved by A. We remark that these definitions make sense when $\mathbb{P}^1$ is replaced by the formal punctured disk. In this setting, a concept equivalent to $(\mathrm{GL}(N), q)$-connections was introduced by Baranovsky and Ginzburg [BG96] while the notion of a formal q-oper is inherent in the work of Frenkel, Reshetikhin, and Semenov-Tian-Shansky on Drinfeld-Sokolov reduction for difference operators [FRSTS98].

We now explain how q-opers can be viewed as the Galois side of a q-Langlands correspondence. The XXZ spin chain model is an integrable model whose dynamical symmetry algebra is the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ [Res87]. Under an appropriate limiting process, it degenerates to the Gaudin model. The model depends on certain twist parameters which can be described by a diagonal matrix Z. We will always assume that Z has distinct eigenvalues. Eigenvectors of the Hamiltonians in the XXZ model can again be found using the Bethe ansatz, and the spectra can be expressed in terms of Bethe equations (see (3.6), (4.10) below).

It turns out that these equations also arise from appropriate q-opers. We consider q-opers with regular singularities on $\mathbb{P}^1 \setminus \{0, \infty\}$. We further assume that the q-oper is *Z-twisted*, where Z is the diagonal matrix appearing in the Bethe equations; this simply means that the underlying q-connection is q-gauge equivalent to the q-connection with matrix Z. (This may be viewed as the quantum analogue of the opers with a double pole singularity at ∞ considered by Feigin, Frenkel, Rybnikov, and Toledano-Laredo in their work on an inhomogeneous version of the Gaudin model [FFTL10, FFR10].) Given a Z-twisted q-oper with regular singularities, we examine a certain associated Miura q-oper. The assumption that this Miura q-oper is "nondegenerate" imposes certain conditions on the zeros of *quantum Wronskians* arising from the q-oper, and these conditions lead to the XXZ Bethe equations. Thus, in type A, we obtain the desired q-Langlands correspondence. It should be noted that in contrast to the results of [AFO], our results do not depend on geometric data related to the quantum K-theory of Nakajima quiver varieties. In particular, there are no restrictions on the dominant weights that can appear in our correspondence.

Our approach has some similarities with the earlier work of Mukhin and Varchenko on discrete opers and the spectra of the XXX model [MV05]. Here, they considered additive difference equations, i.e., equations of the form $f(z+h) = A(z)f(z)$ where A is a G-valued function and $h \in \mathbb{C}^*$ is a fixed parameter. They defined a discrete oper to be the linear difference operator $f(z) \mapsto f(z+h) - A(z)f(z)$ if $A(z)$ had a suitable form. They also introduced a notion of discrete Miura oper and showed that they correspond to solutions of the XXX Bethe ansatz equations. Unlike our q-opers, these discrete opers do not seem to be related to the difference equation version of Drinfeld-Sokolov reduction considered in [FRSTS98].

Since the XXZ model may be viewed as a deformation of the Gaudin model, one would expect that we should recover the Gaudin Bethe equations under an appropriate limit. In fact, by taking this limit in two steps, one can say more. First, a suitable limit takes one to a twisted version of the XXX spin chain, giving rise to a correspondence between the solutions of the Bethe equations for this model and a twisted analogue of the discrete opers of [MV05]. A further limit brings one back to the inhomogeneous Gaudin model and opers with irregular singularity considered in [FFTL10, FFR10].

1.3. **Quantum/classical duality and applications to enumerative geometry.** Quantum/classical duality is a relationship between a quantum and a classical integrable system. Well-known examples are the relationship in type A between the Gaudin model and the rational Calogero-Moser system and between the XXX spin-chain and the rational Ruijsenaars-Schneider model. Both of these can be viewed as limits of the duality between the XXZ spin-chain and the trigonometric Ruijsenaars-Schneider model [HR15, HR12, MTV0906].[1] This duality is given by a transformation relating two sets of generators for the quantum K-theory ring of cotangent bundles of full flag varieties [KPSZ1705]. One set of generators is obtained from the XXZ Bethe equations. One considers certain Bethe equations where the dominant weights all come from the defining representation and then takes symmetric functions on the corresponding Bethe roots. The other generators are functions on a certain Lagrangian subvariety in the phase space for the tRS model.

This correspondence has a direct interpretation in terms of twisted q-opers; indeed, it may be viewed as a special case of the q-Langlands correspondence. As we discussed in the previous section, Bethe equations arise from nondegenerate twisted q-opers. The Bethe roots are precisely those zeros of the quantum Wronskians associated to the q-oper which are not singularities of the underlying q-connection. On the other hand, there is an embedding of the tRS model into the space of twisted q-opers. More precisely, a q-oper structure on a given q-connection (E, A) is determined uniquely by a full flag $\mathcal{L}_\bullet$ of vector subbundles which behave in a specified way with respect to A. A section s generating the line bundle $\mathcal{L}_1$ over $\mathbb{P}^1 \setminus \infty$ may be viewed as an N-tuple of monic polynomials $(s_1, \dots, s_N)$. If these polynomials are all linear, then their constant terms are precisely the momenta in the phase space of the tRS model. Quantum/classical duality is then equivalent to the statement that the Bethe roots and the constant terms of these monic linear polynomial both give coordinates for an appropriate spaces of twisted q-opers.

[1] We refer the reader to Section 4 of [GK13] for more information on quantum/classical duality and additional references.

If the monic polynomials s_i are no longer linear, it is still the case that the Bethe roots and the coefficients of these polynomials are equivalent sets of coordinates for a space of twisted q-opers. It is more complicated to interpret this statement as a duality between the XXZ spin-chain and a classical multiparticle integrable system. However, we do get an application to the quantum K-theory of the cotangent bundles of partial flag varieties. This K-theory ring is again generated by symmetric functions in appropriate Bethe roots. In [RTV1411], Rimanyi, Tarasov, and Varchenko gave another conjectural set of generators for this ring. We show that these generators are precisely those obtained from the coordinates for the set of twisted q-opers coming from the coefficients of the polynomials s_i, thereby proving this conjecture.

1.4. **Structure of the paper.** In Section 2, we recall the relationship between monodromy-free $\mathrm{SL}(N)$-opers with regular singularities on the projective line and Gaudin models [Freb, Frea]. We follow an approach hinted at in [GW11], describing opers in terms of vector bundles instead of principal bundles and obtaining the Bethe equations from Wronskian relations. We also discuss the correspondence between an inhomogeneous version of the Gaudin model and opers with an irregular singularity at infinity.

Next, in Section 3, we consider a q-deformation of opers in the case of $\mathrm{SL}(2)$. We adapt the techniques of the previous section to give a correspondence between twisted q-opers and the Bethe ansatz equations for the XXZ spin chain for $\mathfrak{sl}_2$. In Section 4, we generalize these constructions to $\mathrm{SL}(N)$ and again prove a correspondence between q-opers and the XXZ spin chain model. We then discuss the case of $\mathrm{SL}(3)$ in detail in Section 5.

In Section 6, we consider classical limits of our results. We show that an appropriate limit leads to a correspondence between a twisted analogue of the discrete opers considered in [MV05] and the spectra of a version of the XXX spin chain. By taking a further limit, we recover the relationship between opers with an irregular singularity and the inhomogeneous Gaudin model [FFTL10, FFR10].

Finally, Section 7 is devoted to some geometric implications of the results of this paper. The quantum K-theory ring of the cotangent bundle to the variety of partial flags is known to be described via the Bethe ansatz equations [KPSZ1705]. We find a new set of generators defined in terms of canonical coordinates on an appropriate set of q-opers. These generators turn out to be the same as the conjectural generators given in [RTV1411].

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2. $\mathrm{SL}(N)$-opers with trivial monodromy and regular singularities

2.1. $\mathrm{SL}(2)$ **opers and Bethe equations.**

In this section, we describe a simple reformulation of the results of [Freb, Frea] due to Gaiotto and Witten [GW11].

Definition 2.1. A $\mathrm{GL}(2)$-*oper* on $\mathbb{P}^1$ is a triple $(E, \nabla, \mathcal{L})$, where E is a rank 2 vector bundle on $\mathbb{P}^1$, $\nabla : E \longrightarrow E \otimes K$ is a connection (here K is the canonical bundle), and $\mathcal{L}$ is a line subbundle such that the induced map $\bar{\nabla} : \mathcal{L} \longrightarrow E/\mathcal{L} \otimes K$ is an isomorphism. The triple is called an $\mathrm{SL}(2)$-*oper* if the structure group of the flat $\mathrm{GL}(2)$-bundle may be reduced to $\mathrm{SL}(2)$.

We always assume that the vector bundle E is trivializable.

The oper condition may be checked explicitly in terms of a determinant condition on local sections. Indeed, $\bar{\nabla}$ is an isomorphism in a neighborhood of a given point z if for some (or for any) local section s of $\mathcal{L}$ with $s(z) \neq 0$,

$$s(z) \wedge \nabla_z s(z) \neq 0.$$

Here, $\nabla_z = \iota_{\frac{d}{dz}} \circ \nabla$, where $\iota_{\frac{d}{dz}}$ is the inner derivation by the vector field $\frac{d}{dz}$.

In this section, we will be interested in $\mathrm{SL}(2)$-opers with regular singularities. An $\mathrm{SL}(2)$-oper with regular singularities of weights $k_1, \ldots, k_L, k_\infty$ at the points $z_1, \ldots, z_L, \infty$ is a triple $(E, \nabla, \mathcal{L})$ as above where $\bar{\nabla}$ is an isomorphism everywhere except at each z_i (resp. ∞), where it has a zero of order k_i (resp. k_∞). Concretely, near the point z_i, we have

$$s(z) \wedge \nabla_z s(z) \sim (z - z_i)^{k_i}. \tag{2.1}$$

We will always assume that our opers have trivial monodromy, i.e., that the monodromy of the connection around each z_i is trivial. This means that after an appropriate gauge change, we can assume that the connection is trivial. In terms of this trivialization of E over $\mathbb{P}^1 \setminus \infty$, the line bundle $\mathcal{L}$ is generated over this affine space by the section

$$s = \begin{pmatrix} q_+(z) \\ q_-(z) \end{pmatrix}, \tag{2.2}$$

where $q_\pm(z)$ are polynomials without common roots. The condition (2.1) leads to the following equation on the *Wronskian*:

$$q_+(z)\partial_z q_-(z) - \partial_z q_+(z) q_-(z) = \rho(z), \tag{2.3}$$

where $\rho(z)$ is a polynomial whose zeros are determined by (2.1). After multiplying s by a constant, we may take $\rho(z) = \prod_{i=1}^{L}(z - z_i)^{k_i}$. By applying a constant gauge transformation in $\mathrm{SL}(2, \mathbb{C})$, we may further normalize s so that $\deg(q_-) < \deg(q_+)$ and $q_-(z) = \prod_{i=1}^{l_-}(z - w_i)$ has leading coefficient 1. (More precisely, transforming by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ if necessary allows us to assume that $\deg(q_-) \leq \deg(q_+)$; if the degrees are equal, transforming by an elementary matrix brings us to the case $\deg(q_-) < \deg(q_+)$. The final reduction uses a diagonal gauge change.)

We now make the further assumption that our oper is *nondegenerate*, meaning that none of the z_i's are roots of q_-. It is now an immediate consequence of (2.3) that each root of q_- has multiplicity 1.

Let $k = \sum_{i=1}^{L} k_i$ denote $\deg(\rho)$. An easy calculation using the fact that $\deg(q_-) < \deg(q_+)$ gives $\deg(q_-) + \deg(q_+) = k + 1$; this implies that $\deg(q_-) = l_- \leq k/2$. We now rewrite (2.3) in the equivalent form

$$\partial_z \left(\frac{q_+(z)}{q_-(z)} \right) = -\frac{\rho(z)}{q_-(z)^2}. \tag{2.4}$$

Since the residue at each w_i of the left-hand side of this equation is 0, computing the residues of the right-hand side leads to the conditions

$$\sum_m \frac{k_m}{z_m - w_i} = \sum_{j \neq i} \frac{2}{w_j - w_i}, \qquad i = 1, \ldots, l_-. \tag{2.5}$$

These are the Bethe ansatz equations for the $\mathfrak{sl}_2$-Gaudin model at level $k - 2l_- \geq 0$; they determine the spectrum of this model.

A local section for $\mathcal{L}$ at ∞ is given by

$$\begin{pmatrix} \tilde{q}_+(\tilde{z}) \\ \tilde{q}_-(\tilde{z}) \end{pmatrix} = \tilde{z}^{l_+} \begin{pmatrix} q_+(1/\tilde{z}) \\ q_-(1/\tilde{z}) \end{pmatrix}, \tag{2.6}$$

where $l_+ = \deg(q_+)$. If we set $k_\infty = k - 2l_- = l_+ - l_- - 1$, we obtain

$$\tilde{q}_+(\tilde{z})\partial_{\tilde{z}}\tilde{q}_-(\tilde{z}) - \partial_{\tilde{z}}\tilde{q}_+(\tilde{z})\tilde{q}_-(\tilde{z}) \sim \tilde{z}^{k_\infty}. \tag{2.7}$$

Thus, we have proved the following theorem.

Theorem 2.2. *There is a one-to-one correspondence between the spectrum of the Gaudin model, described by the Bethe equations for dominant weights, and the space of nondegenerate* SL(2)*-opers with trivial monodromy and regular singularities at the points $z_1, \ldots, z_L, \infty$ with weights $k_1, \ldots, k_L, k_\infty$.*

2.2. **Miura opers and the Miura transformation.** The previous theorem raises the natural question of whether one can give a geometric interpretation to solutions of the Bethe equations without assuming that the level $k - 2l_-$ is nonnegative. *Miura opers* provide such an description. A Miura oper is an oper $(E, \nabla, \mathcal{L})$ together with an additional line bundle $\hat{\mathcal{L}}$ preserved by ∇. There may be a finite set of points where $\mathcal{L}$ and $\hat{\mathcal{L}}$ do not span E. It turns out that one can associate to any oper with regular singularities a family of Miura opers parameterized by the flag variety [Freb].

Given a Miura oper, we may choose a trivialization of E so that the line bundle $\hat{\mathcal{L}}$ is generated by the section $\hat{s} = (1, 0)$. We retain our notation for the section $s = \left(\begin{smallmatrix} q_+ \\ q_- \end{smallmatrix}\right)$ generating $\mathcal{L}$, but here, we do not impose any restrictions on $\deg(q_-)$.

Theorem 2.2 can be generalized to give the following theorem, which is proved in a similar way.

Theorem 2.3. *There is a one-to-one correspondence between the set of solutions of the Bethe Ansatz equations* (2.5) *and the set of nondegenerate* SL(2)*-Miura opers with trivial monodromy and regular singularities at the points $z_1, \ldots, z_L, \infty$ with weights at the finite points given by $k_1, \ldots, k_L$.*

We now give a different formulation of SL(2)-opers which shows how the eigenvalues of the Gaudin Hamiltonian can be seen directly from the oper. We will do this by applying several SL(2)-gauge transformations to our trivial connection to reduce it to a canonical form. We start with a gauge change by $g(z) = \begin{pmatrix} q_-(z) & -q_+(z) \\ 0 & q_-^{-1}(z) \end{pmatrix}$; note that $g(z)s(z) = \binom{0}{1}$. The new connection matrix is

$$- (\partial_z g)g^{-1} = -\begin{pmatrix} \partial_z q_-(z) & -\partial_z q_+ \\ 0 & -\frac{\partial_z q_-(z)}{q_-(z)^2} \end{pmatrix} \begin{pmatrix} q_-^{-1}(z) & q_+(z) \\ 0 & q_-(z) \end{pmatrix} = \begin{pmatrix} \frac{-\partial_z q_-(z)}{q_-(z)} & -\rho(z) \\ 0 & \frac{\partial_z q_-(z)}{q_-(z)} \end{pmatrix}. \tag{2.8}$$

Next, the diagonal transformation $\begin{pmatrix} \rho(z)^{-1/2} & 0 \\ 0 & \rho(z)^{1/2} \end{pmatrix}$ brings us to the *Cartan connection*

$$A(z) = \begin{pmatrix} -u(z) & -1 \\ 0 & u(z) \end{pmatrix}, \tag{2.9}$$

where

$$u(z) = -\frac{\partial_z \rho(z)}{2\rho(z)} + \frac{\partial_z q_-(z)}{q_-(z)} = -\sum_m \frac{k_m/2}{z - z_m} + \sum_i \frac{1}{z - w_i}.$$

Finally, we apply the *Miura transformation*: gauge change by the lower triangular matrix $\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$ gives the connection matrix

$$B(z) = \begin{pmatrix} 0 & -1 \\ -t(z) & 0 \end{pmatrix}, \qquad \text{where } t(z) = \partial_z u(z) + u^2(z). \tag{2.10}$$

An explicit computation using the Bethe equations (2.5) gives

$$t(z) = \sum_m \frac{k_m(k_m+2)/4}{(z - z_m)^2} + \sum_m \frac{c_m}{z - z_m},$$

where

$$c_m = k_m \left(\sum_{n \neq m} \frac{k_n/2}{z_m - z_n} - \sum_{i=1}^{l_-} \frac{1}{z_m - w_i} \right).$$

This shows that $t(z)$ does not have any singularities at $z = w_i$; moreover, since the c_m are the eigenvalues of the Gaudin Hamiltonians, it depends only on this spectrum. In particular, the Gaudin eigenvalues can be read off explicitly from the residues of the connection matrix $B(z)$. Note that a horizontal section $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ to the connection in this gauge is determined by a solution to the linear differential equation

$$(\partial_z^2 - t(z)) f_1(z) = 0. \tag{2.11}$$

The differential operator $\partial_z^2 - t(z)$ can be viewed as a projective connection.

2.3. **Generalization to $\mathrm{SL}(N)$: a brief summary.** We now give a brief description of the interpretation of the spectrum of the $\mathfrak{sl}_N$-Gaudin model in terms of $\mathrm{SL}(N)$-opers.

Definition 2.4. A $\mathrm{GL}(N)$-*oper* on $\mathbb{P}^1$ is a triple $(E, \nabla, \mathcal{L}_\bullet)$, where E is a rank n vector bundle on $\mathbb{P}^1$, $\nabla : E \longrightarrow E \otimes K$ is a connection, and $\mathcal{L}_\bullet$ is a complete flag of subbundles such that ∇ maps $\mathcal{L}_i$ into $\mathcal{L}_{i+1} \otimes K$ and the induced maps $\bar{\nabla}_i : \mathcal{L}_i/\mathcal{L}_{i-1} \longrightarrow \mathcal{L}_{i+1}/\mathcal{L}_i \otimes K$ are isomorphisms for $i = 1, \ldots, N-1$. The triple is called an $\mathrm{SL}(N)$-*oper* if the structure group of the flat $\mathrm{GL}(N)$-bundle may be reduced to $\mathrm{SL}(N)$.

As in the $\mathrm{SL}(2)$-case, the fact that the $\bar{\nabla}_i$'s are isomorphisms is equivalent to the nonvanishing of certain determinants involving local sections of $\mathcal{L}_1$. Given a local section s of $\mathcal{L}_1$, for $i = 1, \ldots, N$, let

$$\mathcal{W}_i(s)(z) = \big(s(z) \wedge \nabla_z s(z) \wedge \cdots \wedge \nabla_z^{i-1} s(z)\big)\big|_{\Lambda^i \mathcal{L}_i}$$

Then $(E, \nabla, \mathcal{L}_\bullet)$ is an oper if and only if for each z, there exists a local section of $\mathcal{L}_1$ for which $\mathcal{W}_i(s)(z) \neq 0$ for all i. Note that $\mathcal{W}_1(s) \neq 0$ simply means that s locally generates $\mathcal{L}_1$.

We again will need to relax the isomorphism condition in the above definition to allow the oper to have regular singularities. Recall that the weight lattice for $\mathrm{SL}(N)$ is the free abelian group on the fundamental weights $\omega_1, \ldots, \omega_{N-1}$. Moreover, a weight is dominant if it is a nonnegative linear combination of the ω_i's.

Fix a collection of points $z_1, \ldots, z_L$ and corresponding dominant integral weights $\lambda_1, \ldots, \lambda_L$. Write $\lambda_m = \sum l_m^i \omega_i$. We say that $(E, \nabla, \mathcal{L}_\bullet)$ is an $\mathrm{SL}(N)$-oper with regular singularities of weights $\lambda_1, \ldots, \lambda_L$ at $z_1, \ldots, z_L$ if (E, ∇) is a flat $\mathrm{SL}(N)$-bundle, and each of the $\bar{\nabla}_i$'s is an isomorphism except possibly at z_m, where it has a zero of order l_m^i, and ∞. The conditions at the singularities may be expressed equivalently in terms of a nonvanishing local section. For each j with $1 \leq j \leq N-1$, set $\Lambda_j = \prod_{m=1}^L (z - z_m)^{l_m^j}$ and $\ell_m^j = \sum_{k=1}^j l_m^k$. Then, for $2 \leq i \leq N$,

$$\mathcal{W}_i(s)(z) \sim P_{i-1} := \Lambda_1(z)\Lambda_2(z) \cdots \Lambda_{i-1}(z) = \prod_{m=1}^L (z - z_m)^{\ell_m^{i-1}}. \tag{2.12}$$

As we saw for $\mathrm{SL}(2)$, to get the Bethe equations for nondominant weights, we need to introduce Miura opers. Again, a Miura oper is a quadruple $(E, \nabla, \mathcal{L}_\bullet, \hat{\mathcal{L}}_\bullet)$ where $(E, \nabla, \mathcal{L}_\bullet)$ is an oper with regular singularities and $\hat{\mathcal{L}}_\bullet$ is a complete flag of subbundles preserved by ∇. Given a Miura oper, choose a trivialization of E on $\mathbb{P}^1 \setminus \infty$ such that $\hat{\mathcal{L}}_\bullet$ is the standard flag, i.e., the flag generated by the ordered basis $e_1, \ldots, e_N$. If s is a section generating $\mathcal{L}_1$ on this affine line, consider the following determinants for $i = 1, \ldots, N$:

$$\mathcal{D}_i(s)(z) = e_1 \wedge \cdots \wedge e_{N-i} \wedge s(z) \wedge \nabla_z s(z) \wedge \cdots \wedge \nabla_z^{i-1} s(z).$$

Each of these is a polynomial multiple of the volume form. Note that $\mathcal{D}_N(s)(z) = \mathcal{W}_N(s)(z)$; in particular, $\mathcal{D}_N(s)(z) \neq 0$ away from the z_m's. We will call a Miura oper *nondegenerate* if the orders of the zero of $\mathcal{D}_i(s)$ and $\mathcal{W}_i(s)$ at each z_m are the same and moreover, if $\mathcal{D}_i(s)$ and $\mathcal{D}_k(s)$ for $i \neq k$ both vanish at a point z, then $z = z_m$ for some m.

These conditions may be expressed in a more Lie-theoretic form. Let B be the upper triangular Borel subgroup of $\mathrm{SL}(N)$. Under the usual identification of $\mathrm{SL}(N)/B$ as the variety of complete flags, B corresponds to the standard flag $\mathcal{E}$. If $\mathcal{F}$ is another flag, we say that $(\mathcal{E}, \mathcal{F})$ have relative position w (with w an element of the Weyl group S_N) if $\mathcal{F} = g \cdot \mathcal{E}$ for some g in the double coset BwB. If the relative position is w_0, where w_0 is the longest element given by the permutation $i \mapsto N + 1 - i$ for all i, we say that the flags are in general position.

Given an ordered basis $f = (f_1, \dots, f_N)$ for $\mathbb{C}^N$, let $Q_k(f) = e_1 \wedge \cdots \wedge e_{N-k} \wedge f_1 \wedge \cdots \wedge f_k$. It is immediate that the zeros of the function $k \mapsto Q_k(f)$ depend only on the flag determined by f. (Of course, $Q_N(f)$ is always nonzero, since f is a basis.) Let $\sigma_k = (k\ k+1) \in S_N$.

Lemma 2.5. *Let $\mathcal{F}$ be a flag determined by the ordered basis $f = (f_1, \dots, f_N)$.*

1. *The pair $(\mathcal{E}, \mathcal{F})$ are in general position if and only if $Q_j(f) \neq 0$ for all j.*
2. *The pair $(\mathcal{E}, \mathcal{F})$ have relative position $w_0\sigma_k$ if and only if $Q_k(f) = 0$ and $Q_j(f) \neq 0$ for all $j \neq k$.*

Proof. In both cases, the forward implication is an easy direct calculation and will be omitted. Note that $Q_j(f) \neq 0$ is equivalent to the fact that the projection of $\mathrm{span}(f_1, \dots, f_j)$ onto $\mathrm{span}(e_1, \dots, e_j)$ is an isomorphism. If this is true for all j, then one shows inductively that the basis f can be modified to give a new ordered basis $\hat{f}$ for $\mathcal{F}$ for which the matrix $b = (\,\hat{f}_N\ \hat{f}_{N-1} \dots\ \hat{f}_1\,) \in B$. Thus, $\mathcal{F} = bw_0\mathcal{E}$.

Now, assume that $Q_k(f) = 0$, but the other $Q_j(f)$'s are nonzero. The same argument as above shows that without loss of generality, we may assume that for $j = 1, \dots, k-1$, f_j is a column vector with lowest nonzero component in the $N - j$ place. We may further assume that all other f_i's have bottom $k - 1$ components zero. Since $Q_k(f) = 0$, $(f_k)_{N-k} = 0$. However, $Q_{k+1}(f) \neq 0$ now gives $(f_{k+1})_{N-k} \neq 0$ and $(f_k)_{N-k-1} \neq 0$. It is now clear that the flag $\mathcal{F}$ is determined by an ordered basis $\hat{f}$ for which $b = (\,\hat{f}_N \dots \hat{f}_k\ \hat{f}_{k+1} \dots\ \hat{f}_1\,) \in B$. This means that $\mathcal{F} = bw_0w_k\mathcal{E}$. $\square$

Returning to our Miura oper, recall that $s(z), \nabla_z s(z), \dots, \nabla_z^{N-1} s(z)$ is an ordered basis for the flag $\mathcal{L}(z)$ as long as z is not a singular point. If we denote this basis by $\boldsymbol{s}(z)$, we see that $\mathcal{D}_i(s)(z) = Q_i(\boldsymbol{s}(z))$. The lemma now shows that the fact that the $\mathcal{D}_i(s)$'s have no roots in common outside of regular singularities is equivalent to the statement that the relative position of $(\hat{\mathcal{L}}_\bullet(z), \mathcal{L}_\bullet(z))$ is either w_0 or $w_0\sigma_k$ for some k. Furthermore, $s(z), (z-z_m)^{-l_m^1} \nabla_z s(z), \dots, (z-z_m)^{-l_m^{N-1}} \nabla_z^{N-1} s(z)$ is an ordered basis for $\mathcal{L}_\bullet$ at z_m. Hence, $\mathcal{D}_i(s)(z)$ and $\mathcal{W}_i(s)(z)$ having zeros of the same order at z_m is equivalent to the fact that the flags $\hat{\mathcal{L}}_\bullet(z_m)$ and $\mathcal{L}_\bullet(z_m)$ are in general position.

The determinant conditions for the zeros of $\mathcal{D}_k(s)$ lead to Bethe equations in the same way as before [Freb]:

$$\sum_{i=1}^{L} \frac{\langle \lambda_i, \check{\alpha}_{i_j} \rangle}{w_j - z_i} = \sum_{s \neq j} \frac{\langle \check{\alpha}_{i_s}, \check{\alpha}_{i_j} \rangle}{w_s - w_j} \tag{2.13}$$

where the w_j's are distinct points corresponding to zeros of the determinants $\mathcal{D}(s)$.

We can now state the $\mathrm{SL}(N)$ analogue of Theorem 2.3. Here, λ_∞ is a dominant weight determined by the λ_i's and the α_{i_j}'s.

Theorem 2.6. *There is a one-to-one correspondence between the set of solutions to the Bethe ansatz equations (2.13) and the set of nondegenerate* $\mathrm{SL}(N)$*-Miura opers with trivial monodromy and regular singularities at the points* $z_1, \dots, z_L, \infty$ *with weights* $\lambda_1, \dots, \lambda_L, \lambda_\infty$.

2.4. **Irregular singularities.** In this section, we recall the relationship between opers with irregular as well as regular singularities and an inhomogeneous version of the Gaudin model introduced in [FFTL10, FFR10]. Here, we will only consider the simplest case of a double pole irregularity at ∞. We also restrict the discussion to $\mathrm{SL}(2)$.

Let $(E, \nabla, \mathcal{L})$ be an $\mathrm{SL}(2)$-oper with regular singularities on $\mathbb{P}^1 \setminus \infty$ whose underlying connection is gauge equivalent to $d + \boldsymbol{a}\, dz$, where $\boldsymbol{a} = \mathrm{diag}(a, -a)$ with $a \neq 0$. Changing variables to $1/z$, we see that this connection has a double pole at ∞. It is no longer possible to trivialize the connection algebraically, but it can be trivialized using the exponential transformation $h(z) = e^{\boldsymbol{a} z}$. If we let $\begin{pmatrix} q_+(z) \\ q_-(z) \end{pmatrix}$ be a section generating the line bundle $\mathcal{L}$ (so $q_+(z)$ and $q_-(z)$ are polynomials with no common zeros), then in the trivial gauge, this section becomes

$$s(z) = e^{-\boldsymbol{a} z} \begin{pmatrix} q_+(z) \\ q_-(z) \end{pmatrix}.$$

Note that we cannot assume that $\deg(q_-) < \deg(q_+)$, since the necessary constant gauge changes do not preserve $d + \boldsymbol{a}\, dz$. However, we can assume that q_- is monic: $q_-(z) = \prod_{i=1}^{l_-}(z - w_i)$.

The condition $s(z) \wedge \nabla_z s(z) = \rho(z)$ gives a "twisted" form of the Wronskian:

$$q_+(z)\partial_z q_-(z) - q_-(z)\partial_z q_+(z) + 2a q_+(z) q_-(z) = \rho(z) \tag{2.14}$$

As before, we assume this oper is nondegenerate, i.e., $q_-(z_m) \neq 0$ for all m; again, this implies that the zeros of q_- are simple.

To compute the Bethe ansatz equations, we observe that after multiplying (2.14) by $-e^{-2az}/(q_-(z))^2$, we obtain

$$\partial_z \left(-e^{-2az} \frac{q_+(z)}{q_-(z)} \right) = \frac{e^{-2az}\rho(z)}{q_-(z)^2}. \tag{2.15}$$

Taking residues at each w_i now leads to the inhomogeneous Bethe equations

$$-2a + \sum_m \frac{k_n}{z_n - w_i} = \sum_{j \neq i} \frac{2}{w_j - w_i}, \qquad i = 1, \dots, l_-. \tag{2.16}$$

We thus obtain the following theorem:

Theorem 2.7. *There is a one-to-one correspondence between the set of solutions of the inhomogeneous Bethe equations (2.16) and the set of nondegenerate* $\mathrm{SL}(2)$*-opers with regular singularities at the points* $z_1, \dots, z_L$ *of weights* $k_1, \dots, k_L$ *at the points* $z_1, \dots, z_L$ *and with a double pole with* 2*-residue* $-\boldsymbol{a}$.

There is a similar result for $\mathrm{SL}(N)$; see [FFTL10, FFR10] for the precise statement.

We remark that for the opers considered in this section, there is no longer an entire flag variety of associated Miura opers. Indeed, the only line bundles $\hat{\mathcal{L}}$ preserved by $d + \boldsymbol{a}\,dz$ are those generated by e_1 and e_2. More generally, consider an $\mathrm{SL}(N)$-oper with underlying connection $d + A\,dz$, where A is a diagonal matrix with distinct eigenvalues. The flags $\hat{\mathcal{L}}_\bullet$ preserved by this connection are precisely those generated by ordered bases obtained by permuting the standard basis. Hence, the associated Miura opers are parameterized by the Weyl group.

3. $(\mathrm{SL}(2), q)$-OPERS

3.1. **Definitions.** We now consider a q-deformation of the set-up in the previous section. It involves a difference equation version of connections and opers.

Fix $q \in \mathbb{C}^*$. Given a vector bundle E over $\mathbb{P}^1$, let E^q denote the pullback of E under the map $z \mapsto qz$. We will always assume that E is trivializable. Consider a map of vector bundles $A : E \longrightarrow E^q$. Upon picking a trivialization, the map A is determined by a matrix $A(z)$ giving the linear map $E_z \longrightarrow E_{qz}$ in the given bases. A change in trivialization by $g(z)$ changes the matrix via

$$A(z) \mapsto g(qz)A(z)g^{-1}(z); \tag{3.1}$$

thus, q-gauge change is twisted conjugation. Let $D_q : E \longrightarrow E^q$ be the operator that takes a section $s(z)$ to $s(qz)$. We associate the map A to the difference equation $D_q(s) = As$.

Definition 3.1. A meromorphic $(\mathrm{GL}(N), q)$-*connection* over $\mathbb{P}^1$ is a pair (E, A), where E is a (trivializable) vector bundle of rank N over $\mathbb{P}^1$ and A is a meromorphic section of the sheaf $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(E, E^q)$ for which $A(z)$ is invertible. The pair (E, A) is called an $(\mathrm{SL}(N), q)$-connection if there exists a trivialization for which $A(z)$ has determinant 1.

For simplicity, we will usually omit the word ‘meromorphic’ when referring to q-connections.

Remark 3.2. More generally, if G is a complex reductive group, one can define a meromorphic (G, q)-connection over $\mathbb{P}^1$ as a pair $(\mathcal{G}, A)$ where $\mathcal{G}$ is a principal G-bundle over $\mathbb{P}^1$ and A is a meromorphic section of $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{G}, \mathcal{G}^q)$.

Next, we define a q-analogue of opers. In this section, we will restrict to type A_1.

Definition 3.3. A $(\mathrm{GL}(2), q)$-*oper* on $\mathbb{P}^1$ is a triple $(E, A, \mathcal{L})$, where (E, A) is a $(\mathrm{GL}(2), q)$-connection and $\mathcal{L}$ is a line subbundle such that the induced map $\bar{A} : \mathcal{L} \longrightarrow (E/\mathcal{L})^q$ is an isomorphism. The triple is called an $(\mathrm{SL}(2), q)$-*oper* if (E, A) is an $(\mathrm{SL}(2), q)$-connection.

The condition that $\bar{A}$ is an isomorphism can be made explicit in terms of sections. Indeed, it is equivalent to

$$s(qz) \wedge A(z)s(z) \neq 0$$

for $s(z)$ any section generating $\mathcal{L}$ over either of the standard affine coordinate charts.

From now on, we assume that q is not a root of unity. We want to define a q-analogue of the opers considered in Section 2.4. First, we introduce the notion of a q-oper with regular singularities. Let $z_1, \dots, z_L \neq 0, \infty$ be a collection of points such that $q^{\mathbb{Z}} z_m \cap q^{\mathbb{Z}} z_n = \varnothing$ for all $m \neq n$.

Definition 3.4. A $(\mathrm{SL}(2), q)$-*oper with regular singularities* at the points $z_1, \dots, z_L \neq 0, \infty$ with weights $k_1, \dots k_L$ is a meromorphic $(\mathrm{SL}(2), q)$-oper $(E, A, \mathcal{L})$ for which $\bar{A}$ is an isomorphism everywhere on $\mathbb{P}^1 \setminus \{0, \infty\}$ except at the points z_m, $q^{-1} z_m$, $q^{-2} z_m$, $\dots$, $q^{-k_m+1} z_m$ for $m \in \{1, \dots, L\}$, where it has simple zeros.

The second condition can be restated in terms of a section $s(z)$ generating $\mathcal{L}$ over $\mathbb{P}^1 \setminus \infty$: $s(qz) \wedge A(z) s(z)$ has simple zeros at z_m, $q^{-1} z_m$, $q^{-2} z_m$, $\dots$, $q^{-k_m+1} z_m$ for every $m \in \{1, \dots, L\}$ and has no other finite zeros.

Next, we define twisted q-opers; these are q-analogues of the opers with a double pole singularity considered in Section 2.4. Let $Z = \mathrm{diag}(\zeta, \zeta^{-1})$ be a diagonal matrix with $\zeta \neq \pm 1$.

Definition 3.5. A $(\mathrm{SL}(2), q)$-oper $(E, A, \mathcal{L})$ with regular singularities is called a Z-*twisted* q-*oper* if A is gauge-equivalent to Z^{-1}.

Finally, we will need the notion of a *Miura* q-*oper*. As in the classical case, this is a quadruple $(E, A, \mathcal{L}, \hat{\mathcal{L}})$ where $(E, A, \mathcal{L})$ is a q-oper and $\hat{\mathcal{L}}$ is a line bundle preserved by A.

For the rest of Section 3, $(E, A, \mathcal{L})$ will be a Z-twisted $(\mathrm{SL}(2), q)$-connection with regular singularities at $z_1, \dots, z_L \neq 0, \infty$ having (nonnegative) weights $k_1, \dots k_L$.

3.2. **The quantum Wronskian and the Bethe ansatz.** Choose a trivialization for which the q-connection matrix is Z^{-1}. Since $\mathcal{L}$ is trivial on $\mathbb{P}^1 \setminus \infty$, it is generated by a section

$$s(z) = \begin{pmatrix} Q_+(z) \\ Q_-(z) \end{pmatrix}, \tag{3.2}$$

where $Q_+(z)$ and $Q_-(z)$ are polynomials without common roots. The regular singularity condition on the q-oper becomes an explicit equation for the *quantum Wronskian*:

$$\zeta^{-1} Q_+(z) Q_-(qz) - \zeta Q_+(qz) Q_-(z) = \rho(z) := \prod_{m=1}^{L} \prod_{j=0}^{k_m-1} (z - q^{-j} z_m). \tag{3.3}$$

We can assume that ρ is monic, since we can multiply s by a nonzero constant. We are also free to perform a constant diagonal gauge transformation, since this leaves the q-connection matrix unchanged. Thus, we may assume that Q_- is monic, say $Q_-(z) = \prod_{i=1}^{l_-} (z - w_i)$.

We now restrict attention to *nondegenerate* q-opers. This means the $q^{\mathbb{Z}}$-lattices generated by the roots of ρ and Q_- do not overlap, i.e., $q^{\mathbb{Z}} z_m \cap q^{\mathbb{Z}} w_i = \varnothing$ for all m and i. Note that this condition implies that $w_j \neq q w_i$ for all i, j; if $w_j = q w_i$, then (3.3) shows that w_i would be a common zero of ρ and Q_-.

Evaluating (3.3) at $q^{-1} z$ gives $\rho(q^{-1} z) = \zeta^{-1} Q_+(q^{-1} z) Q_-(z) - \zeta Q_+(z) Q_-(q^{-1} z)$. If we divide (3.3) by this equation and evaluate at the zeros of Q_-, we obtain the following constraints:

$$\frac{\rho(w_i)}{\rho(q^{-1}w_i)} = -\zeta^{-2}\frac{Q_-(qw_i)}{Q_-(q^{-1}w_i)}, \tag{3.4}$$

or more explicitly, setting $k = \sum k_m$,

$$q^k \prod_{m=1}^{L} \frac{w_i - q^{1-k_m} z_m}{w_i - qz_m} = -\zeta^{-2}\prod_{j=1}^{l_-}\frac{qw_i - w_j}{q^{-1}w_i - w_j}. \tag{3.5}$$

Rewriting this equation, we obtain the $\mathfrak{sl}_2$ XXZ Bethe equations (see e.g. [Res87]):

$$\prod_{m=1}^{L} \frac{w_i - q^{1-k_m} z_m}{w_i - qz_m} = -\zeta^{-2}q^{l_- - k}\prod_{j=1}^{l_-}\frac{qw_i - w_j}{w_i - qw_j}, \qquad i = 1, \ldots, l_-. \tag{3.6}$$

We call a solution of the Bethe equations nondegenerate if the $q^{\mathbb{Z}}$ lattices generated by the w_i's and z_m's are disjoint for all i and m. We have proven the following theorem:

Theorem 3.6. *There is a one-to-one correspondence between the set of nondegenerate solutions of the $\mathfrak{sl}_2$ XXZ Bethe equations (3.6) and the set of nondegenerate Z-twisted $(\mathrm{SL}(2), q)$-opers with regular singularities at the points $z_1, \ldots, z_L \neq 0, \infty$ with weights $k_1, \ldots k_L$.*

3.3. **The q-Miura transformation and the transfer matrix.** We now consider the q-Miura transformation which puts the q-connection matrix into a form analogous to (2.10) in the classical setting. As we will see, the eigenvalue of the transfer matrix for the XXZ model will appear explicitly in the q-connection matrix.

First, we consider the gauge change by

$$g(z) = \begin{pmatrix} Q_-(z) & -Q_+(z) \\ 0 & Q_-^{-1}(z)\end{pmatrix}, \tag{3.7}$$

which takes the section $s(z)$ into $g(z)s(z) = \binom{0}{1}$. In this gauge, the q-connection matrix has the form

$$\begin{aligned} A(z) &= \begin{pmatrix} Q_-(qz)\zeta^{-1} & -\zeta Q_+(qz) \\ 0 & \zeta Q_-^{-1}(qz)\end{pmatrix}\begin{pmatrix} Q_-(z) & -Q_+(z) \\ 0 & Q_-^{-1}(z)\end{pmatrix} \\ &= \begin{pmatrix} \zeta^{-1}Q_-(qz)Q_-^{-1}(z) & \rho(z) \\ 0 & \zeta Q_-^{-1}(qz)Q_-(z)\end{pmatrix}, \end{aligned} \tag{3.8}$$

where ρ is the quantum Wronskian.

Before proceeding, we recall that the eigenvalues of the *transfer matrix* [Res1010] for the XXZ model have the form

$$T(z) = \zeta^{-1}\rho(q^{-1}z)\frac{Q_-(qz)}{Q_-(z)} + \zeta\rho(z)\frac{Q_-(q^{-1}z)}{Q_-(z)}. \tag{3.9}$$

For ease of notation, we set $a(z) = \zeta^{-1}Q_-(qz)Q_-^{-1}(z)$, so that $A(z) = \left(\begin{smallmatrix} a(z) & \rho(z) \\ 0 & a^{-1}(z)\end{smallmatrix}\right)$ and $T(z) = \frac{a(z)}{\rho(q^{-1}z)} + a^{-1}(q^{-1}z)\rho(z)$. We now apply the gauge transformation by the matrix

$\left(\begin{smallmatrix} 1 & 0 \\ a(z)/\rho(z) & 1 \end{smallmatrix}\right)$; this brings the q-connection into the form

$$\hat{A}(z) = \begin{pmatrix} 0 & \rho(z) \\ -\rho^{-1}(z) & T(qz)\rho^{-1}(qz) \end{pmatrix}. \tag{3.10}$$

If $\left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right)$ is a solution of the corresponding difference equation, then we have $D_q(f_1) = \rho(z) f_2$ and $D_q(f_2) = -\rho^{-1}(z) f_1 + T(qz)\rho^{-1}(qz) f_2$. Simplifying, we see that f_1 is a solution of the second-order scalar difference equation

$$\left(D_q^2 - T(qz) D_q - \frac{\rho(qz)}{\rho(z)} \right) f_1 = 0. \tag{3.11}$$

Summing up, we have

Theorem 3.7. *Nondegenerate Z-twisted $(\mathrm{SL}(2), q)$-opers with regular singularities at the points $z_1, \ldots, z_n \neq 0, \infty$ with weights $k_1, \ldots k_n$ may be represented by meromorphic q-connections of the form (3.10) or equivalently, by the second-order scalar difference operators (3.11).*

3.4. **Embedding of the tRS model into q-opers.** We now explain a connection between nondegenerate twisted $(\mathrm{SL}(2), q)$-opers and the two particle trigonometric Ruijsenaars-Schneider model. More precisely, we show that the integrals of motion in the tRS model arise from nondegenerate twisted opers with two regular singularities of weight one and with Q_- linear.

Consider Z-twisted opers with two regular singularities $z_\pm$, both of weight one, so $\rho = (z - z_+)(z - z_-)$. For generic q, the degree of the quantum Wronskian equals $\deg(Q_+) + \deg(Q_-)$. Here, we will only look at q-opers for which $\deg(Q_\pm) = 1$, say $Q_- = z - p_-$ and $Q_+ = c(z - p_+)$. Here, c is a nonzero constant for which the quantum Wronskian is monic; an easy calculation shows that $c = q^{-1}(\zeta^{-1} - \zeta)^{-1}$.

Setting the quantum Wronskian equal to ρ gives us the equation

$$z^2 - \frac{z}{q}\left[\frac{\zeta - q\zeta^{-1}}{\zeta - \zeta^{-1}} p_+ + \frac{q\zeta - \zeta^{-1}}{\zeta - \zeta^{-1}} p_- \right] + \frac{p_+ p_-}{q} = (z - z_+)(z - z_-)\,. \tag{3.12}$$

Comparing powers of z on both sides, we obtain

$$\begin{aligned} \frac{\zeta - q\zeta^{-1}}{\zeta - \zeta^{-1}} p_+ + \frac{q\zeta - \zeta^{-1}}{\zeta - \zeta^{-1}} p_- &= q(z_+ + z_-) \\ \frac{p_+ p_-}{q} &= z_+ z_-\,. \end{aligned} \tag{3.13}$$

Upon introducing coordinates ζ_+, ζ_- such that $\zeta = \zeta_+/\zeta_-$ and viewing $\zeta_\pm, p_\pm$ as the positions and momenta in the two particle tRS model, we see that (3.13) are just the trigonometric Ruijsenaars-Schneider equations [KPSZ1705]. In fact, the set of Z-twisted opers with weight one singularities at $z_\pm$ is just the intersection of two Lagrangian subspaces of the two particle tRS phase space: the subspace determined by (3.13) and the subspace with the $\zeta_\pm$ fixed constants satisfying $\zeta = \zeta_+/\zeta_-$. As we will see in Section 7, this construction can be generalized to higher rank.

4. $(\mathrm{SL}(N), q)$-OPERS

4.1. **Definitions.**
We now discuss the generalization of $(\mathrm{SL}(2), q)$-opers to $\mathrm{SL}(N)$.

Definition 4.1. A $(\mathrm{GL}(N), q)$-*oper* on $\mathbb{P}^1$ is a triple $(E, A, \mathcal{L}_\bullet)$, where (E, A) is a $(\mathrm{GL}(N), q)$-connection and $\mathcal{L}_\bullet$ is a complete flag of subbundles such that A maps $\mathcal{L}_i$ into $\mathcal{L}_{i+1}^q$ and the induced maps $\bar{A}_i : \mathcal{L}_i/\mathcal{L}_{i-1} \longrightarrow \mathcal{L}_{i+1}^q/\mathcal{L}_i^q$ are isomorphisms for $i = 1, \ldots, N-1$. The triple is called an $\mathrm{SL}(N)$-*oper* if (E, A) is an $(\mathrm{SL}(N), q)$-connection.

To make this definition more explicit, consider the determinants

$$\Big(s(q^{i-1}z) \wedge A(q^{i-2}z)s(q^{i-2}z) \wedge \cdots \wedge \Big(\prod_{j=0}^{i-2}(A(q^{i-2-j}z)\Big)s(z)\Big)\Big|_{\Lambda^i \mathcal{L}_i^{q^{i-1}}} \tag{4.1}$$

for $i = 1, \ldots, N$, where s is a local section of $\mathcal{L}_1$. Then $(E, A, \mathcal{L}_\bullet)$ is a q-oper if and only if at every point, there exists local sections for which each $\mathcal{W}_i(s)(z)$ is nonzero. It will be more convenient to consider determinants with the same zeros as those in (4.1), but with no q-shifts:

$$\mathcal{W}_i(s)(z) = \Big(s(z) \wedge A(z)^{-1}s(qz) \wedge \cdots \wedge \Big(\prod_{j=0}^{i-2}(A(q^j z)^{-1}\Big)s(q^{i-1}z)\Big)\Big|_{\Lambda^i \mathcal{L}_i}. \tag{4.2}$$

As in the classical setting, we need to relax these conditions to allow for regular singularities. Fix a collection of L points $z_1, \ldots, z_L \neq 0, \infty$ such that the $q^{\mathbb{Z}}$-lattices they generate are pairwise disjoint. We associate a dominant integral weight $\lambda_m = \sum l_m^i \omega_i$ to each z_m. Set $\ell_m^i = \sum_{j=1}^i l_m^j$.

Definition 4.2. An $(\mathrm{SL}(N), q)$-*oper with regular singularities* at the points $z_1, \ldots, z_L \neq 0, \infty$ with weights $\lambda_1, \ldots \lambda_L$ is a meromorphic $(\mathrm{SL}(N), q)$-oper such that each $\bar{A}_i$ is an isomorphism except at the points $q^{-\ell_m^{i-1}}z_m, q^{-\ell_m^{i-1}+1}z_m, \ldots, q^{-\ell_m^i+1}z_m$ for each m, where it has simple zeros.

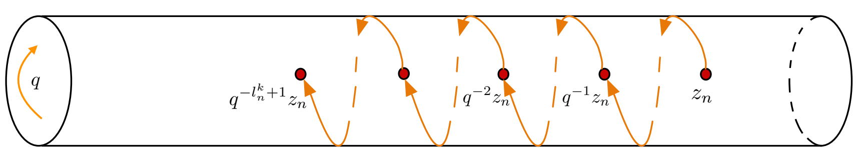


FIGURE 1. Weight of the singularity z_n as q-monodromy around the cylinder ($\mathbb{P}^1$ with 0 and ∞ removed).

In order to express the locations of the roots of the $\mathcal{W}_i(s)$'s, it is convenient to introduce the polynomials

$$\Lambda_i = \prod_{m=1}^{L} \prod_{j=\ell_m^{i-1}}^{\ell_m^i - 1} (z - q^{-j} z_m) \tag{4.3}$$

with zeros precisely where $\bar{A}_i$ is not an isomorphism. We also set

$$P_i = \Lambda_1\Lambda_2\cdots\Lambda_i = \prod_{m=1}^{L}\prod_{j=0}^{\ell_m^i-1}(z-q^{-j}z_m). \tag{4.4}$$

We introduce the notation $f^{(j)}(z) = D_q^j(f)(z) = f(q^jz)$. The zeros of $\mathcal{W}_k(s)$ coincide with those of the polynomial

$$\begin{aligned} W_k(s) &= \Lambda_1\left(\Lambda_1^{(1)}\Lambda_2^{(1)}\right)\cdots\left(\Lambda_1^{(k-2)}\cdots\Lambda_{k-1}^{(k-2)}\right) \\ &= P_1\cdot P_2^{(1)}\cdot P_3^{(2)}\cdots P_{k-1}^{(k-2)}\,. \end{aligned} \tag{4.5}$$

We now define *twisted* q*-opers*. Let $Z = \operatorname{diag}(\zeta_1,\ldots,\zeta_N) \in \mathrm{SL}(N,\mathbb{C})$ be a diagonal matrix with distinct eigenvalues.

Definition 4.3. An $(\mathrm{SL}(N),q)$-oper $(E,A,\mathcal{L}_\bullet)$ with regular singularities is called a Z*-twisted* q*-oper* if A is gauge-equivalent to Z^{-1}.

4.2. **Miura q-opers and quantum Wronskians.** Given a q-oper with regular singularities $(E,A,\mathcal{L}_\bullet)$, we can define the associated *Miura* q*-opers* as quadruples $(E,A,\mathcal{L}_\bullet,\hat{\mathcal{L}}_\bullet)$ where $\hat{\mathcal{L}}_\bullet$ is a complete flag preserved by the q-connection, i.e., A maps $\hat{\mathcal{L}}_i$ into $\hat{\mathcal{L}}_i^q$ for all i. Again, we will primarily be interested in *nondegenerate* Miura q-opers. This means that the flags $(\mathcal{L}_\bullet(z),\hat{\mathcal{L}}_\bullet(z))$ are in general position at all but a finite number of points $\{w_j\}$; moreover, at each w_j, the relative position is $w_0\sigma_k$ for some simple reflection σ_k. Finally, we assume that $q^{\mathbb{Z}}w_i \cap q^{\mathbb{Z}}w_j = \varnothing$ if $i \neq j$ and also that the $q^{\mathbb{Z}}$ lattices generated by the z_m's and w_j's do not intersect. (We remark that these last conditions are stronger than necessary; for example, one may instead specify that $w_j \neq q^iz_m$ for all j and m and for $|i| \leq n$, where n is a positive integer that may be computed explicitly from the weights.)

We now specialize to the case where $(E,A,\mathcal{L}_\bullet)$ is a Z-twisted q-oper. Here, there are only a finite number of possible associated Miura q-opers. Indeed, if we consider the gauge where the matrix of the q-connection is the regular semisimple diagonal matrix Z^{-1}, we see that the only possibilities for $\hat{\mathcal{L}}_\bullet$ are the $N!$ flags given by the permutations of the standard ordered basis $e_1,\ldots,e_N$. (This is analogous to the classical situation. The Miura opers lying above a given oper with regular singularities and trivial monodromy are parametrized by the flag manifold. However, there are only $N!$ Miura opers associated to an oper with regular singularities on $\mathbb{P}^1\setminus\infty$ whose underlying connection if $d + h\,dz$, where $h \in \mathfrak{gl}(N,\mathbb{C})$ is regular semisimple.) It suffices to consider Miura q-opers for the standard flag; indeed, if not, we can gauge change to one where $\hat{\mathcal{L}}_\bullet$ is the standard flag, but where Z is replaced by a Weyl group conjugate.

Let $s(z) = (s_1(z),\ldots,s_N(z))$ be a section generating $\mathcal{L}_1$, where the s_a's are polynomials. We now show that the nondegeneracy of the Miura q-oper may be expressed in terms of quantum Wronskians. Consider the zeros of the determinants

$$\mathcal{D}_k(s) = e_1\wedge\cdots\wedge e_{N-k}\wedge s(z)\wedge Zs(qz)\wedge\cdots\wedge Z^{k-1}s(q^{k-1}z) \tag{4.6}$$

for $k=1,\dots,N$. The arguments of Section 2.3 show that for our q-oper to be nondegenerate, we need the zeros of $\mathcal{D}_k(s)$ in $\cup_m q^{\mathbb{Z}} z_m$ to coincide with those of $\mathcal{W}_k(s)$. Moreover, we want the other roots of $\mathcal{D}_k(s)$ to generate disjoint $q^{\mathbb{Z}}$ lattices. To be more explicit, for $k=1,\dots,N$, we have nonzero constants α_k and polynomials

$$\mathcal{V}_k(z)=\prod_{a=1}^{r_k}(z-v_{k,a})\,, \tag{4.7}$$

for which

$$\det\begin{pmatrix} 1 & \dots & 0 & s_1(z) & \zeta_1 s_1(qz) & \cdots & \zeta_1^{k-1}s_1(q^{k-1}z)\\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & \dots & 1 & s_{N-k}(z) & \zeta_{N-k}s_{N-k}(qz) & \dots & \zeta_{N-k}^{k-1}s_{N-k}(q^{k-1}z)\\ 0 & \dots & 0 & s_{N-k+1}(z) & \zeta_{N-k+1}s_{N-k+1}(qz) & \dots & \zeta_{N-k+1}^{N-k-1}s_{N-k+1}(q^{k-1}z)\\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & \dots & 0 & s_N(z) & \zeta_N s_N(qz) & \cdots & \zeta_N^{k-1}s_N(q^{k-1}z)\end{pmatrix}=\alpha_k W_k\mathcal{V}_k\,; \tag{4.8}$$

moreover $q^{\mathbb{Z}}v_{k,a}$ is disjoint from every other $q^{\mathbb{Z}}v_{i,b}$ and each $q^{\mathbb{Z}}z_m$. Since $\mathcal{D}_N(s)=\mathcal{W}_N(s)$, we have $\mathcal{V}_N=1$. We also set $\mathcal{V}_0=1$; this is consistent with the fact that (4.6) also makes sense for $k=0$, giving $\mathcal{D}_0=e_1\wedge\cdots\wedge e_N$.

We can also rewrite (4.8) as

$$\det_{i,j}\left[\zeta_{N-k+i}^{j-1}s_{N-k+i}^{(j-1)}\right]=\alpha_k W_k\mathcal{V}_k\,, \tag{4.9}$$

where $i,j=1,\dots,k$.

We remark that the nonzero constants $\alpha_1,\dots,\alpha_N$ are normalization constants for the section s and may be chosen arbitrarily by first multiplying s by a nonzero constant and then applying constant gauge changes by diagonal matrices in $\mathrm{SL}(N)$.

4.3. $(\mathrm{SL}(N),q)$**-Opers and the XXZ Bethe ansatz.** We are now ready to prove our main theorem which relates twisted $(\mathrm{SL}(N),q)$-opers to solutions of the XXZ Bethe ansatz equations for $\mathfrak{sl}_N$.

The Bethe equations for the general $\mathfrak{sl}_N$ XXZ spin chain depend on an anisotropy parameter $q\in\mathbb{C}^*$ and twist parameters $\kappa_1,\dots,\kappa_N$ satisfying $\prod\kappa_i=1$. The equations can be written in the following form

$$\frac{\kappa_{k+1}}{\kappa_k}\prod_{s=1}^{L}\frac{q^{\ell_s^k+\frac{k}{2}-\frac{3}{2}}u_{k,a}-z_s}{q^{\ell_s^{k-1}+\frac{k}{2}-\frac{3}{2}}u_{k,a}-z_s}\cdot\prod_{c=1}^{r_{k-1}}\frac{q^{\frac12}u_{k,a}-u_{k-1,c}}{q^{-\frac12}u_{k,a}-u_{k-1,c}}\cdot\prod_{b=1}^{r_k}\frac{q^{-1}u_{k,a}-u_{k,b}}{qu_{k,a}-u_{k,b}}\cdot\prod_{d=1}^{r_{k+1}}\frac{q^{\frac12}u_{k,a}-u_{k+1,d}}{q^{-\frac12}u_{k,a}-u_{k+1,d}}=1 \tag{4.10}$$

for $k=1,\dots N-1$, $a=1,\dots,r_k$. (See, for example, [Res87].) The constants ℓ_m^i are determined by the dominant weights $\lambda_1,\dots,\lambda_L$ as in Section 4.1. We use the convention that $r_0=r_N=0$, so one of the products in the first and last equations is empty.

We remark that there exist many different normalizations of the XXZ Bethe equations in the literature depending on the scaling of the twist parameters. Our current normalization is designed to match the formulas obtained from q-opers.

We say that a solution of the Bethe equations is nondegenerate if $z_s \notin q^{\frac{1-k}{2}} q^{\mathbb{Z}} u_{k,a}$ for all k and a and also that $u_{k,a} \notin q^{\frac{k-k'}{2}} q^{\mathbb{Z}} u_{k',a'}$ unless $k = k'$ and $a = a'$.

For the computations to follow, it will be convenient to introduce the Baxter operators

$$\Pi_k = p_k \prod_{s=1}^{L} \prod_{j=\ell_s^{k-1}}^{\ell_s^k - 1} \left(z - q^{1-\frac{k}{2}-j} z_s\right), \qquad Q_k = \prod_{a=1}^{r_k} (z - u_{k,a}), \qquad k = 1, \dots N-1, \tag{4.11}$$

where the normalization constants $p_k = q^{(\frac{k}{2}-1)\sum_{m=1}^{L} l_k}$ are chosen so that $\Pi_k = \Lambda_k^{(\frac{k}{2}-1)}$. The Bethe equations (4.10) can then be written as

$$\frac{\kappa_{k+1}}{\kappa_k} \frac{\Pi_k^{(\frac{1}{2})} Q_{k-1}^{(\frac{1}{2})} Q_k^{(-1)} Q_{k+1}^{(\frac{1}{2})}}{\Pi_k^{(-\frac{1}{2})} Q_{k-1}^{(-\frac{1}{2})} Q_k^{(1)} Q_{k+1}^{(-\frac{1}{2})}} \Bigg|_{u_{k,a}} = -1, \tag{4.12}$$

where we recall that $f^{(p)}(z) = f(q^p z)$.

We observe that the Baxter operators are remarkably similar to the polynomials Λ_k and $\mathcal{V}_k$ (see (4.3) and (4.7)) which we used to describe the zeros of the quantum Wronskians arising from twisted q-opers. Our main theorem will make this connection precise. We begin by proving four lemmas.

Lemma 4.1. *Suppose that $\kappa_k \notin q^{\mathbb{N}_0} \kappa_{k+1}$ for all k. Then, the system of equations (4.12) is equivalent to the existence of auxiliary polynomials $\widetilde{Q}_k(z)$ satisfying the following system of equations*

$$\kappa_{k+1} Q_k^{(-\frac{1}{2})} \widetilde{Q}_k^{(\frac{1}{2})} - \kappa_k Q_k^{(\frac{1}{2})} \widetilde{Q}_k^{(-\frac{1}{2})} = (\kappa_{k+1} - \kappa_k) Q_{k-1} Q_{k+1} \Pi_k, \tag{4.13}$$

for $k = 1, \dots N-1$. Moreover, these polynomials are unique.

Proof. Set $g(z) = \widetilde{Q}_k(z)/Q_k(z)$ and $f(z) = (\kappa_{k+1} - \kappa_k) Q_{k-1}^{(\frac{1}{2})} Q_{k+1}^{(\frac{1}{2})} \Pi_k^{(\frac{1}{2})}$, so that (4.13) may be rewritten as

$$\kappa_{k+1} g_k^{(1)}(z) - \kappa_k g_k(z) = \frac{f(z)}{Q_k(z) Q_k^{(1)}(z)}. \tag{4.14}$$

We then have the partial fraction decompositions

$$\begin{aligned} \frac{f(z)}{Q_k(z) Q_k^{(1)}(z)} &= h(z) - \sum_a \frac{b_a}{z - u_{k,a}} + \sum_a \frac{c_a}{qz - u_{k,a}}, \\ g_k(z) &= \tilde{g}_k(z) + \sum_a \frac{d_a}{z - u_{k,a}} \end{aligned} \tag{4.15}$$

where $h(z)$ and $\tilde{g}_k(z)$ are polynomials. In order for the residues at each $u_{k,a}$ to match on the two sides of (4.14), one needs

$$d_a = \frac{b_a}{\kappa_k} = \frac{c_a}{\kappa_{k+1}}. \tag{4.16}$$

The second equality is merely the Bethe equations (4.12) in the alternate form

$$\mathrm{Res}_{u_{k,a}}\left[\frac{f(z)}{\kappa_k Q_k(z) Q_k^{(1)}(z)}\right] + \mathrm{Res}_{u_{k,a}}\left[\frac{f^{(-1)}(z)}{\kappa_{k+1} Q_k^{(-1)}(z) Q_k(z)}\right] = 0 \tag{4.17}$$

or

$$\left.\left(\frac{Q_{k-1}^{(\frac{1}{2})} Q_{k+1}^{(\frac{1}{2})} \Pi_k^{(\frac{1}{2})}}{\kappa_k Q_k^{(1)}} + \frac{Q_{k-1}^{(-\frac{1}{2})} Q_{k+1}^{(-\frac{1}{2})} \Pi_k^{(-\frac{1}{2})}}{\kappa_{k+1} Q_k^{(-1)}}\right)\right|_{u_{k,a}} = 0. \tag{4.18}$$

Next, to solve for the polynomial $\tilde{g}_k(z)$, set $\tilde{g}_k(z) = \sum r_i z^i$ and $h(z) = \sum s_i z^i$. We then obtain the equations $r_i(\kappa_{k+1} q^i - \kappa_k) = s_i$. Our assumptions on the κ_j's imply that these equations are always solvable. Thus, there exist polynomials $\widetilde{Q}_k(z)$ satisfying (4.13) if and only if the Bethe equations hold. The uniqueness statement holds since the solutions for the residues d_a and the coefficients of the polynomial $\tilde{g}_k(z)$ are unique. □

Lemma 4.2. *The system of equations* (4.13) *is equivalent to the set of equations*

$$\kappa_{k+1} \mathscr{D}_k^{(-\frac{1}{2})} \widetilde{\mathscr{D}}_k^{(\frac{1}{2})} - \kappa_k \mathscr{D}_k^{(\frac{1}{2})} \widetilde{\mathscr{D}}_k^{(-\frac{1}{2})} = (\kappa_{k+1} - \kappa_k) \mathscr{D}_{k-1} \mathscr{D}_{k+1}\,, \tag{4.19}$$

for the polynomials

$$\mathscr{D}_k = Q_k F_k\,, \qquad \widetilde{\mathscr{D}}_k = \widetilde{Q}_k F_k\,, \tag{4.20}$$

where $F_k = W_k^{(\frac{1-k}{2})}$.

Proof. The F_k's are solutions to the functional equation

$$\frac{F_{k-1} \cdot F_{k+1}}{F_k^{(\frac{1}{2})} \cdot F_k^{(-\frac{1}{2})}} = \Pi_k\,. \tag{4.21}$$

Indeed, since $W_k = P_{k-1}^{(k-2)} W_{k-1}$ and $P_k = \Lambda_k P_{k-1}$, we have

$$\frac{F_{k-1} \cdot F_{k+1}}{F_k^{(\frac{1}{2})} \cdot F_k^{(-\frac{1}{2})}} = \frac{W_{k-1}^{(\frac{2-k}{2})} \cdot W_{k+1}^{(-\frac{k}{2})}}{W_k^{(\frac{2-k}{2})} \cdot W_k^{(-\frac{k}{2})}} = \frac{P_k^{(\frac{k}{2}-1)}}{P_{k-1}^{(\frac{k}{2}-1)}} = \Lambda_k^{(\frac{k}{2}-1)} = \Pi_k. \tag{4.22}$$

The equivalence of (4.13) and (4.19) follows easily from this fact. □

Let $V(\gamma_1, \dots, \gamma_k)$ denote the $k \times k$ Vandermonde matrix (γ_i^j). We recall that this determinant is nonzero if and only if the γ_i's are distinct.

Lemma 4.3. *Suppose that $\gamma_1, \dots, \gamma_{k-1}$ are nonzero complex numbers such that $\gamma_j \notin q^{\mathbb{N}_0}\gamma_k$ for $j < k$. Let $f_1, \dots, f_{k-1}$ be polynomials that do not vanish at 0, and let g be an arbitrary polynomial. Then there exist unique polynomials $f_1, \dots, f_k$ satisfying*

$$g = \det \begin{pmatrix} f_1 & \gamma_1 f_1^{(1)} & \cdots & \gamma_1^{k-1} f_1^{(k-1)} \\ \vdots & \vdots & \ddots & \vdots \\ f_k & \gamma_k f_k^{(1)} & \cdots & \gamma_k^{k-1} f_k^{(k-1)} \end{pmatrix}. \tag{4.23}$$

Moreover, if $g(0) \neq 0$, then $f_k(0) \neq 0$.

Proof. Set $f_j(z) = \sum a_{ji} z^i$ and $g(z) = \sum b_i z^i$, and let F denote the matrix in (4.23). We show that we can solve for the a_{ji}'s recursively. Expanding by minors along the bottom row, we get $g = \sum_{j=1}^{k} (-1)^{k+j} \det F_{k,j} f_k^{(j-1)}$. First, we equate the constant terms. This gives

$$b_0 = a_{k0} \left(\prod_{j=1}^{k-1} a_{j0} \right) \sum_{j=1}^{k} (-1)^{k+j} \gamma_k^{j-1} \det V(\gamma_1, \dots, \gamma_k)_{k,j} = a_{k0} \left(\prod_{j=1}^{k-1} a_{j0} \right) \det V(\gamma_1, \dots, \gamma_k).$$

Since the γ_j's are distinct, the Vandermonde determinant is nonzero. Moreover, $a_{j0} \neq 0$ for $j = 1, \dots, k-1$. Thus, we can solve uniquely for a_{k0}. In particular, if $b_0 = 0$, then $a_{k0} = 0$.

For the inductive step, assume that we have found unique a_{kr} for $r < s$ such that the polynomial equation (4.23) has equal coefficients up through degree $s-1$. We now look at the coefficient of z^s. The only way that a_{ks} appears in this coefficient is through the constant terms of the minors $F_{k,j}$. To be more explicit, equating the coefficient of z^s in (4.23) expresses $c a_{ks}$ as a polynomial in known quantities, where

$$\begin{aligned} c &= \left(\prod_{j=1}^{k-1} a_{j0} \right) \sum_{j=1}^{k} (-1)^{k+j} (q^s \gamma_k)^{j-1} \det V(\gamma_1, \dots, \gamma_{k-1}, q^s \gamma_k)_{k,j} \\ &= \left(\prod_{j=1}^{k-1} a_{j0} \right) \det V(\gamma_1, \dots, \gamma_{k-1}, q^s \gamma_k). \end{aligned}$$

Again, our condition on the γ_j's implies that the Vandermonde determinant is nonzero, so there is a unique solution for a_{ks}. $\square$

In the following lemma, we consider matrices

$$M_{i_1,\dots,i_j} = \begin{pmatrix} \mathfrak{q}_{i_1}^{(\frac{1-j}{2})} & \kappa_{N+1-i_1} \mathfrak{q}_{i_1}^{(\frac{3-j}{2})} & \cdots & \kappa_{N+1-i_1}^{j-1} \mathfrak{q}_{i_1}^{(\frac{j-1}{2})} \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{q}_{i_j}^{(\frac{1-j}{2})} & \kappa_{N+1-i_j} \mathfrak{q}_{i_j}^{(\frac{3-j}{2})} & \cdots & \kappa_{N+1-i_j}^{j-1} \mathfrak{q}_{i_j}^{(\frac{j-1}{2})} \end{pmatrix}, \tag{4.24}$$

where $\mathfrak{q}_1, \dots, \mathfrak{q}_N$ are polynomials. We also set $V_{i_1,\dots,i_j} = V(\kappa_{N+1-i_1}, \dots, \kappa_{N+1-i_j})$.

Lemma 4.4. *Assume that the lattices $q^{\mathbb{Z}}\kappa_k$ are disjoint for distinct k. Given polynomials $\mathscr{D}_k, \widetilde{\mathscr{D}}_k$ for $k = 1, \ldots, N-1$ satisfying (4.19), there exist unique polynomials $\mathfrak{q}_1, \ldots, \mathfrak{q}_N$ such that*

$$\mathscr{D}_k = \frac{\det M_{N-k+1,\ldots,N}}{\det V_{N-k+1,\ldots,N}} \qquad \textit{and} \qquad \widetilde{\mathscr{D}}_k = \frac{\det M_{N-k,N-k+2,\ldots,N}}{\det V_{N-k,N-k+2,\ldots,N}}\,. \tag{4.25}$$

Proof. We begin by observing that since W_k and Q_k do not vanish at 0, $\mathscr{D}_k(0) \neq 0$ for all k. This implies that $\widetilde{\mathscr{D}}_k(0) \neq 0$ for all k as well; otherwise, by (4.19), either $\mathscr{D}_{k-1}$ or $\mathscr{D}_{k+1}$ would vanish at 0.

Now, set $\mathfrak{q}_N = \mathscr{D}_1$ and $\mathfrak{q}_{N-1} = \widetilde{\mathscr{D}}_1$. It is obvious that these are the unique polynomials satisfying (4.25) for $k = 1$ and that $\mathfrak{q}_N(0), \mathfrak{q}_{N-1}(0) \neq 0$. Also, (4.19) gives

$$\kappa_2 \mathfrak{q}_N^{(-\frac{1}{2})} \mathfrak{q}_{N-1}^{(\frac{1}{2})} - \kappa_1 \mathfrak{q}_N^{(\frac{1}{2})} \mathfrak{q}_{N-1}^{(-\frac{1}{2})} = (\kappa_2 - \kappa_1)\, \mathscr{D}_2 \tag{4.26}$$

so $\mathscr{D}_2 = M_{N-1,N}/V_{N-1,N}$.

Next, suppose that for $2 \leq k \leq N-1$, we have shown that there exist unique polynomials $\mathfrak{q}_N, \ldots, \mathfrak{q}_{N-k+1}$ such the formulas for $\mathscr{D}_j$ (resp. $\widetilde{\mathscr{D}}_j$) in (4.25) hold for $1 \leq j \leq k$ (resp. $1 \leq j \leq k-1$). Furthermore, assume that none of these polynomials vanish at 0. We will show that there exists a unique $\mathfrak{q}_{N-k}$ such that the formulas for $\mathscr{D}_{k+1}$ and $\widetilde{\mathscr{D}}_k$ hold and that $\mathfrak{q}_{N-k}(0) \neq 0$. This will prove the lemma.

We use Lemma 4.3 to define $\mathfrak{q}_{N-k}$. In the notation of that lemma, set $f_j = \mathfrak{q}_{N+1-j}^{(\frac{1-k}{2})}$ and $\gamma_j = \kappa_j$ for $1 \leq j \leq k-1$, and set $g = (-1)^{\frac{k(k-1)}{2}}(\det V_{N-k,N-k+2,\ldots,N})\widetilde{\mathscr{D}}_k$. (The sign factor occurs because we have written the rows in reverse order to apply the lemma.) By hypothesis, $f_j(0) \neq 0$ for $1 \leq j \leq k-1$, so there exists a unique f_k satisfying (4.23). Moreover, $g(0) \neq 0$, so $f_k \neq 0$. It is now clear that $\mathfrak{q}_{N-k} = f_{k-1}^{(\frac{k-1}{2})}$ is the unique polynomial satisfying the formula in (4.25) for $\widetilde{\mathscr{D}}_k$. Of course, $\mathfrak{q}_{N-k} \neq 0$.

To complete the inductive step, it remains to show that the formula for $\mathscr{D}_{k+1}$ is satisfied. We make use of the Desnanot-Jacobi/Lewis Carroll identity for determinants. Given a square matrix M, let M^i_j denote the square submatrix with row i and column j removed; similarly, let $M^{i,i'}_{j,j'}$ be the submatrix with rows i and i' and columns j and j' removed. We will apply this identity in the form

$$\det M^1_1 \det M^2_{k+1} - \det M^1_{k+1} \det M^2_1 = \det M^{1,2}_{1,k+1} \det M\,. \tag{4.27}$$

Set $M = M_{N-k,\ldots,N}$. All of the matrices appearing in (4.27) are obtained from matrices of the form (4.24) via q-shifts, multiplication of each row by an appropriate κ_i, or both. In particular, $M^1_{k+1} = M^{(-\frac{1}{2})}_{N-k+1,\ldots,N}$ and $M^2_{k+1} = M^{(-\frac{1}{2})}_{N-k,N-k+2,\ldots,N}$ while the determinants of the other three are given by

(4.28)
$$\det M_1^1 = \kappa_k \left(\prod_{j=1}^{k-1} \kappa_j\right) \det M^{(\frac{1}{2})}_{N-k+1,\dots,N}, \qquad \det M_1^2 = \kappa_{k+1} \left(\prod_{j=1}^{k-1} \kappa_j\right) \det M^{(\frac{1}{2})}_{N-k,N-k+2,\dots,N},$$
$$\det M^{1,2}_{1,k+1} = \left(\prod_{j=1}^{k-1} \kappa_j\right) \det M_{N-k+2,\dots,N}.$$

Upon substituting into (4.27) and dividing by $\prod_{j=1}^{k-1}\kappa_j$, we obtain

(4.29)
$$\kappa_k \det M^{(\frac{1}{2})}_{N-k+1,\dots,N} \det M^{(-\frac{1}{2})}_{N-k,N-k+2,\dots,N} - \kappa_{k+1} \det M^{(-\frac{1}{2})}_{N-k+1,\dots,N} \det M^{(\frac{1}{2})}_{N-k,N-k+2,\dots,N} \\ = \det M_{N-k+2,\dots,N} \det M_{N-k,\dots,N}\,.$$

Finally, dividing both sides by $V_{N-k+1,\dots,N}V_{N-k,N-k+2,\dots,N}$ and applying the inductive hypothesis gives (4.20) multiplied by -1. This is obvious for the left-hand sides. To see that the other sides match, one need only observe that

$$V_{N-k+2,\dots,N}V_{N-k,\dots,N} = (\kappa_k - \kappa_{k+1}) \prod_{1\leq i<j\leq k-1} (\kappa_i-\kappa_j)^2 \prod_{i=1}^{k-1}(\kappa_i-\kappa_k)(\kappa_i-\kappa_{k+1}) \\ = (\kappa_k - \kappa_{k+1}) V_{N-k+1,\dots,N}V_{N-k,N-k+2,\dots,N}. \tag{4.30}$$

□

Note that the first relations from (4.25) can be rewritten as

$$\det_{i,j}\left[\kappa_{k+1-i}^{j-1}\mathfrak{q}_{N-k+i}^{\left(j-\frac{k+1}{2}\right)}\right] = \det_{i,j}\left[\kappa_{k+1-i}^{j-1}\right]\mathscr{D}_k\,, \tag{4.31}$$

We are finally ready to prove our main theorem.

Theorem 4.5. *Suppose that $\kappa_1,\dots,\kappa_N$ generate disjoint $q^{\mathbb{Z}}$-lattices. Then, there is a one-to-one correspondence between nondegenerate solutions of the $\mathfrak{sl}_N$ XXZ Bethe ansatz equations (4.10) with twist parameters κ_i and nondegenerate Z-twisted $(\mathrm{SL}(N),q)$-opers with regular singularities at $z_1,\dots,z_L$ with dominant weight $\lambda_1,\dots,\lambda_L$ provided that*

$$q^{\frac{1-k}{2}}u_{k,a} = v_{k,a} \quad \text{and} \quad \zeta_k = \kappa_{N+1-k} \tag{4.32}$$

for $k=1,\dots,N$. Moreover, the q-oper equations (4.8) become identical to the Bethe equations if one normalizes the section s via

$$\alpha_k = q^{\frac{k-1}{2}r_k}\det V(\kappa_k,\dots,\kappa_1). \tag{4.33}$$

Proof. We have shown that a solution to the Bethe equations is uniquely determined by polynomials $\mathfrak{q}_k$ satisfying (4.31). We will show that after matching the parameters as in the statement and normalizing the section $s(z)$ generating the q-oper, the components s_k

also satisfy these equations, so $s_k = \mathfrak{q}_k$ for all k. Since the twisted q-oper is uniquely determined by s, we obtain the desired correspondence.

After shifting (4.31) by $\frac{k-1}{2}$ and using the definition of $\mathscr{D}_k$ from (4.20), we obtain the equivalent form

$$\det_{i,j}\left[\kappa_{k+1-i}^{j-1}\mathfrak{q}_{N-k+i}^{(j-1)}\right] = \det_{i,j}\left[\kappa_{k+1-i}^{j-1}\right] W_k Q_k^{(\frac{k-1}{2})}. \tag{4.34}$$

On the other hand, rewriting the q-oper relations (4.9) for convenience, we have

$$\det_{i,j}\left[\zeta_{N-k+i}^{j-1}s_{N-k+i}^{(j-1)}\right] = \alpha_k W_k \mathcal{V}_k\,. \tag{4.35}$$

If we set $q^{\frac{1-k}{2}}u_{k,a} = v_{k,a}$, then the roots of $\mathcal{V}$ and $Q_k^{(\frac{k-1}{2})}$ coincide; moreover, the leading terms of the polynomials on the right are the same if one takes $\alpha_k = q^{\frac{k-1}{2}r_k}\det V(\kappa_k,\dots,\kappa_1)$. Thus, if one sets $\zeta_k = \kappa_{N+1-k}$, the two equations are identical.

It only remains to observe that the notions of nondegeneracy are preserved by the transformation (4.32). $\square$

5. Explicit equations for $(\mathrm{SL}(3),q)$-opers

5.1. **A canonical form.** In this section, we illustrate the general theory in the case of SL(3). In particular, we show that the underlying q-connection can be expressed entirely in terms of the Baxter operators and the twist parameters.

We start in the gauge where the connection is given by the diagonal matrix $\mathrm{diag}(\zeta_1^{-1},\zeta_2^{-1},\zeta_3^{-1})$ and the section generating the line bundle $\mathcal{L}_1$ is $s=(s_1,s_2,s_3)$. We now apply a q-gauge change by a certain matrix $g(z)$ mapping s to the standard basis vector e_3:

$$g(z) = \begin{pmatrix}\beta(z) & -\alpha(z) & 0\\ 0 & \beta(z)^{-1} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}s_2(z) & -s_1(z) & 0\\ 0 & \frac{s_3(z)}{s_2(z)} & -1\\ 0 & 0 & \frac{1}{s_3(z)}\end{pmatrix}, \tag{5.1}$$

where $\alpha(z) = \zeta_1^{-1}s_1^{(-1)}s_2 - \zeta_2^{-1}s_1s_2^{(-1)}$ and $\beta(z) = \frac{1}{s_2}(\zeta_2^{-1}s_2^{(-1)}s_3 - \zeta_3^{-1}s_2s_3^{(-1)})$. Applying the q-change formula (3.1) leads to a matrix all of whose entries are expressible in terms of minors of the matrix

$$M_{1,2,3}^{(1)} = \begin{pmatrix}s_1 & \zeta_1 s_1^{(1)} & \zeta_1^2 s_1^{(2)}\\ s_2 & \zeta_2 s_2^{(1)} & \zeta_2^2 s_2^{(2)}\\ s_3 & \zeta_3 s_3^{(1)} & \zeta_3^2 s_3^{(2)}\end{pmatrix}. \tag{5.2}$$

By (4.8), the relations between the Baxter operators and these determinants are given by

$$\det M_3 = \alpha_1\mathcal{V}_1^{(-1)}, \qquad M_{2,3} = \alpha_2 W_2^{(-1)}\mathcal{V}_2^{(-1)} = \alpha_2\Lambda_1^{(-1)}\mathcal{V}_2^{(-1)}, \qquad \text{and}$$
$$\det M_{1,2,3} = \alpha_3 W_3^{(-1)} = \alpha_3\Lambda_1^{(-1)}\Lambda_1\Lambda_2. \tag{5.3}$$

A further diagonal q-gauge change by $\mathrm{diag}(\alpha_3^{-2/3}(\Lambda_1^{(-1)})^{-1}, \alpha_3^{1/3}\Lambda_1^{(-1)}, \alpha_3^{1/3})$ brings us to our desired form:

$$A(z) = \begin{pmatrix} a_1(z) & \Lambda_2(z) & 0 \\ 0 & a_2(z) & \Lambda_1(z) \\ 0 & 0 & a_3(z) \end{pmatrix}, \tag{5.4}$$

where

$$\begin{aligned} a_1 &= \zeta_1^{-1} \frac{\Lambda_1^{(-1)}}{\Lambda_1} \cdot \frac{\det M_{2,3}}{\det M_{2,3}^{(-1)}} = \zeta_1^{-1} \frac{\mathcal{V}_2}{\mathcal{V}_2^{(-1)}}, \\ a_2 &= \zeta_2^{-1} \frac{\Lambda_1}{\Lambda_1^{(-1)}} \cdot \frac{s_3^{(1)}}{s_3} \frac{\det M_{2,3}^{(-1)}}{\det M_{2,3}} = \zeta_2^{-1} \frac{\mathcal{V}_2^{(-1)}}{\mathcal{V}_2} \cdot \frac{\mathcal{V}_1^{(1)}}{\mathcal{V}_1}, \\ a_3 &= \zeta_3^{-1} \frac{s_3}{s_3^{(1)}} = \zeta_3^{-1} \frac{\mathcal{V}_1}{\mathcal{V}_1^{(1)}}. \end{aligned} \tag{5.5}$$

Note that the singularities of the oper and the Bethe roots can be determined from the zeros of the superdiagonal and the diagonal respectively.

5.2. **Scalar difference equations and eigenvalues of transfer matrices.** The first-order system of difference equations $f(qz) = A(z)f(z)$ determined by (5.4) can be expressed as a third-order scalar difference equation. This is accomplished by the q-Miura transformation: a q-gauge change by a lower triangular matrix which reduces $A(z)$ to companion matrix form. (This procedure appears as part of the difference equation version of Drinfeld-Sokolov reduction introduced in [FRSTS98].)

In the XXZ model, the eigenvalues of the SL(3)-transfer matrices for the two fundamental weights are [BHK02, FH15]

$$\begin{aligned} T_1 &= a_1^{(2)}\Lambda_1\Lambda_2^{(1)} + a_2^{(1)}\Lambda_1\Lambda_2^{(2)} + a_3\Lambda_1^{(1)}\Lambda_2^{(2)}, \\ T_2 &= a_1^{(1)}a_2^{(1)}\Lambda_1\Lambda_2 + a_1^{(1)}a_3\Lambda_1^{(1)}\Lambda_2 + a_2a_3\Lambda_1^{(1)}\Lambda_2^{(1)}. \end{aligned} \tag{5.6}$$

Just as for SL(2), these eigenvalues appear in the coefficients of the scalar difference equation associated to our twisted q-oper. Indeed, a simple calculation shows that the system

$$\begin{aligned} f_1^{(1)} &= a_1 f_1 + \Lambda_2 f_2 \\ f_2^{(1)} &= a_2 f_2 + \Lambda_1 f_3 \\ f_3^{(1)} &= a_3 f_3 \end{aligned} \tag{5.7}$$

is equivalent to

$$\Lambda_1\Lambda_2\Lambda_2^{(1)} \cdot f_1^{(3)} - \Lambda_2\, T_1 \cdot f_1^{(2)} + \Lambda_2^{(2)}\, T_2 \cdot f_1^{(1)} - \Lambda_1^{(1)}\Lambda_2^{(1)}\Lambda_2^{(2)} \cdot f_1 = 0\,. \tag{5.8}$$

6. Scaling limits: From q-opers to opers

In this section, we consider classical limits of our results. We will take the limit from q-opers to opers in two steps. The first will give rise to a correspondence between the spectra of a twisted version of the XXX spin chain and a twisted analogue of the discrete opers of [MV05]. By taking a further limit, we recover the relationship between opers with an irregular singularity and the inhomogeneous Gaudin model [FFTL10, FFR10].

First, we introduce an exponential reparameterization of q, the singularities, and the Bethe roots: $q = e^{R\epsilon}$, $z_s = e^{R\sigma_s}$, and $v_{k,a} = e^{R\upsilon_{k,a}}$. We also set $\tilde{\ell}_s^k = \ell_s^k + \frac{k}{2} - \frac{3}{2}$. We now take the limit of the XXZ Bethe equations (4.10) as R goes to 0. This limit brings us to the XXX Bethe equations

(6.1)

$$\frac{\kappa_{k+1}}{\kappa_k}\prod_{s=1}^{L}\frac{\upsilon_{k,a}+\ell_s^k\epsilon-\sigma_s}{\upsilon_{k,a}+\ell_s^{k-1}\epsilon-\sigma_s}\cdot\prod_{c=1}^{r_{k-1}}\frac{\upsilon_{k,a}-\upsilon_{k-1,c}+\frac{1}{2}\epsilon}{\upsilon_{k,a}-\upsilon_{k-1,c}-\frac{1}{2}\epsilon}\cdot\prod_{b=1}^{r_k}\frac{\upsilon_{k,a}-\upsilon_{k,b}-\epsilon}{\upsilon_{k,a}-\upsilon_{k,b}+\epsilon}\cdot\prod_{d=1}^{r_{k+1}}\frac{\upsilon_{k,a}-\upsilon_{k+1,d}+\frac{1}{2}\epsilon}{\upsilon_{k,a}-\upsilon_{k+1,d}-\frac{1}{2}\epsilon}=1\,.$$

Geometrically, we identify $\mathbb{C}^*$ with an infinite cylinder of radius R^{-1} and view this cylinder as the base space of our twisted q-oper. We then send the radius to infinity, thereby arriving at a twisted version of the discrete opers of Mukhin and Varchenko [MV05].

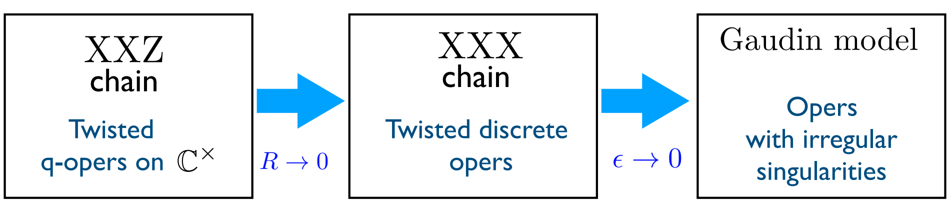
The second limit takes us from the XXX spin chain to the Gaudin model. In order to do this, we set $\kappa_i = e^{\epsilon \mathsf{K}_i}$, and let ϵ go to 0. As expected, we obtain the Bethe equations for the inhomogeneous Gaudin model, i.e., the higher rank analogues of (2.16):

(6.2)

$$\mathsf{K}_{k+1}-\mathsf{K}_k+\sum_{s=1}^{L}\frac{l_s^k}{\upsilon_{k,a}-\sigma_s}+\sum_{c=1}^{r_{k-1}}\frac{1}{\upsilon_{k,a}-\upsilon_{k-1,c}}-\sum_{b\neq a}^{r_k}\frac{2}{\upsilon_{k,a}-\upsilon_{k,b}}+\sum_{d=1}^{r_{k+1}}\frac{1}{\upsilon_{k,a}-\upsilon_{k+1,d}}=0\,.$$

Note that the difference of the twists K_i can be identified with the monodromy data of the connection $A(z)$ at infinity.

We have thus established the following hierarchy between integrable spin chain models and oper structures.



7. Quantum K-theory of Nakajima quiver varieties and q-opers

7.1. **The quantum K-theory ring for partial flag varieties.** As we discussed in the introduction, integrable models play an important role in enumerative geometry. For example, consider the XXZ spin chain for $\mathfrak{sl}_N$ where the dominant weights at the marked points z_m all correspond to the defining representation, i.e., $\lambda_m = (1, 0, 0, \dots, 0)$. Recall that cotangent bundles to partial flag varieties are particular case of quiver varieties of type A (see Fig. 2).

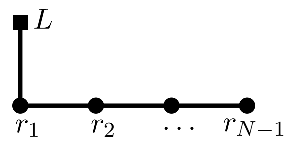


FIGURE 2. The cotangent bundle to the partial flag variety $T^*\mathbb{F}l_{\boldsymbol{\mu}}$

It follows from work of Nakajima [Nak01] that the space of localized equivariant K-theory of such a cotangent bundle can be identified with an appropriate weight space in the corresponding XXZ model; moreover, the span of all such weight spaces for partial flag varieties of $\mathrm{SL}(N)$ is endowed with a natural action of the quantum group $U_q(\mathfrak{sl}_N)$.

In [KPSZ1705], it was established that the Bethe algebra for this XXZ model—the algebra generated by the Q-operators of the XXZ spin chain—can be entirely described in terms of enumerative geometry. The equivariant quantum K-theory of the cotangent bundle to a partial flag variety has generators which are quantum versions of tautological bundles. It is shown in [KPSZ1705] that the eigenvalues of these quantum tautological bundles are the symmetric functions in the Bethe roots, so that the quantum K-theory may be identified with the Bethe algebra. Moreover, the twist parameters κ_{i+1}/κ_i and the inhomogeneity (or evaluation) parameters z_m are identified with the Kähler parameters of the quantum deformation and the equivariant parameters respectively.

In the case of complete flag varieties, the authors of [KPSZ1705] found another set of generators which allows the identification of the quantum K-theory ring with the algebra of functions on a certain Lagrangian subvariety in the phase space for the trigonometric Ruijsenaars-Schneider model. The formulas used to establish this (see Proposition 4.4 of [KPSZ1705]) are strikingly similar to the equations (4.8) describing nondegenerate twisted q-opers. Let us normalize the section $s(z)$ in the definition of a twisted q-oper so that all of its components are monic polynomials:

$$s_a(z) = \prod_{i=1}^{\rho_a} (z - w_{a,i}) \,, \qquad a = 1, \dots, N. \tag{7.1}$$

If we restrict to the space of q-opers for which all these polynomials have degree one, then their roots may be viewed as coordinates. These coordinates may be identified with the momenta of the dual tRS model whereas the coordinates of the tRS model correspond bijectively to the twist (Kähler) parameters κ_{i+1}/κ_i [KPSZ1705].

These observations lead us to the following theorem about the equivariant quantum K-theory of cotangent bundles to partial flag varieties. This result was conjectured by Rimanyi, Tarasov, and Varchenko; see Conjecture 13.15 in [RTV1411].

Theorem 7.1. *Let X be the cotangent bundle of the $\mathrm{GL}(L)$ partial flag variety $T^*\mathbb{F}l_{\boldsymbol{\mu}}$ labeled by the vector $\boldsymbol{\mu} = (r_{N-1} - r_{N-2}, \ldots, r_1 - r_2, L - r_1)$ where $r_1, \ldots, r_{N-1}$ and L are the dimensions of the vector spaces corresponding to the nodes of the A_{N-1} quiver and the framing on the first node in Fig. 2 respectively. Let T be a maximal torus in $\mathrm{GL}(L)$.*

Then the T-equivariant quantum K-theory of X is given by the algebra

$$QK_T(X) = \frac{\mathbb{C}\left[\boldsymbol{p}^{\pm 1}, \boldsymbol{\kappa}^{\pm 1}, \boldsymbol{a}^{\pm 1}, q^{\pm\frac{1}{2}}\right]}{\left(\det M(z) = \det V_{1,\ldots,N} \cdot \Pi(z)^{\left(\frac{1-N}{2}\right)}\right)}\,, \tag{7.2}$$

where $\boldsymbol{\kappa} = (\kappa_1, \ldots, \kappa_N)$ are the quantum deformation parameters, $\boldsymbol{a} = (a_1, \ldots, a_L)$ are the equivariant parameters of the action of T on X,

$$\boldsymbol{p} = \{p_{a,i}\}\,, \quad i = 1, \ldots, \rho_a, \quad a = 1, \ldots, N-1\,, \tag{7.3}$$

are the coefficients of the polynomials

$$s_a(z) = \prod_{i=1}^{\rho_a} (z - w_{a,i}) = \sum_{i=0}^{\rho_a} (-1)^i p_{a,i}\, z^{N-i}\,, \tag{7.4}$$

where $\rho_k = r_k - r_{k-1}$, $\Pi(z) = \prod\limits_{s=1}^{L}(z - a_s)$, and the matrix M is given by

$$M = \begin{pmatrix} s_1^{\left(\frac{1-N}{2}\right)} & \kappa_1 s_1^{\left(\frac{3-N}{2}\right)} & \cdots & \kappa_1^{N-1} s_1^{\left(\frac{N-1}{2}\right)} \\ \vdots & \vdots & \ddots & \vdots \\ s_N^{\left(\frac{1-N}{2}\right)} & \kappa_N s_N^{\left(\frac{3-N}{2}\right)} & \cdots & \kappa_N^{N-1} s_N^{\left(\frac{N-1}{2}\right)} \end{pmatrix}. \tag{7.5}$$

The ideal in (7.2) depends on the auxiliary variable z, and both sides of the equation are polynomials of degree L in z. Thus, the quantum K-theory ring is determined by L relations.

The case where X is a complete flag variety, so that $L = N$ and $\rho_1 = \cdots = \rho_{N-1} = 1$, was investigated in [KPSZ1705]. Here, the determinantal relation in (7.2) yields the equations of motion of the N-body trigonometric Ruijsenaars-Schneider model.

We would like to emphasize that the space of q-opers which is described by the system of equations (4.8) contains the K-theory of X (7.2) as a subspace. In particular, one identifies the singularities $z_1, \ldots, z_L$ of the q-oper with the equivariant parameters $a_1, \ldots, a_L$ of the action of the maximal torus of $\mathrm{GL}(L)$ on X, so that $\Pi = W_N = \Lambda_1$. (For $s > 1$, $l_s^k = 0$, so $\Lambda_s = 1$.)

Proof. We prove this by combining two theorems. First, we will use Theorem 3.4 in [KPSZ1705], where the quantum K-theory of Nakajima quiver varieties was defined using quasimaps [CFKM14, Oko1512] from the base curve of genus zero to the quiver variety. The second ingredient is Theorem 4.5 from this paper in the special case when the dominant weights at all oper singularities correspond to the defining representation, so that

$l_s^1 = 1$ for all s and the other l_s^k vanish. Here, the Bethe ansatz equations (4.10) are given by

$$\begin{aligned}
&\frac{\kappa_2}{\kappa_1}\prod_{s=1}^{L}\frac{u_{1,a}-a_s}{q^{-1}u_{1,a}-a_s}\cdot\prod_{b=1}^{r_1}\frac{q^{-1}u_{1,a}-u_{1,b}}{qu_{1,a}-u_{1,b}}\cdot\prod_{d=1}^{r_2}\frac{q^{\frac12}u_{1,a}-u_{2,d}}{q^{-\frac12}u_{1,a}-u_{2,d}}=1\,,\\
&\frac{\kappa_{k+1}}{\kappa_k}\prod_{c=1}^{r_{k-1}}\frac{q^{\frac12}u_{k,a}-u_{k-1,c}}{q^{-\frac12}u_{k,a}-u_{k-1,c}}\cdot\prod_{b=1}^{r_k}\frac{q^{-1}u_{k,a}-u_{k,b}}{qu_{k,a}-u_{k,b}}\cdot\prod_{d=1}^{r_{k+1}}\frac{q^{\frac12}u_{k,a}-u_{k+1,d}}{q^{-\frac12}u_{k,a}-u_{k+1,d}}=1\,,\\
&\frac{\kappa_N}{\kappa_{N-1}}\prod_{c=1}^{r_{N-2}}\frac{q^{\frac12}u_{N,a}-u_{N-1,c}}{q^{-\frac12}u_{N,a}-u_{N-1,c}}\cdot\prod_{b=1}^{r_{N-1}}\frac{q^{-1}u_{N-1,a}-u_{N-1,b}}{qu_{N-1,a}-u_{N-1,b}}=1\,,
\end{aligned}\tag{7.6}$$

where $r_k = \rho_1 + \cdots + \rho_k$ and k runs from 2 to $N-2$ in the middle equation.

The system (7.6) coincides with the Bethe equations from Theorem 3.4 of [KPSZ1705] up to the identification of Bethe roots and twists. This latter set of Bethe equations describes the relations in the quantum K-theory of X, where the Bethe roots $v_{k,a}$ are the Chern roots of the k-th tautological bundle over X and the other variables are identified with the geometry of X as in the statement of the theorem.

We have proven in Theorem 4.5 that equations (7.6) can be written as (4.31). For $k = N$ and for the dominant weights above, this gives

$$\det M_{1,\dots,N}(z) = \det V_{1,\dots,N} W_N(z)^{\left(\frac{1-N}{2}\right)} = \det V_{1,\dots,N}\Pi(z)^{\left(\frac{1-N}{2}\right)}\,. \tag{7.7}$$

This statement completes the proof. □

7.2. **The trigonometric RS model in the dual frame.** The trigonometric Ruijsenaars-Schneider model enjoys *bispectral duality.* This may be described in geometric language as follows. For a given quiver variety X of type A, there are two dual realizations of the tRS model. The first was explained for SL(2) in Section 3.4. Here, the twist variables $\boldsymbol{\kappa}$ play the role of particle positions; their conjugate momenta $p_{\boldsymbol{\kappa}} = (p_{\kappa_1}, \dots, p_{\kappa_N})$ are defined as

$$p_{\boldsymbol{\kappa}} = \exp\left(\frac{\partial \mathcal{Y}}{\partial \log \boldsymbol{\kappa}}\right)\,, \tag{7.8}$$

where $\mathcal{Y}$ is the so-called Yang-Yang function which depends on the Bethe roots $v_{k,a}$ as well as all other parameters. The Yang-Yang function serves as a potential for equations (7.6) [NS09b, NS09a], i.e., the k-th equation is given by

$$\exp\left(\frac{\partial \mathcal{Y}}{\partial \log v_{k,a}}\right) = 1\,, \qquad a = 1,\dots r_k\,, \quad k = 1,\dots,N-1\,. \tag{7.9}$$

(See [GK13, BKK15, KPSZ1705] for more details.)

The other realization—the *3d-mirror* or *spectral/symplectic dual* description—involves a mirror quiver variety $X^\vee$ and the associated dual Yang-Yang function $\mathcal{Y}^\vee$. (The construction of the mirror is discussed in [GK13, AO].) Under the mirror map, the Kähler parameters $\boldsymbol{\kappa}$ are interchanged with the equivariant parameters $\boldsymbol{a}$; the same holds for the conjugate momenta $p_{\boldsymbol{\kappa}}$ and $p_{\boldsymbol{a}}$. Therefore, the variables $\boldsymbol{a}, p_{\boldsymbol{a}}$ can be viewed as the canonical degrees of freedom in the dual tRS model; this has been studied in the context of enumerative geometry in [KZ, BLZZ16]. In particular, such a duality was demonstrated between the XXZ spin chain whose Bethe equations describe the equivariant quantum K-theory of the quiver variety from Fig. 2 and the L-body tRS model whose coordinates are the equivariant parameters $(a_1, \dots, a_L)$ of the maximal torus for $\mathrm{GL}(L)$. This result allows us to construct a natural embedding of the intersection of two Lagrangian cycles inside the tRS phase space into the space of q-opers with the first fundamental weight at each regular singularity.

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