QUANTUM RACAH COEFFICIENTS AND SUBREPRESENTATION SEMIRINGS

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Abstract. Let G be a group and A a G-algebra. The subrepresentation semiring of A is the set of subrepresentations of A endowed with operations induced by the algebra operations. The introduction of these semirings was motivated by a problem in material science. Typically, physical properties of composite materials are strongly dependent on microstructure. However, in exceptional situations, exact relations exist which are microstructure-independent. Grabovsky has constructed an abstract theory of exact relations, reducing the search for exact relations to a purely algebraic problem involving the product of SU(2)-subrepresentations in certain endomorphism algebras. We have shown that the structure of the associated semirings can be described explicitly in terms of Racah coefficients. In this paper, we prove an analogous relationship between Racah coefficients for the quantum algebra $\breve{U}_q(\mathfrak{sl}_2)$ and semirings for endomorphism algebras of representations of $\breve{U}_q(\mathfrak{sl}_2)$. We generalize the construction of subrepresentation semirings to the Hopf algebra setting. For $\breve{U}_q(\mathfrak{sl}_2)$, we compute these semirings for the endomorphism algebra of an arbitrary complex finite-dimensional representation. When the representation is irreducible, we show that the subrepresentation semiring can be described explicitly in terms of the vanishing of q-Racah coefficients. We further show that q-Racah coefficients can be defined entirely in terms of the multiplication of subrepresentations.

1. Introduction

Recent work of Grabovsky, Milton, and Sage has produced an unexpected application of the quantum theory of angular momentum to material science[8, 17]. Moreover, this work has offered a new perspective on $6j$-symbols. In studying a problem in composite materials, the following situation has arisen. Let V be a representation of SU(2), and consider the matrix algebra $\mathrm{End}(V)$ consisting of the linear operators $V \to V$. Suppose that X and Y are subrepresentations of $\mathrm{End}(V)$. Then the subspace $XY = \mathrm{span}\{xy \mid x \in X, y \in Y\}$ is also a subrepresentation. In fact, this product makes the set $\mathcal{E}(V)$ of subrepresentations of $\mathrm{End}(V)$ into a semiring, called a subrepresentation semiring. Solving the physical problem reduces to the algebraic problem of computing the structure constants of these semirings, which have been shown in [17] to have an intimate relationship with Racah coefficients. This paper shows that there is an analogous relationship between Racah coefficients for the quantum algebra $\breve{U}_q(\mathfrak{sl}_2)$ and the product of subrepresentations in the semiring $\mathcal{E}(V)$, where V is now a representation of $\breve{U}_q(\mathfrak{sl}_2)$.

We begin by describing how classical $6j$-coefficients arise in the theory of composite materials. Typically, physical properties of composites such as conductivity and elasticity are strongly dependent on the microstructure of the composite. We consider the set of all possible values of a given physical property for composites made out of fixed materials taken in fixed proportions. This will be a subset of an appropriate tensor space, and generically this subset will have nonempty interior. However, in exceptional circumstances, this set degenerates to a surface, called an exact relation. These relations represent fundamental physical invariances. Finding them is of great importance in both theory and applications because they describe microstructure-independent situations. To give an illustration from elasticity, Hill has shown that a mixture of isotropic materials with constant shear modulus is isotropic and has the same shear modulus[9, 10].

The classical approach to finding exact relations has suffered from the drawback of relying heavily on the details of the physical context. In the late 1990's, Grabovsky and Grabovsky, Sage, and Milton developed an abstract theory of exact relations which has been able, not only to find many new exact relations, but also to give complete lists of rotationally invariant exact relations for three-dimensional thermopiezoelectric composites[6, 8]. In particular, we obtain all exact relations for conductivity, elasticity, and piezoelectricity as special cases. This general theory has been successful by reducing the search for exact relations to purely algebraic questions involving the representation theory of $\mathrm{SO}(3)$.

We briefly sketch how this is accomplished. We consider a physical property which is described by elements of the real symmetric tensor space $\mathrm{Sym}(\mathcal{T})$, i.e. the set of symmetric linear operators $\mathcal{T} \to \mathcal{T}$ where

$\mathcal{T}$ is a representation of SO(3). For example, $\mathcal{T}$ is $\mathbf{R}^3$ for conductivity and $\mathrm{Sym}(\mathbf{R}^3)$ for elasticity. Milton has shown how to associate to any (rotationally invariant) exact relation surface a subrepresentation of $\mathrm{Sym}(\mathcal{T})$[15]. It turns out that conditions for a subrepresentation Π to determine an exact relation can be given in terms of the multiplication of subrepresentations in $\mathrm{End}(\mathcal{T})$. A necessary condition is that Π must satisfy the equation $(\Pi\mathcal{A}\Pi)_{\mathrm{sym}} \subset \Pi$, where $\mathcal{A}$ is a fixed subrepresentation determined by the physical context and $X_{\mathrm{sym}} = (X + X^t) \cap \mathrm{Sym}(\mathcal{T})$[8]. Similar, but more complicated, sufficient conditions have also been found. Thus, the search for exact relations has been reduced in large part to understanding the product of subrepresentations of $\mathrm{End}(\mathcal{T})$.

These considerations motivated us to introduce subrepresentation semirings in [17] in the context of a group G acting by algebra automorphisms on an algebra A. In section two of this paper, we generalize this construction to Hopf algebras. Given a Hopf algebra H and an H-module algebra A (i.e. an algebra and H-module whose H-action is compatible with the ring structure), we show that the set of submodules of A is a semiring.

In [17], we showed how Racah coefficients arose in the computation of the semirings $\mathcal{E}(V)$ for an arbitary finite-dimensional representation of SU(2). Consider the product of subrepresentations induced by the composition of linear maps $\mathrm{Hom}(V_k, V_l) \otimes \mathrm{Hom}(V_j, V_k) \to \mathrm{Hom}(V_j, V_k)$, where V_j denotes the irreducible representation with total angular momentum quantum number j. We showed that if $V_a \subset \mathrm{Hom}(V_j, V_k)$ and $V_b \subset \mathrm{Hom}(V_k, V_l)$, then $V_c \subset V_b V_a$ if and only if the Racah coefficient $W(jkcb; al)$ is nonzero. The computation of the structure constants for $\mathcal{E}(V_j)$ is just a special case. We also showed that Racah coefficients can be defined entirely in terms of the multiplication of subrepresentations. Finally, we used these results to give explicit computations of the structure constants of $\mathcal{E}(V)$ for general V.

It is not at all obvious a priori that Racah coefficients should arise in this context. By definition, $W(jkcb; al) = 0$ means that two embeddings $V_c \to V_j \otimes V_k \otimes V_b$ given by different iterations of the Clebsch-Gordan formula are orthogonal. The fact that this is equivalent to the nonexistence of a nontrivial intertwining map $V_c \to V_b V_a$ is a special property of the representation theory of SU(2) (and also of the quantum algebras $\breve{U}_q(\mathfrak{sl}_2)$). This does not hold even for simply reducible groups, whose representation theory is very similar to that of SU(2), down to the existence of Clebsch-Gordan and Racah coefficients satisfying the usual formal properties.

In section 3, we prove quantum group analogues of our results for SU(2). We work with the quantum algebra $\breve{U}_q(\mathfrak{sl}_2)$ with deformation parameter $q > 0$. We show that the structure coefficients for the semiring $\mathcal{E}(V)$ where V is irreducible (or for the more general product map described below in equation (7)) are zero or one depending on whether a certain q-Racah coefficient vanishes or not. We prove that the q-Racah coeffients can moreover be defined entirely in terms of the multiplication of subrepresentations. We conclude the paper by computing the semiring $\mathcal{E}(V)$, where V is an arbitrary finite-dimensional representation of $\breve{U}_q(\mathfrak{sl}_2)$.

2. H-module algebras and subrepresentation semirings

Let H be a Hopf algebra over a field F with comultiplication Δ, counit ϵ, and antipode S, and let A be an H-module algebra. This means that the H-action on A is compatible with the algebra structure on A; in other words, the multiplication map $A \otimes A \to A$ and the inclusion $F \to A$ are H-module maps. Explicitly, we have $h \cdot (ab) = \sum (h'a)(h''b)$ and $h \cdot (1_A) = \epsilon(h) 1_A$ for all $h \in H$ and $a, b \in A$. Here, we use Sweedler's sigma notation for the comultiplication $\Delta(h) = \sum h' \otimes h''$.

The concept of an H-module algebra generalizes two familiar algebraic objects. If G is a group, an algebra A is called a G-algebra if the group acts on A by algebra automorphisms. This is equivalent to A being an FG-module algebra. Similarly, given a Lie algebra L with universal enveloping algebra $U(L)$, a $U(L)$-module algebra is just an algebra on which L acts by derivations.

Since ring multiplication in A is well-behaved with respect to the H-action, it is natural to investigate the relationship between the submodules of A and ring multiplication. For example, one can study the H-invariant ideals and subalgebras of A, i.e. those submodules which are also ideals or subalgebras of A. At an even more basic level, given submodules X and Y, if follows from the definition of an H-algebra that the subspace $XY = \operatorname{span}\{xy \mid x \in X, y \in Y\}$ is also a submodule, and we would like to better understand the product of submodules.

We now introduce the submodule semiring associated to an H-module algebra. Let $S_H(A)$ be the set of all H-submodules of A. The usual subspace addition together with the product defined above make this set into an additively idempotent semiring, with additive and multiplicative identities $\{0\}$ and $F = F1_A$ respectively. Note that the semiring multiplication is determined by the products of indecomposable submodules. We can thus define structure constants $C^W_{U,V}$ for $S_H(A)$, where

U, V, and W are indecomposable, by setting $C_{U,V}^W = 1$ if $W \subset UV$ and 0 otherwise. Of course, we need only consider irreducible submodules if A is completely reducible.

Inclusion gives a partial order on the semiring $S_H(A)$, and the supremum of a collection of submodules $\{X_i\}$ is just $\sum X_i$. Accordingly, $S_H(A)$ becomes a complete idempotent semiring.

Let $\phi : A \to B$ be a homomorphism of H-module algebras. It is clear that the map $S_H(\phi) : S_H(A) \to S_H(B)$ given by $S_H(\phi)(X) = \phi(X)$ is a semiring morphism. This correspondence is a functor:

Theorem 2.1. *The correspondence S_H is a functor from the category of H-module algebras to the category of complete idempotent semirings.*

Although we will not discuss invariant ideals and subalgebras of H-module algebras in the present paper, we remark that there is an intimate relationship between them and certain classes of ideals and subsemirings of the submodule semiring. An ideal I of a semiring is called subtractive if $x \in I$ and $x + y \in I$ implies that $y \in I$. Imposing this condition eliminates various pathologies caused by the lack of additive inverses. For example, a two-sided ideal is the kernel of a semiring morphism if and only if it is subtractive[5]. It can be shown that there is a bijective correspondence between H-invariant ideals (left, right, or two-sided) of A and subtractive ideals (of the appropriate type) of $S_H(A)$ which contain their suprema. An analogous statement is true for invariant subalgebras.

From now on, we will focus on one class of H-module algebras, endomorphism algebras. Let V be a finite-dimensional representation of H, and consider the central simple algebra $A = \mathrm{End}(V)$ consisting of all linear maps $V \to V$. The natural H-action given by the formula

$$(h \cdot f)(v) = \sum h' f(S(h'')v) \tag{1}$$

makes $\mathrm{End}(V)$ into an H-module algebra[11]. We denote the semiring $S_H(\mathrm{End}(V))$ by $\mathcal{E}(V)$.

Let us give some examples of this construction.

Examples. 1. If V is one-dimensional, then $\mathrm{End}(V)$ is the trivial H-module. It follows that $\mathcal{E}(V)$ is the Boolean semiring $\mathbf{B} = \{0, 1\}$ with $1 + 1 = 1$.

2. Let $V_{1/2}$ be the standard complex representation of $\mathrm{SU}(2)$, or equivalently, of the universal enveloping algebra $U(\mathfrak{sl}(2))$. The $\mathrm{SU}(2)$-algebra $\mathrm{End}(V_{1/2})$ decomposes into the direct sum of irreducible representations $\mathbf{C} \oplus V_1$, and the four element commutative semiring $\mathcal{E}(V_{1/2})$ is determined by $V_1^2 = \mathrm{End}(V_{1/2})$.

3. Let V be a two-dimensional irreducible representation of the quantized enveloping algebra $\breve{U}_q(\mathfrak{sl}_2)$, where q is not a root of unity. The semiring $\mathcal{E}_{\breve{U}_q(\mathfrak{sl}_2)}(V)$ is isomorphic to $\mathcal{E}_{\mathrm{SU}(2)}(V_{1/2})$.

These semirings have been studied in [16] and [17] in the case of group algebras and universal enveloping algebras of complex semisimple Lie algebras. For these Hopf algebras, it is possible to say quite a lot about the invariant ideals and subalgebras of $\mathrm{End}(V)$. We briefly recall these results. There is a straightforward bijective correspondence between the invariant left and right ideals and the subrepresentations of V. The situation is much more complicated for subalgebras, and we restrict attention to irreducible V. With this hypothesis, the invariant subalgebras are semisimple of a very special type. When the base field is algebraically closed, there is an explicit parameterization of the invariant subalgebras. This classification (in the group algebra case) shows that the invariant subalgebras encapsulate complicated information about the group G and V involving both how V can be expressed as an induced representation $\mathrm{Ind}_H^G(W)$ and how W can be factored into a tensor product of projective representations. For more details and examples, see [16] and [17]. It seems likely that analogues of these results hold for other Hopf algebras.

Before proceeding, we will need a more general notion of the product of submodules. Let A, B, and C be three H-modules together with an H-map $A \otimes B \to C$. We now define a multiplication map $S_H(A) \times S_H(B) \to S_H(C)$ as before; here $S_H(X)$ is the additive monoid of submodules of X. Composition of linear maps provides an illustration of this construction. Given finite-dimensional representations U and V, the space $\mathrm{Hom}(U, V)$ of linear maps $U \to V$ becomes an H-module via the action (1). We call the set of its submodules $\mathcal{H}(U, V)$. If W is a third module, then composition gives an H-map $\mathrm{Hom}(V, W) \otimes \mathrm{Hom}(U, V) \to \mathrm{Hom}(U, W)$, thus inducing the product $\mathcal{H}(V, W) \otimes \mathcal{H}(U, V) \to \mathcal{H}(U, W)$. We remark that the natural map $V \otimes U^* \cong \mathrm{Hom}(U, V)$ is an H-isomorphism. It is not true in general that $\mathrm{Hom}(U, V) \cong U^* \otimes V$ unless H is cocommutative.

3. Subrepresentation semirings for $\breve{U}_q(\mathfrak{sl}_2)$ and q-Racah coefficients

3.1. **Preliminaries.** There are several variants of the quantized enveloping algebra of $\mathfrak{sl}_2(\mathbf{C})$. It will be most convenient for us to work with the quantum algebra $\breve{U}_q(\mathfrak{sl}_2)$. However, the results will also hold for $U_q(\mathfrak{sl}_2)$, since it can be embedded as a Hopf subalgebra of $\breve{U}_q(\mathfrak{sl}_2)$.

(We follow the notation of Klimyk and Schmüdgen's book[14].) The algebra $\breve{U}_q(\mathfrak{sl}_2)$ is generated by E, F, K, and K^{-1} subject to the relations $KEK^{-1} = qE$, $KFK^{-1} = q^{-1}F$, and $[E,F] = \frac{K^2-K^{-2}}{q-q^{-1}}$. We assume the deformation parameter q is not a root of unity. The Hopf algebra structure maps are determined by $\Delta(K) = K \otimes K$, $\Delta(E) = E \otimes K + K^{-1} \otimes E$, $\Delta(F) = F \otimes K + K^{-1} \otimes F$, $\epsilon(K) = 1$, $\epsilon(E) = \epsilon(F) = 0$, $S(K) = K^{-1}$, $S(E) = -qE$, and $S(F) = -q^{-1}F$.

We recall the standard facts about the representation theory of $\breve{U}_q(\mathfrak{sl}_2)$[14, 13, 2]. Every finite dimensional representation is completely reducible and a sum of weight spaces, i.e eigenspaces of K. For every $j \in \frac{1}{2}\mathbf{Z}_{\geq 0}$, there are four irreducible representations of dimension $2j+1$, one for each fourth root of unity ω. We call this index set $\mathfrak{J}$. We use the convention that if we refer to an element of $\mathfrak{J}$ by an upper case letter J, then the angular momentum quantum number will be given by the corresponding lower case letter j. Also, if $J = (j, \omega)$, then $J^{-1} = (j, \omega^{-1})$. The irreducible $V_{j\omega}$ has highest weight ωq^j and is obviously the tensor product of $V_j \overset{\text{def}}{=} V_{j1}$ with one of the four characters of $\breve{U}_q(\mathfrak{sl}_2)$. We say that $V_{j\omega}$ is of type ω. It has a basis of weight vectors $v_m^{j\omega}$ for $m = -j, -j+1, \ldots, j$ with $K \cdot v_m^{j\omega} = \omega q^m v_m^{j\omega}$. Since the dual space $V_{j\omega}^*$ has highest weight $\omega^{-1}q^j$, we see that $V_{j\omega}^*$ is isomorphic to $V_{j\omega^{-1}}$ We also note that the tensor product of irreducibles is multiplicity free, and the Clebsch-Gordan formula holds for $\breve{U}_q(\mathfrak{sl}_2)$:

$$V_{k\omega} \otimes V_{l\omega'} \cong \sum_{j=|k-l|}^{k+l} V_{j(\omega\omega')}. \tag{2}$$

We call the triple (klj) admissible if j satisfies the triangle inequality conditions appearing in this sum. More generally, we say that (KLJ) is admissible if in addition $\omega_K \omega_L = \omega_J$.

We now assume that q is a positive real number. In this case, $\breve{U}_q(\mathfrak{sl}_2)$ is a Hopf $*$-algebra via $E^* = F$, $F^* = E$, and $K^* = K$, and the representations $V_{j\omega}$ are $*$-representations. This means that $V_{j\omega}$ is endowed with a scalar product under which the action of X^* on $V_{j\omega}$ is just the adjoint of the action of X. The weight vectors for $V_{j\omega}$ can be normalized in a way entirely analogous to the situation for SU(2). Thus, the representation is given concretely in terms of the orthonormal basis $\{v_m^{j\omega}\}$ by

$$\begin{aligned} E \cdot v_m^{j\omega} &= \omega([j-m][j+m-1])^{1/2} v_{m+1}^{j\omega}, \\ F \cdot v_m^{j\omega} &= \omega([j+m][j-m+1])^{1/2} v_{m-1}^{j\omega}, \\ K \cdot v_m^{j\omega} &= \omega q^m v_m^{j\omega}, \end{aligned} \tag{3}$$

where $[a] = [a]_q$ is the q-number $\frac{q^a - q^{-a}}{q - q^{-1}}$. We call such a basis a Clebsch-Gordan or CG basis.

The quantum Clebsch-Gordan coefficients (CGC's) are the matrix elements of the unitary map realizing the isomorphism of equation (2) for type 1 representations. Explicitly,

$$v^a_m = \sum_{m_1+m_2=m} {}_qC^{jka}_{m_1m_2m} v^j_{m_1} \otimes v^k_{m_2}. \tag{4}$$

The quantum Racah coefficients are constants $W_q(jkcb; al)$, parametrized by six irreducibles, which encode the associativity of triple tensor products. Iterating the Clebsch-Gordan formula gives two bases for $V_j \otimes V_k \otimes V_b$, one from $(V_j \otimes V_k) \otimes V_b \cong (\sum V_a) \otimes V_b$ and the other from $V_j \otimes (V_k \otimes V_b) \cong V_j \otimes (\sum V_l)$. Racah coefficients are obtained from the unitary change of basis matrix. They vanish unless (kja), (lkb), (ljc), and (bac) are all admissible. The Racah coefficients can be expressed in terms of products of CGC's:

$$\begin{aligned}([2j+1][2b+1])^{\frac{1}{2}} W_q(lkca; bj)\, {}_qC^{ljc}_{m_1m_2m} \\ = \sum_s {}_qC^{bac}_{(m_1+s)(m_2-s)m}\, {}_qC^{lkb}_{m_1s(m_1+s)}\, {}_qC^{kaj}_{s(m_2-s)(m_2)}.\end{aligned} \tag{5}$$

This does in fact define $W_q(lkca; bj)$ because with the above admissibility conditions, it is always possible to choose appropriate projection quantum numbers for which ${}_qC^{ljc}_{m_1m_2m} \neq 0$. The Racah coefficient $W_q(jkcb; al)$ is nonzero precisely when the $\breve{U}_q(\mathfrak{sl}_2)$-map

$$V_c \to V_a \otimes V_b \to (V_j \otimes V_k) \otimes V_b \cong V_j \otimes (V_k \otimes V_b) \to V_j \otimes V_l \to V_c \tag{6}$$

is nonzero[3].

3.2. **The product of subrepresentations – the irreducible case.** In order to compute the structure constants of the semiring $\mathcal{E}(V)$ for general V, we will need to understand the multiplication map

$$\mathcal{H}(V_K, V_L) \otimes \mathcal{H}(V_J, V_K) \to \mathcal{H}(V_J, V_L). \tag{7}$$

We assume for the moment that the representations have type 1. Accordingly, let V_a and V_b be subrepresentations of $\mathrm{Hom}(V_j, V_k)$ and $\mathrm{Hom}(V_k, V_l)$ respectively. The product V_bV_a is a homomorphic image of $V_b \otimes V_a$ and is thus multiplicity free. If the irreducible V_c is a component of V_bV_a, then it is simultaneously a component of $V_b \otimes V_a$ and $\mathrm{Hom}(V_j, V_l) \cong V_l \otimes V_j$. In other words, (bac) and (ljc) are both admissible. However, the converse is not true. In fact, analogously to the situation for SU(2), V_c is a component of V_bV_a if and only if a certain $6j$-coefficient is nonzero.

Theorem 3.1. *The quantum Racah coefficient $W_q(jkcb;al)$ is nonzero if and only if V_a, V_b, and V_c are subrepresentations of Hom(V_j,V_k), Hom(V_k,V_l), and V_bV_a respectively. In particular, if $V_{a\omega_A}\in\mathcal{H}(V_J,V_K)$ and $V_{b\omega_B}\in\mathcal{H}(V_K,V_L)$, then*

$$V_{b\omega_B}V_{a\omega_A}=\bigoplus_{\{c|W_q(jkcb;al)\neq 0\}}V_{c(\omega_B\omega_A)}. \tag{8}$$

Corollary 3.2. *If V_a, V_b, and V_c are subrepresentations of End$(V_{j\omega_j})$ (whose components automatically have type 1), then*

$$V_bV_a=\bigoplus_{\{c|W_q(jjcb;aj\neq 0\}}V_c. \tag{9}$$

Remarks. 1. In terms of $6j$-coefficients, the condition of the theorem is that

$$\begin{Bmatrix} j & k & a\\ b & c & l\end{Bmatrix}_q\neq 0.$$

2. The description of a nontrivial zero of $W_q(jkcb;al)$ (i.e. a zero that is not caused by admissibility requirements) given by the usual definition is rather complicated. It says that the two embeddings $V_c\to V_j\otimes V_k\otimes V_b$ given by different iterations of the Clebsch-Gordan formula are orthogonal. The interpretation provided by the theorem is conceptually much simpler.

Corollary 3.3. *The semirings $\mathcal{E}(V_{j\omega})$ are commutative of order 2^{2j+1} and are independent of ω. Their structure constants C^c_{ab} are invariant under all permutations of the indices.*

Proof. The commutativity and statement about the structure constants follows from the fact that the quantum $6j$-symbol $\left\{\begin{smallmatrix} j & j & a\\ b & c & j\end{smallmatrix}\right\}_q$ for fixed j has S_3 symmetry[14]. The rest is clear. □

Before proceeding to the proof, we observe that the theorem describes a connection between two seemingly quite different kinds of intertwining maps. By definition, the Racah coefficient $W_q(jkcb;al)$ is nonzero if and only if there is a nonzero intertwiner given by the recoupling map (6). The theorem says this is the case if and only if a nonzero intertwiner exists $V_c\to V_bV_a$. This is not a general algebraic fact, but rather a special property of SU(2) and the quantum algebras $\breve{U}_q(\mathfrak{sl}_2)$. Indeed, this does not hold even for groups whose representation theory is formally similar to that of SU(2). More precisely, let G

be a simply reducible group. This means that G is a compact group with the following properties:

(1) All the irreducible representations of G are self-dual.
(2) If V and W are irreducible, then $V \otimes W$ is multiplicity free.

This class of groups was introduced by Wigner, who showed that it is possible to define $6j$-coefficients in this context satisfying the customary properties[21, 22]. For example, the $6j$-coefficients are symmetric under column and triad permutation as well as satisfying the usual orthogonality relations and the Biedenharn-Elliot identity[19]. However, the analogue of theorem 3.1 is not true[18].

We now turn to the proof of the theorem. We will need an explicit $\breve{U}_q(\mathfrak{sl}_2)$-isomorphism between $V_{j\omega}$ and its dual. This is given by the map $\phi_{j\omega} : V_{j\omega} \longrightarrow V^*_{j\omega^{-1}}$ defined by $v^{j\omega}_m \mapsto (-q\omega^2)^m v^{j\omega^{-1}*}_{-m}$, where $\{v^{j\omega^{-1}*}_m\}$ is the basis dual to the standard basis for $V_{j\omega^{-1}}$.

Lemma 3.4. *The map $\phi_{j\omega}$ is an isomorphism of $\breve{U}_q(\mathfrak{sl}_2)$-modules.*

Proof. A direct calculation shows that $\phi_{j\omega}(Xv^{j\omega}_m)(v^{j\omega^{-1}}_n) = (-q\omega^2)^m (v^{j\omega^{-1}*}_{-m})(S(X)v^{j\omega^{-1}}_n)$, where X is one of the standard generators of $\breve{U}_q(\mathfrak{sl}_2)$. □

The elements $w^{j\omega}_m = (-q\omega^2)^m v^{j\omega^{-1}*}_{-m}$ are the CG basis vectors for $V^*_{j\omega^{-1}}$. For future reference, we note that

$$w^{j\omega}_m(v^{j\omega^{-1}}_n) = \delta_{m,-n}(-q\omega^2)^m \tag{10}$$

We now restrict attention to type 1 representations. It is easy to see that the general case will follow immediately. We use the product basis $\{v^k_m \otimes w^j_s\}$ for $\mathrm{Hom}(V_j, V_k)$, where we identify $\mathrm{Hom}(V_j, V_k)$ and $V_k \otimes V^*_j$ under the canonical isomorphism.

Let V_c be an irreducible component of $\mathrm{Hom}(V_j, V_l)$, and let $\{z^c_m(l, j)\}$ be the image of the CG basis of V_c under the composition $V_c \to V_l \otimes V^*_j \to \mathrm{Hom}(V_j, V_l)$. Explicitly,

$$z^c_m(l, j) = \sum_{m_1+m_2=m} {}_qC^{ljc}_{m_1m_2m} v^l_{m_1} \otimes w^j_{m_2}. \tag{11}$$

We assume without loss of generality that the four triples (kja), (lkb), (ljc), and (bac) are all admissible; if this does not hold, then $W_q(jkcb; al)$ is a structural zero while V_bV_a either is undefined or does not contain V_c for trivial reasons. Accordingly, we now suppose that V_a, V_b, and V_c are components of $\mathrm{Hom}(V_j, V_k)$, $\mathrm{Hom}(V_k, V_l)$, and $\mathrm{Hom}(V_j, V_l)$ respectively and, in addition, V_c is a component of $V_b \otimes V_a$. The image of the CG basis for V_c in $V_b \otimes V_a$ under the projection to V_bV_a is given by

$$\xi_m^c = \sum_{p_1+p_2=m} {}_qC^{bac}_{p_1p_2m} z^b_{p_1}(l,k) z^a_{p_2}(k,j). \tag{12}$$

The vectors $\{\xi_m^c\}$ must be related to the CG basis $\{z_m^c(l,j)\}$ given in (11) by a scalar multiple ${}_qR^{jkl}_{abc}$ independent of m, so that

$$\xi_m^c = {}_qR^{jkl}_{abc} z^c_m(l,j). \tag{13}$$

Multiplying out (12) and using (10), we get

$$\xi_m^c = \sum_{p_1,p_2,s_1,s_2,t_1,t_2} \delta_{t_1,-s_2}(-q)^{s_2} {}_qC^{bac}_{p_1p_2m} {}_qC^{lkb}_{s_1s_2p_1} {}_qC^{kja}_{t_1t_2p_2} v^l_{s_1} \otimes w^l_{t_2}. \tag{14}$$

Equating the coefficient of the basis element $v^l_{m_1} \otimes w^j_{m_2}$ on both sides of (13) gives the formula

$$ {}_qR^{jkl}_{abc} {}_qC^{ljc}_{m_1m_2m} = \sum_s (-q)^s {}_qC^{bac}_{(m_1+s)(m_2-s)m} {}_qC^{lkb}_{m_1 s(m_1+s)} {}_qC^{kja}_{(-s)m_2(m_2-s)}. \tag{15}$$

This summation is just another form of the defining expression for quantum Racah coefficients (5). Indeed, we have

$$\begin{aligned} {}_qC^{kja}_{(-s)m_2(m_2-s)} &= (-1)^{k+s} q^{-s} ([2a+1]_q/[2j+1]_q)^{\frac{1}{2}} {}_{q^{-1}}C^{kaj}_{(-s)(s-m_2)(-m_2)} \\ &= (-1)^{2k+s+a-j} q^{-s} ([2a+1]_q/[2j+1]_q)^{\frac{1}{2}} {}_qC^{kaj}_{s(m_2-s)(m_2)}, \end{aligned}$$

where we have used symmetries of the quantum CGC's. (See for example [20] or theorem 3.62 of [2].) Inserting in (15) and using (5) and the fact that $(-1)^{2k+2s} = 1$, we obtain

$$ {}_qR^{jkl}_{abc} {}_qC^{ljc}_{m_1m_2m} = (-1)^{a-j}([2a+1][2b+1])^{\frac{1}{2}} W_q(lkca;bj) {}_qC^{ljc}_{m_1m_2m}.$$

Consequently, we get

$$ {}_qR^{jkl}_{abc} = (-1)^{a-j}([2a+1][2b+1])^{\frac{1}{2}} W_q(lkca;bj), \tag{16}$$

and thus V_c is a component of V_bV_a precisely when $W_q(lkca;bj) \neq 0$. Since $W_q(jkcb;al) = (-1)^{l+a-j-b} W_q(lkca;bj)$, this is equivalent to $W_q(jkcb;al) \neq 0$. (We state the theorem in terms of the latter Racah coefficient to make it match the SU(2)-analogue of the theorem given in [17].)

It is an immediate consequence of (16) that we can use the coefficients ${}_qR^{jkl}_{abc}$ as a starting point for defining $6j$-coefficients instead of the usual procedure involving the associativity of tensor products:

Theorem 3.5. *The Racah coefficients for $\breve{U}_q(\mathfrak{sl}_2)$ can be defined in terms of matrix multiplication of subrepresentations.*

3.3. **The generic semirings.** We define a family of semirings $\mathcal{C}_j$ for $j \in \frac{1}{2}\mathbf{Z}_{\geq 0}$ as follows. The elements of $\mathcal{C}_j$ are the set of subsets of $\{0, \ldots, 2j\}$ with idempotent addition given by set union. Multiplication is determined by the triangle inequality condition $\{a\}\{b\} = \{i \mid 0 \leq i \leq 2j, |a-b| \leq i \leq a+b\}$. The zero element is the empty set, and the multiplicative identity is $\{0\}$. This is a commutative semiring of order 2^{2j+1}. Generically, the semirings $\mathcal{E}(V_J)$ are isomorphic to $\mathcal{C}_j$.

Theorem 3.6. *The semiring $\mathcal{E}(V^q_{j\omega})$ is isomorphic to $\mathcal{C}_j$ for all but finitely many q.*

To prove this, we recall the connection between quantum Racah coefficients and basic hypergeometric series. The basic hypergeometric function ${}_4\phi_3$ is defined as the infinite series

$${}_4\phi_3(a_1, a_2, a_3, a_4; b_1, b_2, b_3; q, z) = \sum_{n=0}^{\infty} \frac{(a_1, q)_n (a_2, q)_n (a_3, q)_n (a_4, q)_n}{(b_1, q)_n (b_2, q)_n (b_3, q)_n (q, q)_n} z^n,$$

where $(a;q)_n = (1-a)(1-aq)\ldots(1-aq^{n-1}); (a;q)_0 = 1$ is the q-shifted factorial. We will be interested in the case when $a_i = q^{s_i}$ and $b_i = q^{t_i}$ for integers s_i and t_i. This series terminates if $s_i \leq 0$ with any nonpositive t_j strictly smaller than the largest nonpositive s_i[4].

Given the Racah coefficient $W_q(jkcb; al)$, let us set $\beta_1 = \min\{j+k+c+b, j+b+a+l, k+c+a+l\}$ with β_2 and β_3 the remaining two parameters, and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ any permutation of $(j+k+a, a+b+c, j+c+l, k+b+l)$. Then we have the formula:

(17)
$$\begin{aligned} W_q(jkcb; al) = \kappa(jkcbal) {}_4\phi_3(&q^{2(\alpha_1-\beta_1)}, q^{2(\alpha_2-\beta_1)}, q^{2(\alpha_3-\beta_1)}, q^{2(\alpha_4-\beta_1)}; \\ &q^{-2(\beta_1+1)}, q^{2(\beta_2-\beta_1+1)}, q^{2(\beta_3-\beta_1+1)}; q^2, q^2), \end{aligned}$$

where $\kappa(jkcbal)$ is a rational function of q all of whose zeros and poles are at roots of unity or zero[12, 14, 2]. This means that the zeros of $W_q(jkcb; al)$ are just the positive real zeros of the rational function given by the hypergeometric factor of (17).

It now follows from theorem 3.1 that the semiring $\mathcal{E}(V^q_{j\omega})$ is isomorphic to $\mathcal{C}_j$ except when q is a zero of one of the hypergeometric functions appearing in equation (17) for $W_q(jjcb; aj)$ as a, b, and c vary over the finitely many nontrivial possibilities. There are a finite number of such zeros, and the theorem follows.

The smallest examples in which the subrepresentation semiring $\mathcal{E}(V_j^q)$ is not isomorphic to $\mathcal{C}_j$ occur for $j = 2$. (By contrast, the $j = 3/2$ semiring for SU(2) is not isomorphic to $\mathcal{C}_{3/2}$[17].) There are two exceptional cases:

(1) V_2^q is not a component of $V_2^q V_2^q$ when q is one of the two positive roots of the irreducible polynomial $q^{20} - q^{18} - q^{14} - q^{12} + q^{10} - q^8 - q^6 - q^2 + 1$.
(2) V_3^q is not in $V_3^q V_3^q$ when q is one of the two positive roots of $q^8 - q^6 - q^4 - q^2 + 1$.

3.4. **The general case.** Let V be a finite-dimensional representation of $\breve{U}_q(\mathfrak{sl}_2)$. The representation V can be expressed as $V = \bigoplus_J \mathbf{C}^{r_J} \otimes V_J$, where $\mathbf{C}^{r_J}$ is a trivial $\breve{U}_q(\mathfrak{sl}_2)$-module and $r_J = 0$ for all but finitely many J. We have the isomorphism of H-module algebras

$$\begin{aligned} \mathrm{End}(V) &\cong \bigoplus_J \mathrm{Hom}(\mathbf{C}^{r_J} \otimes V_J, \mathbf{C}^{r_K} \otimes V_K) \\ &\cong \bigoplus_J \mathrm{Hom}(\mathbf{C}^{r_J}, \mathbf{C}^{r_K}) \otimes \mathrm{Hom}(V_J, V_K), \end{aligned} \tag{18}$$

where again the first factor is a trivial module.

The endomorphism algebra $\mathrm{End}(V)$ is no longer multiplicity free and the number of subrepresentations isomorphic to V_C is in bijective correspondence to the projective space $\mathbf{CP}^n$, where n is the multiplicity of V_C. To get homogeneous coordinates for a given subrepresentation X_C isomorphic to V_C, we need an analogue of equation (4) defining CGC's for arbitrary irreducible representations:

$$v_m^A = \sum_{m_1+m_2=m} (\omega_J)^{2m_2}\, {}_qC^{jka}_{m_1 m_2 m} v^J_{m_1} \otimes v^K_{m_2}. \tag{19}$$

This follows immediately from (4) and the lemma:

Lemma 3.7. *The map $V_{j\omega} = V_j \otimes V_{0\omega} \to V_{0\omega} \otimes V_j$ given by $v_m^{j\omega} \mapsto \omega^{2m} v_0^{0\omega} \otimes v_m^j$ is a $\breve{U}_q(\mathfrak{sl}_2)$-module isomorphism.*

If V_C is a subrepresentation of $\mathrm{Hom}(V_J, V_L)$, then the corresponding CG basis is $z_m^C(L, J) = \sum_{m_1,m_2} (\omega_L)^{2m_2}\, {}_qC^{lja}_{m_1 m_2 m} v^L_{m_1} \otimes w^{J^{-1}}_{m_2}$. If in addition V_A, V_B, and V_C are components of $\mathrm{Hom}(V_J, V_K)$, $\mathrm{Hom}(V_K, V_L)$, and $V_B \otimes V_A$ respectively, then the image of the CG basis for V_C under the projection to $V_B V_A$ given by

$$\xi_m^C = \sum_{p_1+p_2=m} (\omega_B)^{2p_2}\, {}_qC^{bac}_{p_1 p_2 m} z^B_{p_1}(L, K) z^A_{p_2}(K, J)$$

must be related to $z_m^C(L,J)$ by a scalar multiple independent of m. It turns out that this scalar is just ${}_qR^{jkl}_{abc}$. The verification amounts to showing that the additional factors of fourth roots of unity introduced by (4) and the intertwining map $V_K^* \to V_{K^{-1}}$ of Lemma 3.4 cancel each other out.

We obtain homogeneous coordinates $x_{KJ} \in \mathrm{Hom}(\mathbf{C}^{r_J}, \mathbf{C}^{r_K})$ for X_A in $\mathrm{End}(V)$ from the decomposition (18) by setting $z_m^A(X) = \sum_{KJ} x_{KJ} \otimes z_m^A(K,J)$. We can now completely determine the structure constants for $\mathcal{E}(V)$.

Theorem 3.8. *Let X_A and Y_B be irreducible subrepresentations of $\mathcal{E}(V)$, isomorphic to V_A and V_B respectively, with homogeneous coordinates x_{JK} and y_{JK}. Then $Y^B X^A$ contains a copy of V_C if and only if the coefficients*

$$z_{JL} = \sum_K y_{LK} x_{KJ\,q} R^{jkl}_{abc} \tag{20}$$

are not all zero, where ${}_qR^{jkl}_{abc}$ is the nonzero multiple of $W_q(jkcb;al)$ defined in (16)*. In this case, the z_{JL} are the homogeneous coordinates for the unique subrepresentation isomorphic to V_C.*

Proof. Since $Y_B X_A$ is multiplicity free, it contains at most one copy of V_C. We assume without loss of generality that (BAC) is admissible.

The image of the CG basis for $Y_B \otimes X_A$ in $Y^b X^a$ is given by the vectors

$$\chi_m^C = \sum_{p_1,p_2} (\omega_L)^{2p_2}\, {}_qC^{bac}_{p_1p_2m} \Big(\sum_{LQ} y_{LQ} \otimes z^B_{p_1}(L,Q)\Big)\Big(\sum_{KJ} x_{KJ} \otimes z^A_{p_2}(K,J)\Big). \tag{21}$$

The only terms that contribute to the sum have $Q = K$. Rearranging and substituting (12), we get

$$\begin{aligned} \chi_m^C &= \Big(\sum_{LKJ} y_{LK} x_{KJ}\Big) \otimes \xi_m^C(L,J) \\ &= \sum_{LJ} \Big(\sum_K y_{LK} x_{KJ\,q} R^{jkl}_{abc}\Big) \otimes z_m^C(L,J) \end{aligned} \tag{22}$$

as desired. ∎

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