FLAT G-BUNDLES AND REGULAR STRATA FOR REDUCTIVE GROUPS

CHRISTOPHER L. BREMER AND DANIEL S. SAGE

Abstract. Let $\hat{G}$ be an algebraic loop group associated to a reductive group G. A fundamental stratum is a triple consisting of a point x in the Bruhat-Tits building of $\hat{G}$, a nonnegative real number r, and a character of the corresponding depth r Moy-Prasad subgroup that satisfies a non-degeneracy condition. The authors have shown in previous work how to associate a fundamental stratum to a formal flat G-bundle and used this theory to define its slope. In this paper, the authors study fundamental strata that satisfy an additional regular semisimplicity condition. Flat G-bundles that contain regular strata have a natural reduction of structure to a (not necessarily split) maximal torus in $\hat{G}$, and the authors use this property to compute the corresponding moduli spaces. This theory generalizes a natural condition on algebraic connections (the GL_n case), which plays an important role in the global analysis of meromorphic connections and isomonodromic deformations.

1. Introduction

The study of meromorphic connections on algebraic curves (or equivalently, flat $\mathrm{GL}_n(\mathbb{C})$-bundles) often reduces to the analysis of the associated formal connections at each pole. This local-to-global approach has proven to be especially effective when the principal part of the connection at any irregular singular point has a diagonalizable leading term with distinct eigenvalues. For example, the first significant progress on the isomonodromy problem for irregular singular differential equations came in a 1981 paper of Jimbo, Miwa, and Ueno, in which they imposed this condition at the singularities [13]. Also in this context, Boalch has constructed well-behaved moduli spaces of connections on $\mathbb{P}^1$ with given formal isomorphism classes at the singularities and has further exhibited the isomonodromy equations as an integrable system on an appropriate Poisson manifold [1]. Analogous results hold for flat G-bundles, where G is a complex reductive group [10]. Other aspects of the monodromy map for flat G-bundles of this type have been studied in [6, 2].

While there is a very satisfactory picture of this type of connection, the conditions imposed are quite restrictive. Indeed, such connections necessarily have integral slope at each singularity whereas the slope of a rank n formal connection at an irregular singular point can be any positive rational number with denominator at most n. Moreover, many connections of particular interest are not of this type. Recall that in the GL_n case of the geometric Langlands program, the role of Galois representations is played by monodromy data associated to flat connections: over a smooth complex curve X or the formal punctured disk $\Delta^\times = \mathrm{Spec}(F)$ depending on whether one is in the global or local context. By analogy with the classical

The research of the second author was partially supported by a grant from the Simons Foundation (#281502).

situation, one expects that connections corresponding to cuspidal representations will not have regular semisimple leading terms. For example, Frenkel and Gross have constructed a rigid flat G-bundle (for any reductive G) which corresponds to the Steinberg representation at 0 and a certain "small" supercuspidal representation at ∞ [11]. When $G = GL_2(\mathbb{C})$, this is just the classical Airy connection. Here, the leading term at the irregular singular point at ∞ is nilpotent.

In [3], the authors generalized Boalch's results mentioned above to a much wider class of meromorphic connections. This was done through the introduction of a new notion of the "leading term" of a formal connection in terms of a geometric version of the theory of fundamental strata familiar from p-adic representation theory (see, for example, [8]). Let $F = \mathbb{C}((z))$ be the field of formal Laurent series with ring of integers $\mathfrak{o} = \mathbb{C}[[z]]$. A $\mathrm{GL}_n(F)$-stratum is a triple (P, r, β) with $P \subset \mathrm{GL}_n(F)$ a parahoric subgroup, r a nonnegative integer, and β a functional on the quotient of congruent subalgebras $\mathfrak{P}^r/\mathfrak{P}^{r+1}$. The stratum is fundamental if β satisfies a certain nondegeneracy condition. Let $(\hat{V}, \hat{\nabla})$ be a rank n connection over the formal punctured disk $\Delta^\times = \mathrm{Spec}(F)$. After fixing a trivialization for $\hat{V}$, the matrix of the connection $[\hat{\nabla}]$ is an element of $\mathfrak{gl}_n(F)\frac{dz}{z}$. In particular, it induces a functional on $\mathfrak{gl}_n(F)$ via taking the residue of the trace form. We say that $(\hat{V}, \hat{\nabla})$ contains the stratum (P, r, β) if this functional kills $\mathfrak{P}^{r+1}$ and induces β on the quotient space. Every connection contains a fundamental strata, and each such stratum should be viewed as a "correct" leading term of the connection. For example, a stratum determines the slope of an irregular connection if and only if it is fundamental. The Frenkel-Gross connection does not contain a fundamental stratum at the irregular singular point with respect to the usual filtration (with $P = GL_n(\mathfrak{o})$), but it does with respect to a certain Iwahori subgroup.

The key property that allows one to construct smooth moduli spaces of global connections is for the corresponding formal connections to contain *regular strata*. These are fundamental strata which are centralized in a graded sense by a possibly nonsplit maximal torus $S \subset \mathrm{GL}_n(F)$. For example, if $[\nabla] = (M_{-r}z^{-r} + M_{-r+1}z^{-r+1} + \dots)\frac{dz}{z}$ with $M_i \in \mathfrak{gl}_n(\mathbb{C})$ and M_{-r} regular semisimple, then it contains a regular stratum $(\mathrm{GL}_n(\mathfrak{o}), r, \beta)$ centralized by the diagonal torus. Only certain conjugacy classes of maximal tori can centralize a regular stratum, and only *uniform* maximal tori–maximal tori which are the product of some number of copies of $E^\times$ for some field extension E of F–are considered in [3, 4]. The Frenkel-Gross connection for $\mathrm{GL}_n(\mathbb{C})$ contains a regular stratum centralized by a maximal torus isomorphic to $F[z^{1/n}]^\times$. If $(\hat{V}, \hat{\nabla})$ contains a regular stratum (P, r, β) centralized by S, then we show that its matrix is gauge-equivalent to an element of $\mathfrak{s}\frac{dz}{z}$, where $\mathfrak{s} = \mathrm{Lie}(S)$. In fact, we construct a certain affine subvariety of $\mathfrak{s}_{-r}/\mathfrak{s}_{-1}$ called the variety of S-formal types of depth r, which admits a free action of $\hat{W}_S$, the relative affine Weyl group of S. (Here, $\mathfrak{s}_k$ is the k-th piece of the natural filtration on $\mathfrak{s}$.) We then show that the moduli space of such connections is isomorphic to the set of $\hat{W}_S$-orbits.

In [3, 4], we generalize Boalch's results to meromorphic connections on $\mathbb{P}^1$ which contain regular strata at each irregular singular point. In particular, consider meromorphic connections (V, ∇) with singularities at $\mathbf{y} = (y_1, \dots, y_m)$ and which contain regular strata (P_i, r_i, β_i) centralized by S_i at each y_i. We then construct a Poisson manifold $\widetilde{\mathcal{M}}(\mathbf{y}, \mathbf{S}, \mathbf{r})$ of such connections with given "framing data". If $\mathbf{A}$ is an

m-tuple of formal types with the combinatorics determined by $\mathbf{S}$ and $\mathbf{r}$, we also construct the space $\mathcal{M}(\mathbf{y}, \mathbf{A})$ (resp. $\widetilde{\mathcal{M}}(\mathbf{y}, \mathbf{A})$) of framable (resp. framed) connections with the specified formal types. The variety $\mathcal{M}(\mathbf{y}, \mathbf{A})$ is the symplectic reduction of the symplectic manifold $\widetilde{\mathcal{M}}(\mathbf{y}, \mathbf{A})$ via a torus action. The constructions of all of these spaces are automorphic, in the sense that they are realized as the symplectic or Poisson reduction of products of smooth varieties determined by local data. Finally, the monodromy map and the formal types map induce orthogonal foliations on $\widetilde{\mathcal{M}}(\mathbf{y}, \mathbf{S}, \mathbf{r})$. Thus, the fibers of the monodromy map are the leaves of an integrable system on $\widetilde{\mathcal{M}}(\mathbf{y}, \mathbf{S}, \mathbf{r})$ determined by the isomondromy equations while the connected components of the $\widetilde{\mathcal{M}}(\mathbf{y}, \mathbf{A})$'s are the symplectic leaves of $\widetilde{\mathcal{M}}(\mathbf{y}, \mathbf{S}, \mathbf{r})$.

The goal of this paper is to develop the local theory necessary to obtain similar results for flat G-bundles. In particular, we generalize the theory of regular strata and its application to formal G-bundles. Our starting point is the geometric theory of fundamental strata for reductive groups [5], which we review in Section 2. Given any point x in the Bruhat-Tits building $\mathcal{B}$ for $G(F)$, Moy and Prasad have defined a decreasing $\mathbb{R}$-filtration $(\mathfrak{g}_{x,r})$ on $\mathfrak{g}(F)$ with a discrete number of steps [15, 16]. A stratum is a triple (x, r, β) where $x \in \mathcal{B}$, $r \in \mathbb{R}_{\geq 0}$, and β is a functional on the r-th step $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$ in the filtration. In [5], we show that every flat G-bundle contains a fundamental stratum and the stratum depth r is the same for all of them. We thus obtain a new invariant for formal flat G-bundles called the slope. These results are the geometric analogue of Moy and Prasad's theorem on the existence of minimal K-types for admissible representations of p-adic groups [15, 16].

Intuitively, regular strata are fundamental strata that satisfy a graded version of regular semisimplicity. Regular strata do not appear in the p-adic theory, though they have some points in common with the semisimple strata considered for p-adic classical groups in [9, 18]. As a preliminary, we first study points in the building compatible with a given Cartan subalgebra. A point x is compatible with the Cartan subalgebra $\mathfrak{s}$ if the restriction of the filtration given by x to $\mathfrak{s}$ is the unique Moy-Prasad filtration on $\mathfrak{s}$. If $\mathcal{A}_0 \subset \mathcal{B}$ is a fixed rational apartment, Theorems 3.11 and 3.13 give existence and classification results for Cartan subalgebras graded compatible with a given point in $\mathcal{A}_0$. We apply these results in Corollary 3.15 to classify the set of points in $\mathcal{A}_0$ compatible with some conjugate of $\mathfrak{s}$.

In the following section, we introduce the concept of an S-regular stratum (x, r, β), where S is a maximal torus in $G(F)$. Roughly speaking, this means that x is compatible with the associated Cartan subalgebra $\mathfrak{s}$ and that every representative of β has connected centralizer a suitable conjugate of S. The existence of an S-regular stratum is a restrictive condition. Recalling that the classes of maximal tori in $\hat{G}$ correspond bijectively to the conjugacy classes in the Weyl group W, we show in Corollary 4.10 that it can only occur when S corresponds to a regular conjugacy class in W. For example, when $G = \mathrm{GL}_n$, such maximal tori are the uniform maximal tori and tori of the form $S' \times F^\times$ where S' is uniform in $\mathrm{GL}_{n-1}(\mathbb{C})$. Combining this with Corollary 3.15, we obtain a description of all points in $\mathcal{A}_0$ which can support a regular stratum for a given conjugacy class of maximal tori.

Finally, in Section 5, we study the category $\mathcal{C}(S, r)$ of formal flat G-bundles which contain an S-regular stratum of slope r and an associated category $\mathcal{C}^{\mathrm{fr}}_x(S, r)$, depending on a choice of compatible point x, of framed flat bundles. (When S is split, we take $S = T(F)$ and only allow x to be the vertex corresponding to $G(\mathfrak{o})$.) We show that the framed categories are independent of the choice of x. The moduli space of $\mathcal{C}_x^{\mathrm{fr}}(S,r)$ can be viewed as the set $\mathbf{A}(S,r)$ of S-formal types of depth r–a certain affine open (when $r>0$) subset of $\mathfrak{s}_{-r}^\vee/\mathfrak{s}_{0+}^\vee$ endowed with a free action of the relative affine Weyl group $\hat{W}_S$. Theorem 5.1 states that any $\hat{\nabla}$ containing an S-regular stratum is gauge-equivalent to a flat G-bundle determined by a formal type in $\mathbf{A}(S,r)$. More precisely, the forgetful deframing functor $\mathcal{C}_x^{\mathrm{fr}}(S,r)\to\mathcal{C}(S,r)$ induces the quotient map $\mathbf{A}(S,r)\to\mathbf{A}(S,r)/\hat{W}_S$ on moduli spaces.

We expect that the results on meromorphic connections in [3, 4] can be generalized to meromorphic flat G-bundles containing regular strata at each irregular singular point. We are also hopeful that these results will be of use in the geometric Langlands program. In particular, we anticipate that there is an interpretation of fundamental strata for representations of affine Kac-Moody algebras and that representations containing regular strata should correspond to formal flat G-bundles containing regular strata.

2. Preliminaries

Let k be an algebraically closed field of characteristic 0, and let G be a connected reductive group over k with Lie algebra $\mathfrak{g}$. Fix a maximal torus $T\subset G$ with corresponding Cartan subalgebra $\mathfrak{t}$. Let $N=N(T)$ be the normalizer of T, so that the Weyl group W of G is isomorphic to N/T. The set of roots with respect to T will be denoted by Φ. Given $\alpha\in\Phi$, $U_\alpha\subset G$ is the associated root subgroup and $\mathfrak{u}_\alpha\subset\mathfrak{g}$ is the weight space for $\mathfrak{t}$ corresponding to α. We will write Z for the center of G and $\mathfrak{z}$ for its Lie algebra. We fix a nondegenerate invariant symmetric bilinear form $\langle,\rangle$ on $\mathfrak{g}$ throughout. Finally, $\mathrm{Rep}(G)$ denotes the category of finite-dimensional representations of G over k.

Let $F=k((z))$ be the field of formal Laurent series over k with ring of integers $\mathfrak{o}=k[[z]]$, and let $\Delta^\times=\mathrm{Spec}(F)$ be the formal punctured disk. We denote the Euler differential operator on F by $\tau=z\frac{d}{dz}$. We set $\hat{G}=G(F)$ and $\hat{\mathfrak{g}}=\mathfrak{g}\otimes_k F$; note that $\hat{G}$ represents the functor sending a k-algebra R to $G(R((z)))$. We will use the analogous notation $\hat{H}$ and $\hat{\mathfrak{h}}$ for any algebraic group H over k. Similarly, if V is a representation of G, then $\hat{V}=V\otimes F$ will denote the corresponding representation of $\hat{G}$.

The Bruhat-Tits building and the enlarged building of $\hat{G}$ will be denoted by $\bar{\mathcal{B}}$ and $\mathcal{B}$ respectively. If $x\in\mathcal{B}$, we denote the corresponding parahoric subgroup (resp. subalgebra) by $\hat{G}_x$ (resp. $\hat{\mathfrak{g}}_x$). The standard apartment in $\mathcal{B}$ associated to the split rational torus $\hat{T}=T(F)$ is an affine space isomorphic to $X_*(T)\otimes_{\mathbb{Z}}\mathbb{R}$. If $\mathbb{R}\subset k$, then points in $\mathcal{A}_0$ may be viewed as elements of $\mathfrak{t}_{\mathbb{R}}$. The map $\mathcal{A}_0\to\mathfrak{t}_{\mathbb{R}}$ is induced by evaluating cocharacters at 1. We write $\tilde{x}\in\mathfrak{t}_{\mathbb{R}}$ for the image of $x\in\mathcal{A}_0$. If $x\in X_*(T)\otimes_{\mathbb{Z}}\mathbb{Q}$, then $\tilde{x}\in\mathfrak{t}$ is defined for any k.

Convention. If the notation $\tilde{x}$ is used for $x\in\mathcal{A}_0$, then either k contains $\mathbb{R}$ or x is a rational point of $\mathcal{A}_0$ (so $\tilde{x}\in\mathfrak{t}_{\mathbb{Q}}$).

Let $\bar{F}$ be an algebraic closure of F. Later in the paper, we will need to consider elements of $T(\bar{F})$ of the form z^v with $v\in\mathfrak{t}_{\mathbb{Q}}$. Recall that $\bar{F}$ is generated by m-th roots of z. Suppose that $u\in\bar{F}$ satisfies $u^m=z$. If $v\in\mathfrak{t}_{\frac{1}{m}\mathbb{Z}}$, one can define z^v as the unique element of $T(F[u])$ satisfying $\chi(z^v)=u^{d\chi(mv)}$ for all $\chi\in X^*(T)$. This, of course, depends on the choice of m and u, but it is well-defined up to multiplication by $\xi^v\in T$, where ξ is an m^{th} root of unity. Since $\xi^v\in T$, and in particular is fixed by $\mathrm{Gal}(\bar{F}/F)$ and killed by τ, it will follow that all results that involve z^v will be independent of this choice. For convenience, we may assume that all elements of this form are defined in terms of a coherent set of uniformizers for the finite extensions of $\bar{F}$, i.e., a choice of elements $u_m \in \bar{F}$ for each $m \in \mathbb{N}$ satisfying $u_m^m = z$ and such that if $m'|m$, then $u_m^{m/m'} = u_{m'}$.

2.1. **Moy-Prasad filtrations.** If V is any finite-dimensional representation of G, then any point $x \in \mathcal{B}$ induces a decreasing $\mathbb{R}$-filtration $\{\hat{V}_{x,r}\}$ of $\hat{V}$ by $\mathfrak{o}$-lattices called the Moy-Prasad filtration [12, 15]. Since $g\hat{V}_{x,r} = \hat{V}_{gx,r}$, it suffices to recall the definition for $x \in \mathcal{A}_0$, where it can be constructed in terms of a grading on $V \otimes_k k[z, z^{-1}]$. If $\chi \in X^*(T)$ and V_χ is the corresponding weight space, then the r-th graded subspace is given by

$$\hat{V}_{x,\mathcal{A}_0}(r) = \bigoplus_{\langle \chi, x\rangle + m = r} V_\chi z^m \subset \hat{V}. \tag{1}$$

The grading depends on the choice of apartment. However, since we only use gradings with respect to $\mathcal{A}_0$, we usually write $\hat{V}_x(r)$ for $\hat{V}_{x,\mathcal{A}_0}(r)$. For any $r \in \mathbb{R}$, define

$$\hat{V}_{x,r} = \prod_{s \geq r} \hat{V}_{x,\mathcal{A}_0}(s) \subset \hat{V}; \quad \hat{V}_{x,r+} = \prod_{s > r} \hat{V}_{x,\mathcal{A}_0}(s) \subset \hat{V}.$$

The collection of lattices $\{\hat{V}_{x,r}\}$ is the Moy-Prasad filtration on $\hat{V}$ associated to x. The set $\mathrm{Crit}_x(V)$ of *critical numbers* of V at x is the discrete, $\mathbb{Z}$-invariant subset of $\mathbb{R}$ for which $\hat{V}_{x,r}/\hat{V}_{x,r+} \cong \hat{V}_{x,\mathcal{A}_0}(r) \neq \{0\}$. It is easy to see that the sets of critical numbers associated to the adjoint and coadjoint representations coincide and are symmetric around 0.

There is also a corresponding filtration $\{\hat{G}_{x,r}\}_{r \in \mathbb{R}_{\geq 0}}$ of the parahoric subgroup $\hat{G}_x = \hat{G}_{x,0}$ for $x \in \mathcal{B}$. If one sets $\hat{G}_{x,r+} = \bigcup_{s>r} \hat{G}_{x,s}$, then $\hat{G}_{x+} = \hat{G}_{x,0+}$ is the pro-unipotent radical of $\hat{G}_x$. For $r > 0$, there is a natural isomorphism $\hat{G}_{x,r}/\hat{G}_{x,r+} \cong \hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+}$ [15]. On the other hand, $\hat{G}_x/\hat{G}_{x+}$ is isomorphic to a reductive, maximal rank subgroup of G. For $x \in \mathcal{A}_0$, we give an explicit isomorphism. Let $H_x \subset G$ be the subgroup generated by T and the root subgroups U_α such that $d\alpha(\tilde{x}) \in \mathbb{Z}$. (If $\mathbb{C} \subset k$, H_x is the connected centralizer of $\exp(2\pi i\tilde{x}) \in G$.) There is a homomorphism $\theta'_x : H_x \to \hat{G}_x$ defined on the generators of H_x via $T \hookrightarrow T(\mathfrak{o})$ and $\theta'_x(U_\alpha(c)) = U_\alpha(cz^{-\alpha(\tilde{x})})$ for $c \in k$. The induced map $\theta_x : H_x \to \hat{G}_x/\hat{G}_{x+}$ is an isomorphism [5]. It is easy to see that the group H_x acts on $\hat{V}_x(r)$ for any r and θ_x intertwines the representations $\hat{V}_x(r)$ and $\hat{V}_{x,r}/\hat{V}_{x,r+}$.

We collect the basic properties of these filtrations in the following proposition.

Proposition 2.1. *Take $V \in \mathrm{Rep}(G)$, and fix $x \in \mathcal{A}_0$ and $r \in \mathbb{R}$.*

1. *The space $\hat{V}_x(r)$ is the eigenspace corresponding to the eigenvalue r in $\hat{V}$ for the differential operator $\tau + \tilde{x}$.*
2. *An element $v \in \hat{V}$ lies in $\hat{V}_{x,r}$ if and only if $(\tau + \tilde{x})(v) - rv \in \hat{V}_{x,r+}$.*
3. *The set $\hat{V}_x(r)$ constitutes a full set of coset representatives for the coset space $\hat{V}_{x,r}/\hat{V}_{x,r+}$.*
4. *If $X \in \hat{\mathfrak{g}}_x(s)$, then $\mathrm{ad}(X)(\hat{V}_x(r)) \subset \hat{V}_x(r+s)$.*

If E is a degree e extension of F, then these gradings and filtrations extend naturally to $V(E)$ by setting the valuation of the uniformizer in E to be $1/e$.

For example, if $E = F((u))$ with $u^e = z$, then $\mathfrak{t}(E)(m/e) = u^m\mathfrak{t}$ for $m \in \mathbb{Z}$. Proposition 2.1 remains true if one interprets τ as $\frac{1}{e}u\frac{d}{du}$, its unique extension to E. If $\Gamma = \mathrm{Gal}(E/F)$, then $V_{x,r} = V(E)^{\Gamma}_{x,r}$ and similarly for the gradings.

We will frequently need to compare Moy-Prasad filtrations on $\hat{G}$ with filtrations on nonsplit maximal tori. For a torus, there is a unique Moy-Prasad filtration on the maximal bounded subgroup and the Cartan subalgebra. If S is a maximal torus with Lie algebra $\mathfrak{s}$, then we may define graded and filtered pieces by extending scalars to a finite splitting field E and conjugating the analogous data for $T(E)$ and $\mathfrak{t}(E)$. To be more explicit, if $g \in G(E)$ satisfies $\mathrm{Ad}(g^{-1})\mathfrak{s}(E) = \mathfrak{t}(E)$ and $\Gamma = \mathrm{Gal}(E/F)$, then $\mathfrak{s}(r) = \big(\mathrm{Ad}(g)(\mathfrak{t}(E)(r))\big)^{\Gamma}$, $\mathfrak{s}_r = \big(\mathrm{Ad}(g)(\mathfrak{t}(E)_r)\big)^{\Gamma}$, and $S_r = \big(g(T(E))_r)g^{-1}\big)^{\Gamma}$. These definitions do not depend on the choice of E or g. Indeed, if one takes another diagonalizer g' and takes E big enough so that $g, g' \in G(E)$, then $g' = gn$ for $n \in N(E)$. Independence now follows, since $N(E)$ fixes the grading and filtrations on $\mathfrak{t}(E)$ and $T(\mathfrak{o}_E)$. Observe that this definition rescales the index on filtrations for nonsplit groups constructed in [16] by a factor of $1/e$, but it is effectively the same as that appearing in [7, Section 10] and [12, Section 5].

Moy-Prasad filtrations are well-behaved under duality. If W is an $\mathfrak{o}$-module, let $W^\vee$ be its smooth (k-linear) dual. Note that if $V \in \mathrm{Rep}(G)$, then there is a $\hat{G}$-isomorphism $\widehat{(V^\vee)} = V^\vee \otimes F \xrightarrow{\kappa} (\hat{V})^\vee$, $\kappa(\alpha)(v) = \mathrm{Res}\, \alpha(v)\frac{dz}{z}$; we will abuse notation slightly by denoting both by $\hat{V}^\vee$. However, $\hat{V}^\vee_{x,r}$ will always mean $(\hat{V}^\vee)_{x,r}$. We recall the following facts. (See [5] for more details.)

Proposition 2.2. *Fix $V \in \mathrm{Rep}(G)$, $x \in \mathcal{B}$, and $r \in \mathbb{R}$.*

(1) *The isomorphism κ restricts to give $\hat{G}_x$-isomorphisms $\hat{V}^\vee_{x,-r} \cong \hat{V}^\perp_{x,r+}$ and $\hat{V}^\vee_{x,-r+} \cong \hat{V}^\perp_{x,r}$.*

(2) *There is a natural $\hat{G}_x$-invariant perfect pairing*

$$\hat{V}^\vee_{x,-r}/\hat{V}^\vee_{x,-r+} \times \hat{V}_{x,r}/\hat{V}_{x,r+} \to k,$$

which induces the isomorphism $(\hat{V}_{x,r}/\hat{V}_{x,r+})^\vee \cong \hat{V}^\vee_{x,-r}/\hat{V}^\vee_{x,-r+}$.

(3) *There are $\hat{G}_x$-isomorphisms $(\hat{V}_{x,r})^\vee \cong \hat{V}^\vee/\hat{V}^\vee_{x,-r+}$ and $(\hat{V}_{x,r+})^\vee \cong \hat{V}^\vee/\hat{V}^\vee_{x,-r}$.*

(4) *Suppose that V is endowed with a nondegenerate G-invariant symmetric bilinear form $(,)$. Then, $(,)_{\frac{dz}{z}} \overset{def}{=} \mathrm{Res}((,)\frac{dz}{z}$ induces $\hat{G}_x$-isomorphisms $\hat{V}^\vee_{x,-r} \cong \hat{V}_{x,-r}$ and $\hat{V}^\vee_{x,-r+} \cong \hat{V}_{x,-r+}$; in particular, $(\hat{V}_{x,r}/\hat{V}_{x,r+})^\vee \cong \hat{V}_{x,-r}/\hat{V}_{x,-r+}$.*

2.2. **Formal flat G-bundles and strata.** A formal principal G-bundle $\mathcal{G}$ is a principal G-bundle over the formal punctured disk $\Delta^\times$. The G-bundle $\mathcal{G}$ induces a tensor functor from $\mathrm{Rep}(G)$ to the category of formal vector bundles via $V \mapsto V_{\mathcal{G}} = \mathcal{G} \times_G V$, and this tensor functor uniquely determines $\mathcal{G}$. Formal principal G-bundles are trivializable, so we may always choose a trivialization $\phi : \hat{G} \to \mathcal{G}$. Moreover, there is a left action of $\hat{G}$ on the set of trivializations of $\mathcal{G}$.

A flat structure on a principal G-bundle is a formal derivation ∇ that determines a compatible family of flat connections on $V_{\mathcal{G}}$ for all $V \in \mathrm{Rep}(G)$. In terms of a fixed trivialization ϕ for $\mathcal{G}$, ∇ acts on $V_{\mathcal{G}}$ as the operator $d + [\nabla]_\phi\wedge$, where d is the ordinary exterior derivative and $[\nabla]_\phi \in \Omega^1_F(\hat{\mathfrak{g}})$ is the *matrix* of ∇ in the trivialization ϕ. Since $\Omega^1_F(\hat{\mathfrak{g}}) \cong \Omega^1_F(\hat{\mathfrak{g}}^\vee)$ via the choice of invariant form on fg and $\Omega^1_F(\hat{\mathfrak{g}}^\vee) \cong \hat{\mathfrak{g}}^\vee$ canonically, we can view $[\nabla]_\phi$ as a functional on $\hat{\mathfrak{g}}$. The group $\hat{G}$ acts on $[\nabla]_\phi$ by gauge transformations, namely

$$[\nabla]_{g\phi} = g \cdot [\nabla]_\phi = \mathrm{Ad}^*(g)([\nabla]) - (dg)g^{-1}. \tag{2}$$

The right-invariant Maurer-Cartan form $(dg)g^{-1}$ lies in $\Omega^1_F(\hat{\mathfrak{g}})$. Note that if ι_τ is the inner derivation by τ, then we can write $[\nabla]_\phi = [\nabla_\tau]_\phi \frac{dz}{z}$, where $[\nabla_\tau]_\phi = \iota_\tau[\nabla]_\phi \in \hat{\mathfrak{g}}$. The flat G-bundle $(\mathcal{G}, \nabla)$ is called regular singular if the flat connection $V_\mathcal{G}$ is regular singular for each $V \in \mathrm{Rep}(G)$; otherwise, it is irregular singular.

We now recall some results from the theory of minimal K-types (or fundamental strata) for formal flat G-bundles developed in [5]. Given $x \in \mathcal{B}$ and a nonnegative real number r, a G-stratum of depth r is a triple (x, r, β) with $\beta \in (\hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+})^\vee$. We say that $\tilde{\beta} \in \hat{\mathfrak{g}}^\vee_{x,-r}$ is a representative for β if the coset $\tilde{\beta} + \hat{\mathfrak{g}}^\vee_{x,-r+}$ corresponds to β under the isomorphism $\hat{\mathfrak{g}}^\vee_{x,-r}/\hat{\mathfrak{g}}^\vee_{x,-r+} \cong (\hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+})^\vee$. If $x \in \mathcal{A}_0$, we let $\tilde{\beta}_0$ denote the unique homogeneous representative in $\hat{\mathfrak{g}}^\vee_x(-r)$. The loop group $\hat{G}$ acts on the set of strata with $g \cdot (x, r, \beta)$ the stratum determined by gx, r, and the coset $\mathrm{Ad}^*(g)(\tilde{\beta}) + \hat{\mathfrak{g}}^\vee_{gx,-r+}$.

A stratum is called *fundamental* if β is a semistable point of the $\hat{G}_x/\hat{G}_{x+}$-representation $(\hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+})^\vee$; equivalently, the corresponding coset $\tilde{\beta} + \hat{\mathfrak{g}}^\vee_{x,-r+}$ does not contain a nilpotent element. This can only occur when $r \in \mathrm{Crit}_x(\mathfrak{g})$. If $x \in \mathcal{A}_0$, then a stratum is nonfundamental if and only if the homogeneous representative $\tilde{\beta}_0$ is nilpotent.

Given $x \in \mathcal{A}_0$, we say that the flat G-bundle $(\mathcal{G}, \nabla)$ contains the stratum (x, r, β) with respect to the trivialization ϕ for $\mathcal{G}$ if $[\nabla]_\phi - \tilde{x}\frac{dz}{z} \in \hat{\mathfrak{g}}^\perp_{x,r+}$ and the coset $\left([\nabla]_\phi - \tilde{x}\frac{dz}{z}\right) + \hat{\mathfrak{g}}^\vee_{x,-r+}$ determines the functional $\beta \in (\hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+})^\vee$. Note that if $r > 0$, then β is the functional determined by $[\nabla]_\phi$; moreover, if x' and x have the same image in $\bar{\mathcal{B}}$, then (x', r, β) is also contained in $(\mathcal{G}, \nabla)$.

Given a flat G-bundle $(\mathcal{G}, \nabla)$, we say that its *slope* is the infimum of the depths of the strata contained in it. In [5], it is shown that this infimum is actually attained and that it is a rational number. More precisely, we have the following theorem.

Theorem 2.3. [5, Theorem 3.15] *Every flat G-bundle $(\mathcal{G}, \nabla)$ contains a fundamental stratum $(x, \mathrm{slope}(\mathcal{G}), \beta)$, where x is an optimal point in $\mathcal{A}_0$ in the sense of* [15]*; the slope is positive if and only if $(\mathcal{G}, \nabla)$ is irregular singular. Moreover, the following statements hold.*

(1) *If $(\mathcal{G}, \nabla)$ contains the stratum (y, r', β'), then $r' \geq \mathrm{slope}(\mathcal{G})$.*
(2) *If $(\mathcal{G}, \nabla)$ is irregular, a stratum (y, r', β') contained in $(\mathcal{G}, \nabla)$ is fundamental if and only if $r' = \mathrm{slope}(\mathcal{G})$.*

In particular, the slope is an *optimal number*–a critical number for an optimal point in $\mathcal{A}_0$.

For future reference, we recall the following lemma from [5] describing the calculus for change of trivialization on strata contained in $\mathcal{G}$.

Lemma 2.4. [5, Lemma 3.4]

(1) *If $n \in \hat{N}$, $[\nabla]_{n\phi} - \widetilde{nx}\frac{dz}{z} \in \mathrm{Ad}^*(n)([\nabla]_\phi - \tilde{x}\frac{dz}{z}) + \hat{\mathfrak{t}}_{0+}\frac{dz}{z}$.*
(2) *If $X \in \hat{\mathfrak{u}}_\alpha \cap \hat{\mathfrak{g}}_{x,\ell}$, then*
$$[\nabla]_{\exp(X)\phi} - \tilde{x}\frac{dz}{z} \in \mathrm{Ad}^*(\exp(X))([\nabla]_\phi - \tilde{x}\frac{dz}{z}) - \ell X\frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{x,\ell+}.$$
(3) *If $p \in \hat{G}_x$, then $[\nabla]_{p\phi} - \tilde{x}\frac{dz}{z} \in \mathrm{Ad}^*(p)([\nabla]_\phi - \tilde{x}\frac{dz}{z}) + \hat{\mathfrak{g}}^\vee_{x+}$.*

(4) *If* $p \in \hat{G}_{x,\ell}$ *for* $\ell > 0$*, then* $[\nabla]_{p\phi} - \tilde{x}\frac{dz}{z} \in \mathrm{Ad}^*(p)([\nabla]_\phi - \tilde{x}\frac{dz}{z}) + \hat{\mathfrak{g}}^\vee_{x,\ell}$.

Remark 2.5. Applying part (3) of the lemma to $[\nabla]_\phi = 0$, we see that if $p \in \hat{G}_x$, then $\tau(p)p^{-1} \in \mathrm{Ad}(p)(\tilde{x}) - \tilde{x} + \hat{\mathfrak{g}}_{x+} \subset \hat{\mathfrak{g}}_x$. This fact will be used throughout the paper.

3. Compatible Filtrations

Intuitively, one can view a fundamental stratum contained in a flat G-bundle as a nondegenerate "leading term" of the derivation ∇. The goal of this paper is to study flat G-bundles containing strata corresponding to regular semisimple leading terms. In order to do this, we need to study filtrations that are compatible with the natural filtration on a maximal torus in $\hat{G}$.

By [14, Lemma 2], there is a bijection between the set of conjugacy classes of Cartan subalgebras in $\hat{\mathfrak{g}}$ (resp. maximal tori in $\hat{G}$) and the set of conjugacy classes in W. We briefly recall the correspondence. Let $\bar{\Gamma} \cong \hat{Z}$ be the absolute Galois group of F. If $\mathfrak{s} \subset \hat{\mathfrak{g}}$ is a Cartan subalgebra, then there exists $g \in G(\bar{F})$ such that $\mathrm{Ad}(g)\mathfrak{t}(\bar{F}) = \mathfrak{s}(\bar{F})$, so $\rho \mapsto g^{-1}\rho(g)$ is a 1-cocycle of $\bar{\Gamma}$ with values $N(\bar{F})$. In fact, since $H^1(F, \hat{G}) = 1$ (as G is connected reductive and F has cohomological dimension 1), all 1-cocycles in $N(\bar{F})$ are of this form, and such a cocycle coming from $h \in G(\bar{F})$ gives rise to the Cartan subalgebra $(\mathrm{Ad}(h)(\mathfrak{t}(\bar{F})))^{\bar{\Gamma}}$. The induced map gives a bijection between $H^1(F, \hat{N})$ and the conjugacy classes of Cartan subalgebras. Moreover, $H^1(F, \hat{N})$ is isomorphic (as pointed sets) to the set of conjugacy classes of W; the image of the above cocycle is the class of $g^{-1}\sigma(g)T(\bar{F}) \in W$, where σ is a fixed topological generator of $\bar{\Gamma}$. In particular, $\mathrm{Ad}(g^{-1})$ intertwines the action of σ on $\mathfrak{s}(\bar{F})$ with $w \circ \sigma$ on $\mathfrak{t}(\bar{F})$. Of course, if E is a finite splitting field for $\mathfrak{s}$, then one can take $g \in G(E)$ and $\mathrm{Ad}(g^{-1})$ again intertwines a fixed generator σ for $\mathrm{Gal}(E/F) \cong \mathbb{Z}_{[E:F]}$ with $w \circ \sigma$. Since w and σ commute, it follows that the order of w divides $[E : F]$.

Definition 3.1. Let γ denote a conjugacy class in W. We say that a Cartan subalgebra $\mathfrak{s} \subset \hat{\mathfrak{g}}$ (resp. a maximal torus $S \subset \hat{G}$) is of type γ if the conjugacy class of $\mathfrak{s}$ (resp. S) corresponds to γ as above.

For the remainder of this section, we fix a maximal torus S type γ, and choose a representative $w \in W$ for γ. Let $E = k((z^{1/e}))$ be a splitting field of $\mathfrak{s}$, and let $\sigma \in \Gamma = \mathrm{Gal}(E/F)$ be a fixed generator. We then take $g \in G(E)$ such that $\mathrm{Ad}(g^{-1})\mathfrak{s}(E) = \mathfrak{t}(E)$ and $g^{-1}\sigma(g)T(E) = w$. We call such a g a *w-diagonalizer* of S.

3.1. **Compatible gradings and filtrations.**

Let S be a maximal torus in $\hat{G}$ with Cartan subalgebra $\mathfrak{s}$.

Definition 3.2. We say that a point $x \in \mathcal{B}$ is *compatible* with $\mathfrak{s}$ if the filtration induced by x on $\mathfrak{s}$ is the (rescaled) Moy-Prasad filtration on $\mathfrak{s}$, i.e., $\mathfrak{s}_r = \hat{\mathfrak{g}}_{x,r} \cap \mathfrak{s}$ for all r. If $x \in \mathcal{A}_0$, x is *graded compatible* with $\mathfrak{s}$ if $\mathfrak{s}(r) = \hat{\mathfrak{g}}_x(r)$ for all r and $\mathfrak{s}(0) \subset \hat{\mathfrak{t}}(0) = \mathfrak{t}$.

Remark 3.3. Similarly, we say that $x \in \mathcal{B}$ is compatible with the maximal torus S if $S_r = \hat{G}_{x,r} \cap S$ for all $r \geq 0$. It is easy to see that x is compatible with S if and only if it is compatible with $\mathfrak{s}$.

It is obvious that every point in $\mathcal{A}_0$ is graded compatible (hence compatible) with $\hat{\mathfrak{t}}$. Conversely, if $\mathfrak{s}'$ is a split Cartan subalgebra graded compatible with $x \in \mathcal{A}_0$, then $\dim \mathfrak{s}'(0) = \dim \mathfrak{t}$ and $\mathfrak{s}'(0) \subset \mathfrak{t}$, so $\mathfrak{s}' = \hat{\mathfrak{t}}$. As we will see later, nonsplit Cartan subalgebras can be graded compatible with points in $\mathcal{A}_0$. One can also show that if $\mathfrak{s}$ is any Cartan subalgebra compatible with each point of $\mathcal{A}_0$, then $\mathfrak{s} = \hat{\mathfrak{t}}$. Indeed, it follows from the definitions that $\mathfrak{s}_0 \subset \cap_{x \in \mathcal{A}_0} \hat{\mathfrak{g}}_{x,r} = \hat{\mathfrak{t}}_0$, and since $\mathfrak{s}_0$ contains a regular semisimple element, $\mathfrak{s} = \hat{\mathfrak{t}}$.

The goal of this section is to show that if $x \in \mathcal{A}_0$ is compatible with S, then there exists a $\hat{G}_x$-conjugate of S which is graded compatible with x. We begin with several equivalent formulations of graded compatibility.

Lemma 3.4. *The following statements are equivalent:*

(1) *The point $x \in \mathcal{A}_0$ is graded compatible with $\mathfrak{s}$;*
(2) $(\tau + \mathrm{ad}(\tilde{x}))(\mathfrak{s}) \subset \mathfrak{s}$*;*
(3) *If $g \in G(E)$ is a w-diagonalizer, then $\mathrm{Ad}(g^{-1})(\tilde{x} + \tau(g)g^{-1}) \in \mathfrak{t}(E)$; and*
(4) *If $g \in G(E)$ is a w-diagonalizer, then $\mathrm{Ad}(g^{-1})(\tilde{x} + \tau(g)g^{-1}) \in \mathfrak{t}(\mathfrak{o}_E)$.*

Proof. Suppose x is graded compatible with $\mathfrak{s}$. Then, $\mathfrak{s}(r) = \mathfrak{s} \cap \hat{\mathfrak{g}}_x(r)$ implies that there is a topological basis for $\mathfrak{s}$ consisting of eigenvectors for $\tau + \mathrm{ad}(\tilde{x})$, proving the second statement.

Next, observe that

$$(3) \qquad (\tau + \mathrm{ad}(\tilde{x}))(\mathrm{Ad}(g)X) = \mathrm{Ad}(g)\left(\left[\tau + \mathrm{ad}\left(\mathrm{Ad}(g^{-1})(\tilde{x}) + g^{-1}\tau(g)\right)\right](X)\right)$$

for $X \in \mathfrak{g}(E)$. If $(\tau + \mathrm{ad}(\tilde{x}))(\mathfrak{s}) \subset \mathfrak{s}$, then applying $\mathrm{Ad}(g^{-1})$ of this equation to $X \in \mathfrak{t}(E)$ gives

$$(4) \qquad \left[\tau + \mathrm{ad}\left(\mathrm{Ad}(g^{-1})(\tilde{x}) + g^{-1}\tau(g)\right)\right](\mathfrak{t}(E)) \subset \mathfrak{t}(E).$$

Since $\tau\mathfrak{t}(E) \subset \mathfrak{t}(E)$, it follows that $\mathrm{ad}\left(\mathrm{Ad}(g^{-1})(\tilde{x}) + g^{-1}\tau(g)\right)(\mathfrak{t}(E)) \subset \mathfrak{t}(E)$. Therefore, $\mathrm{Ad}(g^{-1})(\tilde{x}) + g^{-1}\tau(g) \in \mathfrak{t}(E)$.

Now, assume that $\mathrm{Ad}(g^{-1})(\tilde{x}) + g^{-1}\tau(g) \in \mathfrak{t}(E)$. We see that the differential operator $\tau + \mathrm{ad}\left(\mathrm{Ad}(g^{-1})(\tilde{x}) + g^{-1}\tau(g)\right)$ restricts to τ on $\mathfrak{t}(E)$, so the r-eigenspace of the former on $\mathfrak{t}(E)$ is $\mathfrak{t}(E)(r)$. Applying (3) and Proposition 2.1 gives $\mathrm{Ad}(\mathfrak{t}(E)(r)) = \mathfrak{s}(E) \cap \mathfrak{g}(E)_x(r)$, and $\mathfrak{s}(r) = \mathfrak{s} \cap \hat{\mathfrak{g}}_x(r)$ follows by taking Galois fixed points.

It remains to show the equivalence of the last two statements. One direction is trivial, so assume that $Y = \mathrm{Ad}(f^{-1})(\tilde{x} + \tau(f)f^{-1}) \in \mathfrak{t}(E)$. By the Iwasawa decomposition, we can write $f^{-1} = put$, where $p \in G(\mathfrak{o}_E)$, $u \in U(E)$ (with U the unipotent radical of B), and $t \in T(E)$. Before calculating Y, we make several observations. First, the fact that $\tau(z^m)z^{-m} = m$ implies that $\tau(t)t^{-1} = t^{-1}\tau(t) \in \mathfrak{t}(\mathbb{Q}) + z^{1/e}\mathfrak{t}(\mathfrak{o}_E)$. Setting $X = \mathrm{Ad}((ut))[\tilde{x} + \tau((ut)^{-1})(ut)]$, we see that $X = \mathrm{Ad}(u)(\tilde{x} - \tau(t)t^{-1} - u^{-1}\tau(u)) \in \tilde{x} - \tau(t)t^{-1} + \mathfrak{u}(E) \subset \mathfrak{g}(\mathfrak{o}_E) + \mathfrak{u}(E)$.

Next, write $p^{-1} = p_1 p_2$ with $p_1 \in G$ and p_2 in the first congruence subgroup of $G(\mathfrak{o}_E)$ with respect to $z^{1/e}$ (i.e., $G(E)_{o+}$). We obtain $\tau(p^{-1})p = \tau(p_1)p_1^{-1} + \mathrm{Ad}(p_1)(\tau(p_2)p_2^{-1}) = \tau(\log(p_2)) \in z^{1/e}\mathfrak{g}(\mathfrak{o}_E)$. Since $Y = \mathrm{Ad}(p)(X + \tau(p^{-1})p)$, we see that $\mathrm{Ad}(p)(X + \mathfrak{g}(\mathfrak{o}_E)) = Y + \mathfrak{g}(\mathfrak{o}_E)$. Because $X + \mathfrak{g}(\mathfrak{o}_E)$ contains the ad-nilpotent element $-\tau(u)u^{-1}$, $Y + \mathfrak{g}(\mathfrak{o}_E)$ also contains an ad-nilpotent element.

Suppose that $Y \notin \mathfrak{z}(E) + \mathfrak{t}(\mathfrak{o}_E)$. Let $n > 0$ be the smallest integer such that $Y \in \mathfrak{z}(E) + z^{-n}\mathfrak{t}(\mathfrak{o}_E)$. This means that there exists a root α such that $\alpha(Y) \in z^{-n}\mathfrak{o}_E \setminus z^{1-n}\mathfrak{o}_E$. Thus, the action of $\mathrm{ad}(Y)$ on the root subalgebra $\mathfrak{u}(E)_\alpha$ is non-nilpotent. Furthermore, if $Y' \in Y + \mathfrak{g}(\mathfrak{o}_E)$ and $Z \in \mathfrak{u}_\alpha$, then $(\mathrm{ad}(Y'))^n(Z) \in$

$(\mathrm{ad}(Y))^n(Z) + \mathfrak{g}(\mathfrak{o}_E) = \alpha(Y)^n Z + \mathfrak{g}(\mathfrak{o}_E)$ and hence is nonzero. Thus, no element of $Y + \mathfrak{g}(\mathfrak{o}_E)$ is ad-nilpotent, a contradiction.

Accordingly, $X \in \mathrm{Ad}(p^{-1})(Y + \mathfrak{g}(\mathfrak{o}_E)) \subset \mathfrak{z}(E) \cap \mathfrak{g}(\mathfrak{o}_E)$. This means that $X \in (\mathfrak{z}(E) \cap \mathfrak{g}(\mathfrak{o}_E)) \cap (\mathfrak{u}(E) + \mathfrak{g}(\mathfrak{o}_E)) = \mathfrak{g}(\mathfrak{o}_E)$, so $Y \in \mathfrak{g}(\mathfrak{o}_E)$ as well. □

Lemma 3.5. *A Cartan subalgebra $\mathfrak{s}$ is compatible with $x \in \mathcal{A}_0$ if and only if $\mathfrak{s}(E)_r = \mathfrak{g}(E)_{x,r} \cap \mathfrak{s}(E)$ for all $r \in \mathbb{R}$.*

Proof. The reverse implication follows by taking Galois invariance of the equations $\mathfrak{s}(E)_r = \mathfrak{s}(E) \cap \mathfrak{g}(E)_{x,r}$.

Now, suppose that S is compatible with x. Let $\mathrm{tr} : \mathfrak{s}(E) \to \mathfrak{s}$ be the trace map, so that $\eta_i(X) = \frac{1}{e} z^{i/e} \mathrm{tr}(z^{-i/e} X)$ is the projection onto the ξ^i-eigenspace for σ. Since $z^{i/e}\mathfrak{s}(E)_r = \mathfrak{s}(E)_{r+\frac{1}{e}}$ and $z^{i/e}\mathfrak{g}(E)_{x,r} = \mathfrak{g}(E)_{x,r+\frac{1}{e}}$, we obtain $\eta_i(\mathfrak{s}(E)_r) \subset z^{i/e}\mathfrak{s}_{r-\frac{1}{e}} \subset z^{i/e}\mathfrak{g}_{x,r-\frac{1}{e}} \subset \mathfrak{g}(E)_{x,r}$. The action of Γ is completely reducible, so $\mathfrak{s}(E)_r \subset \mathfrak{g}(E)_{x,r}$. On the other hand, suppose that there exists $X \in (\mathfrak{s}(E) \cap \mathfrak{g}(E)_{x,r}) \setminus \mathfrak{s}(E)_r$. The same must be true for $\eta_i(X)$ for some i. We then obtain $z^{-i/e}\eta_i(X) \in (\mathfrak{s} \cap \mathfrak{g}_{x,r-\frac{1}{e}}) \setminus \mathfrak{s}_{r-\frac{1}{e}}$, a contradiction. □

If $S' \subset \hat{G}$ is a maximal split torus compatible with $x \in \mathcal{A}_0$, an elementary version of the argument given below in Proposition 4.5 shows that there exists $g \in \hat{G}_x$ such that $g^{-1}S'g = \hat{T}$. If x is compatible with S, it follows from this and the previous lemma that there exists $p \in G(E)_x$ satisfying $p^{-1}S(E)p = T(E)$. We will need a refinement of this statement.

Lemma 3.6. *Suppose that $S \subset \hat{G}$ is a maximal torus that splits over E. If $x \in \mathcal{A}_0$ is compatible with $\mathfrak{s}$, then there exists $q \in G(E)_x$ such that $q^{-1}S(E)q = T(E)$ and $q^{-1}\sigma(q) \in N$.*

Proof. With $p \in G(E)_x$ as defined in the preceding paragraph, we will construct $s \in T(E)_{0+}$ such that $q = ps$ satisfies $q^{-1}\sigma(q) \in N$. Noting that $p^{-1}\sigma(p) \in \hat{N}(E) \cap G(E)_x$, we can write $p^{-1}\sigma(p) = tn$ with $n \in N$ and $t \in T(E)_{0+}$. An easy induction gives $\sigma^j(p) = p\prod_{i=0}^{j-1}\sigma^i(tn)$ for $j \geq 0$ (the product taken in increasing order); since $\sigma^e(p) = p$, we obtain $1 = \prod_{i=0}^{e-1}\sigma^i(tn)$. It follows that $\prod_{i=0}^{e-1}(w \circ \sigma)^e(t) \in T \cap G(E)_{x+} = \{1\}$.

Set $M = T(E)_{0+}$, and view it as a $\mathbb{Z}/e\mathbb{Z}$-module with $\bar{1}$ acting as $w \circ \sigma$. If h denotes the norm map for this action, the previous paragraph shows that $h(t) = 1$. We will show that $H^1(\mathbb{Z}/e\mathbb{Z}, M) = \{1\}$. Assuming this, there exists $s \in M$ such that $s^{-1}(w \circ \sigma)(s) = t^{-1}$. We then see that $(ps)^{-1}\sigma(ps) = s^{-1}w \circ \sigma(s)p^{-1}\sigma(p) = n$.

The exponential map gives an equivariant isomorphism $\mathfrak{t}(E)_{0+} \to M$. Thus, to show that $H^1(\mathbb{Z}/e\mathbb{Z}, M) = \{1\}$, it suffices to check that $H^1(\mathbb{Z}/e\mathbb{Z}, \mathfrak{t}(E)_{0+}) = \{0\}$. Recalling that a w-diagonalizer intertwines this action with the usual Galois action on $\mathfrak{s}(E)_{0+}$, we are reduced to showing that $H^1(\Gamma, \mathfrak{s}(E)_{0+}) = \{0\}$. If X is in the kernel of the norm map, there exists an element $Y' \in \mathfrak{s}(E)$ such that $\sigma(Y') - Y' = X$ by Hilbert's Theorem 90. Letting $Y \in \mathfrak{s}(E)_{0+}$ be the projection of Y obtained by killing graded terms in nonnegative degree, one has $\sigma(Y) - Y = X$ as desired. □

We define $\pi_{\mathfrak{s}}$ to be the orthogonal projection of $\hat{\mathfrak{g}}$ onto $\mathfrak{s}$ with respect to the F-bilinear invariant pairing $\langle , \rangle$ on $\hat{\mathfrak{g}}$ obtained by extending scalars. The invariance of the form makes it clear that $\pi_{\mathrm{Ad}(g)(\mathfrak{s})} = \mathrm{Ad}(g) \circ \pi_{\mathfrak{s}} \circ \mathrm{Ad}(g^{-1})$ for any $g \in \hat{G}$. We will analyze this map in detail in Section 4.

Theorem 3.7. *Suppose that $x \in \mathcal{A}_0$ is compatible with the Cartan subalgebra $\mathfrak{s}$. Then, there exists $q \in \hat{G}_x$ such that x is graded compatible with $q^{-1}\mathfrak{s}q$.*

Proof. Applying Lemma 3.6, choose $p \in G(E)_x$ such that $p^{-1}S(E)p = T(E)$ and $p^{-1}\sigma(p) = n \in N$. In particular, p is a w-diagonalizer of S for $w = nT \in W$. Furthermore, $\tau(p)p^{-1} \in \hat{\mathfrak{g}}_x$, since $\sigma(p) = pn$ and $\sigma(\tau(p)p^{-1}) = (\tau(p)n)(n^{-1}p^{-1}) = \tau(p)p^{-1}$. Finally,

$$\tau(p)p^{-1} + \tilde{x} \in \mathrm{Ad}(p)(\tilde{x}) + \hat{\mathfrak{g}}_{x+}. \tag{5}$$

This follows from Lemma 2.4(3) by setting $[\nabla]_\phi$ to be the zero form and applying the inner derivation τ.

If $\tau(p)p^{-1} + \tilde{x} \in \mathfrak{s}$, then x is graded compatible with $\mathfrak{s}$ by Lemma 3.4(3). Therefore, it is enough to show that there exists $q \in \hat{G}_{x+}$ (so that $qp \in G(E)_x$ will be a w-diagonalizer for qSq^{-1} with $(qp)^{-1}\sigma(qp) = n$) such that $\tau(qp)(qp)^{-1} + \tilde{x} \in \mathrm{Ad}(q)(\mathfrak{s}(E))$. Equivalently, if $q' = p^{-1}qp$,

$$q'^{-1}\tau(q') + \mathrm{Ad}(q'^{-1})(\mathrm{Ad}(p^{-1})\left[\tau(p)p^{-1} + \tilde{x}\right]) \in \mathfrak{t}(E).$$

If we set $X = \mathrm{Ad}(p^{-1})\left[\tau(p)p^{-1} + \tilde{x}\right] - \tilde{x}$, we see from (5) that $X \in \mathrm{Ad}(p)(\hat{\mathfrak{g}}_{x+}) \subset \mathfrak{g}(E)_{x+}$. Also, since $\tau(p)p^{-1} \in \hat{\mathfrak{g}}_x$, $X + \tilde{x} \in \mathrm{Ad}(p^{-1})(\hat{\mathfrak{g}}_x)$. We have thus reduced to the following general problem: Given $X \in \mathfrak{g}(E)_{x+}$ such that $X + \tilde{x} \in \mathrm{Ad}(p^{-1})(\hat{\mathfrak{g}}_x)$, find $q' \in p^{-1}\hat{G}_{x+}p$ such that $\mathrm{Ad}(q'^{-1})(X + \tilde{x}) + q'^{-1}\tau(q') \in \mathfrak{t}(E)$.

We will construct q' recursively. First, observe that since $\mathrm{Ad}(p^{-1})$ intertwines the Galois action with the twisted Galois action generated by $\sigma' = \mathrm{Ad}(n)\circ\sigma$, $X + \tilde{x}$ is fixed by σ'. This action preserves the x-filtration, so $\tilde{x}$ and X are individually fixed. Thus, $X + \tilde{x} \in \mathrm{Ad}(p^{-1})(\mathfrak{s}_0) + \mathrm{Ad}(p^{-1})(\hat{\mathfrak{g}}_{x,\ell_0})$ with $\ell_0 > 0$.

We now apply an inductive argument. Suppose that $Y \in \mathfrak{g}(E)_{x+}$ satisfies $Y + \tilde{x} \in \mathrm{Ad}(p^{-1})(\mathfrak{s}_0) + \mathrm{Ad}(p^{-1})(\hat{\mathfrak{g}}_{x,\ell})$ for $\ell > 0$ a critical number. Let $Y + \tilde{x} = \pi_{\mathfrak{t}}(Y + \tilde{x}) + \sum_{\psi\in\Phi} Y_{\psi,\ell}$ where $Y_{\psi,\ell} \in \mathfrak{u}_\psi \cap \mathfrak{g}(E)_{x,\ell}$. Since $\mathrm{Ad}(p^{-1})\pi_{\mathfrak{s}} = \pi_{\mathfrak{t}}\mathrm{Ad}(p^{-1})$, it follows that $\pi_{\mathfrak{t}}(Y + \tilde{x}) \in \mathrm{Ad}(p^{-1})(\mathfrak{s}_0)$ and hence $\sum_{\psi\in\Phi} Y_{\psi,\ell} \in \mathrm{Ad}(p^{-1})(\hat{\mathfrak{g}}_{x,\ell})$.

Write $q_{\psi,\ell} = \exp(-\frac{1}{\ell}Y_{\psi,\ell})$. Proposition 2.1 now gives

$$\begin{aligned}\mathrm{Ad}(q_{\psi,\ell}^{-1})(Y + \tilde{x}) + q_{\psi,\ell}^{-1}\tau(q_{\psi,\ell}) &\in -\left[\mathrm{ad}(\tilde{x})(\tfrac{1}{\ell}Y_{\psi,\ell}) + \tau(\tfrac{1}{\ell}Y_{\psi,\ell})\right] + Y + \tilde{x} + \mathfrak{g}(E)_{x,\ell+}\\ &= Y + \tilde{x} - Y_{\psi,\ell} + \mathfrak{g}(E)_{x,\ell+}.\end{aligned}$$

Set $q_\ell = \exp(-\sum_{\psi\in\Phi}\frac{1}{\ell}Y_{\psi,\ell}) \in \exp(\mathrm{Ad}(p^{-1})(\hat{\mathfrak{g}}_{x,\ell})) = p^{-1}\hat{G}_{x,\ell}p$. It is evident from the calculation above that $\mathrm{Ad}(q_\ell^{-1})(Y + \tilde{x}) + q_\ell^{-1}\tau(q_\ell) \in \mathrm{Ad}(p^{-1})(\mathfrak{s}_0) + \mathfrak{g}(E)_{x,\ell'}$ for some critical number $\ell' > \ell$.

In order to show that $Y' = \mathrm{Ad}(q_\ell^{-1})(Y + \tilde{x}) + q_\ell^{-1}\tau(q_\ell) - \tilde{x}$ satisfies the inductive hypothesis, first observe that $\mathrm{Ad}(q_\ell^{-1})(Y)$, $q_\ell^{-1}\tau(q_\ell)$, and $\mathrm{Ad}(q_\ell^{-1})(\tilde{x}) - \tilde{x}$ all lie in $\hat{\mathfrak{g}}(E)_{x+}$, so the same holds for Y'. Since we already know that $Y' + \tilde{x} \in \mathrm{Ad}(p^{-1})(\mathfrak{s}_0) + \mathrm{Ad}(p^{-1})(\mathfrak{g}(E)_{x,\ell'})$, to show that $Y' + \tilde{x} \in \mathrm{Ad}(p^{-1})(\hat{\mathfrak{g}}_{x,\ell'})$, it suffices to show that $\mathrm{Ad}(q_\ell^{-1})(Y + \tilde{x})$ and $q_\ell^{-1}\tau(q_\ell)$ both lie in $\mathrm{Ad}(p^{-1})(\hat{\mathfrak{g}}_x)$. It is clear that this holds for the first expression, since $q_\ell^{-1} \in p^{-1}\hat{G}_{x,\ell}p$. A direct calculation, using the same fact along with the observation that $\tau(p)p^{-1} = -p\tau(p^{-1}) \in \hat{\mathfrak{g}}_x$, proves the statement for $q_\ell^{-1}\tau(q_\ell)$.

We thus obtain an increasing sequence $\ell_i > 0$ of critical numbers for the x-filtration and $q_{l_i} \in \hat{G}_{x,\ell_i}$ for which $q' = \lim(q_{\ell_m}\cdots q_{\ell_1})$ satisfy the desired condition. ∎

3.2. **Compatible points.** The goal of this section is to describe the collection of points in $\mathcal{A}_0$ that are compatible with some Cartan subalgebra of type γ. We begin by showing that if $\mathfrak{s}$ is graded compatible with x, then the w-diagonalizer g can be chosen to have very special properties. In particular, this construction gives a well-behaved embedding of a Cartan subalgebra of type γ in $\hat{\mathfrak{g}}$. We then use this to classify $x \in \mathcal{A}_0$ which are graded compatible with $\mathfrak{s}$. Finally, we find all points in $\mathcal{A}_0$ which are compatible with some conjugate of $\mathfrak{s}$. We remark that in [12], Goresky, Kottwitz, and MacPherson construct a particular Cartan subalgebra $\mathfrak{s}$ of type γ and a point x in $\mathcal{A}_0$ which is graded compatible with $\mathfrak{s}$.

Lemma 3.8. *Suppose $x \in \mathcal{A}_0$ is graded compatible with $\mathfrak{s}$. Then $y \in \mathcal{A}_0$ is graded compatible with $\mathfrak{s}$ if and only if $\tilde{x} - \tilde{y} \in \mathfrak{s}(0)$.*

Proof. Assume that $x, y \in \mathcal{A}_0$ are graded compatible with $\mathfrak{s}$. If $s \in \mathfrak{s}(r)$, then $\operatorname{ad}(\tilde{x})(s) = \operatorname{ad}(\tilde{y})(s)$, i.e., $\operatorname{ad}(\tilde{x} - \tilde{y})(s) = 0$. We may construct a topological basis for $\mathfrak{s}$ consisting of elements of $\mathfrak{s}(r)$ for $r \in \frac{1}{e}\mathbb{Z}$. It follows that $\operatorname{ad}(\tilde{x} - \tilde{y})(\mathfrak{s}) = \{0\}$, so $\tilde{x} - \tilde{y} \in \mathfrak{s}$. Write $\tilde{x} - \tilde{y} = \sum_{r \gg -\infty} s_r$ with $s_r \in \mathfrak{s}(r)$. We obtain $\sum_{r \gg -\infty} r s_r = (\tau + \operatorname{ad}(\tilde{x}))(\tilde{x} - \tilde{y}) = \operatorname{ad}(\tilde{x})(\tilde{x} - \tilde{y}) = [\tilde{x}, \tilde{y}]$. Similarly, $\sum_{r \gg -\infty} r s_r = (\tau + \operatorname{ad}(\tilde{y}))(\tilde{x} - \tilde{y}) = \operatorname{ad}(\tilde{y})(\tilde{x} - \tilde{y}) = [\tilde{y}, \tilde{x}]$. This implies that $s_r = 0$ unless $r = 0$, so $\tilde{x} - \tilde{y} \in \mathfrak{s}(0)$.

The converse is immediate from Lemma 3.4. □

Lemma 3.9. *The point $x \in \mathcal{A}_0$ is graded compatible with $\mathfrak{s}$ if and only if $\mathfrak{s}(0) \subset \mathfrak{t}$ and there exists a splitting field E for $\mathfrak{s}$ and a w-diagonalizer $g \in G(E)$ of S such that $-\tau(g)g^{-1} \in (\tilde{x} + \mathfrak{s}(0)) \cap \mathfrak{t}_{\mathbb{Q}}$.*

Proof. Let $f \in G(E')$ be a w-diagonalizer of S. If E is a finite extension of E', it is obvious that any element of the left coset $fT(E)$ is also a w-diagonalizer. We will construct a finite extension E of E' and $g \in fT(E)$ that satisfies the property in the statement.

We retain the notation of the proof of Lemma 3.4 with f replacing g, so $f^{-1} = put$ and $Y = \operatorname{Ad}(f^{-1})(\tilde{x} + \tau(f)f^{-1}) \in \mathfrak{t}(\mathfrak{o}_{E'})$. Write $Y = t_0 + t'$, with $t_0 \in \mathfrak{t}(E')(0) = \mathfrak{t}$ and $t' \in \mathfrak{t}(E')_{0+}$. Replacing f with $fe^{v'}$, where $v' \in \mathfrak{t}(E')_{0+}$ satisfies $\tau(v') = -t'$, we may assume that $Y = t_0 \in \mathfrak{t}$.

We now construct $v \in \mathfrak{t}_{\mathbb{Q}}$ such that $g = fz^v$ satisfies $\tilde{x} + \tau(g)g^{-1} \in \mathfrak{s}(0)$. If $ev \in \mathfrak{t}_{\mathbb{Z}}$ and E is an extension of E' with $[E : F] = e$, then $z^v \in T(E)$. Write $\tilde{w} = f^{-1}\sigma(f) \in N(E')$. Note that we can decompose $\tilde{w} = t_1\tilde{w}_2$, where $\tilde{w}_2 \in N$ and $t_1 \in T(E')$. It follows that $\tau(\tilde{w})\tilde{w}^{-1} = \tau(t_1)t_1^{-1} \in \mathfrak{t}_{\mathbb{Q}} + \mathfrak{t}(E')_{0+}$. Moreover, using the facts that σ commutes with τ and fixes $\mathfrak{t}$,

$$\begin{aligned}
\tau(\tilde{w})\tilde{w}^{-1} &= \operatorname{Ad}(f^{-1})\left(-\tau(f)f^{-1}) + \sigma\left(\tau(f)f^{-1}\right)\right) \\
&= \operatorname{Ad}(f^{-1})\left(-\left[\tilde{x} + \tau(f)f^{-1}\right] + \sigma\left[\tilde{x} + \tau(f)f^{-1}\right]\right) \\
&= -Y + w \circ \sigma(Y) = -Y + w \cdot Y \in \mathfrak{t}.
\end{aligned}$$

We deduce that $\tau(\tilde{w})\tilde{w}^{-1} \in \mathfrak{t}_{\mathbb{Q}}$. Choose a $\mathbb{Q}$-linear projection $\mathfrak{t} \to \mathfrak{t}_{\mathbb{Q}}$, and let $v \in \mathfrak{t}_{\mathbb{Q}}$ be the image of Y. Then, $\tau(\tilde{w})\tilde{w}^{-1} = v - w \cdot v$.

Set $g = fz^v$, so that $\tau(g)g^{-1} = \tau(f)f^{-1} + \operatorname{Ad}(f)(v)$. We obtain

$$\begin{aligned}
\sigma\left[\tau(g)g^{-1}\right] &= \sigma(\tau(f)f^{-1}) + \sigma(\operatorname{Ad}(f)(v)) \\
&= (\tau(f)f^{-1} + \operatorname{Ad}(f)(\tau(\tilde{w})\tilde{w}^{-1}) + \operatorname{Ad}(f)(w \cdot v) \\
&= \tau(f)f^{-1} + \operatorname{Ad}(f)((t - w \cdot v) + w \cdot v) = \tau(g)g^{-1}.
\end{aligned}$$

It follows that $\tilde{x} + \tau(g)g^{-1} \in (\mathfrak{s}(E)(0))^{\Gamma} = \mathfrak{s}(0)$.

It remains to show that $\tau(g)g^{-1} \in \mathfrak{t}_{\mathbb{Q}}$. First, note that Y is conjugate to $\tilde{x} + \tau(g)g^{-1} \subset \mathfrak{t}+\mathfrak{s}(0) \subset \mathfrak{t}$ by hypothesis. Next, recall from the proof of Lemma 3.4 that $Y \in \mathrm{Ad}((p_1p_2)^{-1})(\tilde{x} + q + n + z^{1/e}\mathfrak{g}(\mathfrak{o}_E))$ for some $q \in \mathfrak{t}_{\mathbb{Q}}$ and $n \in \mathfrak{u}$, where $p_1 \in G$ and $p_2 \in G(E)_{o,0+}$. Since the projection $\mathfrak{g}(\mathfrak{o}_E) \to \mathfrak{g}$ restricts to the identity on $\mathfrak{g}$ and $Y \in \mathfrak{t}$, applying the projection shows that Y is conjugate in G to $\tilde{x}+q+n \in \mathfrak{b}$. The latter element is semisimple, so it is B-conjugate to an element of $\mathfrak{t}$, which must clearly be $\tilde{x} + q$. It follows that $\tilde{x} + \tau(g)g^{-1}$ and $\tilde{x} + q$ are $G(\mathfrak{o}_E)$-conjugate, so the same is true for q and $\tau(g)g^{-1}$. However, two elements of $\mathfrak{t}$ are conjugate only if they are in the same W-orbit, and $W \cdot \mathfrak{t}_{\mathbb{Q}} = \mathfrak{t}_{\mathbb{Q}}$. Thus, $\tau(g)g^{-1} \in \mathfrak{t}_{\mathbb{Q}}$.

We now prove the converse. Since $\tilde{x}+\tau(g)g^{-1} \in \mathfrak{s}$, $\mathrm{Ad}(g^{-1})(\tilde{x}+\tau(g)g^{-1}) \in \mathfrak{t}(E)$, and the result follows from Lemma 3.4. □

Lemma 3.10. *Suppose that $n \in \hat{N}$ determines the element $w \in \hat{W}$. For all $x \in \mathcal{A}_0$, $u = \mathrm{Ad}(n)(\tilde{x}) - \tau(n)n^{-1} \in \widetilde{wx} + \hat{\mathfrak{t}}_{0+}$. Moreover, if $u \in \mathfrak{t}_{\mathbb{R}}$ and V is a finite-dimensional representation of G, then $n\hat{V}_x(r) = \hat{V}_{wx}(r)$.*

Proof. The fact that $u \in \widetilde{wx} + \hat{\mathfrak{t}}_{0+}$ is proved in [5, Lemma 2.3]. Now, assume that $u \in \mathfrak{t}_{\mathbb{R}}$ (so $u = \widetilde{wx}$), and take $X \in V_x(r)$. Applying Lemma 2.1 gives

$$\begin{aligned}\tau(nX) + u(nX) &= \tau(n)n^{-1}(nX) + n(\tau(X)) + u(nX)\\ &= n\tau(X) + (\mathrm{Ad}(n)\tilde{x})(nX) = n((\tau + \tilde{x})(X))\\ &= r(nX).\end{aligned}$$

Therefore, $n\hat{V}_x(r) \subset \hat{V}_{wx}(r)$. A similar argument shows the reverse inclusion. □

We now show that every conjugacy class of Cartan subalgebra in $\hat{\mathfrak{g}}$ has a representative $\mathfrak{s}$ that is graded compatible with a point $x \in \mathcal{A}_0$. Given $w \in W$, set $\mathfrak{t}^w := \{X \in \mathfrak{t} \mid wX = X\}$.

Theorem 3.11. *There exists a maximal torus S of type γ and a representative $w \in W$ such that $\mathfrak{s}$ is graded compatible with some $x \in \mathcal{A}_0$ and satisfies $\mathfrak{s}(0) = \mathfrak{t}^w$.*

Proof. Let $n' \in N$ be a finite order coset representative for $w' \in \gamma$; this exists by a theorem of Tits [19]. The element n' is semisimple and commutes with $T^{w'}$. It follows that n' and $T^{w'}$ are contained in a maximal torus T'. Choose $h \in G$ such that $T = hT'h^{-1}$. Since n' has finite order, there exists $t \in \mathfrak{t}(\mathbb{Q})$ such that $\exp(-2\pi i t) = \mathrm{Ad}(h)(n')$. Write $g = z^{-t}h$. It is clear that $g^{-1}\sigma(g) = n'$, so g is a w'-diagonalizer of a maximal torus $S \subset \hat{G}$ with type γ.

Let x be the image of t in $\mathcal{A}_0$; equivalently, $\tilde{x} = \widetilde{\tau(g)g^{-1}}$. Therefore, $\mathrm{Ad}(g^{-1})(\tilde{x}+\tau(g)g^{-1}) \in \mathrm{Ad}(g^{-1})(\mathfrak{z}) \subset \mathfrak{t}$. By Lemma 3.4, x is graded compatible with $\mathfrak{s}$.

It will be shown in Theorem 3.13 below that $\mathfrak{s}(0) = \mathrm{Ad}(h)(\mathfrak{t}^{w'})$. A standard argument now shows that there exists $n_0 \in N$ such that $\mathfrak{s}(0) = \mathrm{Ad}(n_0)(\mathfrak{t}^{w'})$. Indeed, $\mathfrak{t}$ and $\mathrm{Ad}(h)(\mathfrak{t})$ are two Cartan subalgebras in $Z(\mathfrak{s}(0))$, so there exists $k \in Z(\mathfrak{s}(0))$ such that $\mathrm{Ad}(kh)(\mathfrak{t}) = \mathfrak{t}$. One may then take $n_0 = kh$. Setting $w = w_0w'w_0^{-1} \in \gamma$ with $w_0 = n_0T \in W$, we have $\mathfrak{s}(0) = \mathfrak{t}^w$ as desired. □

Remark 3.12. The existence of a Cartan subalgebra $\mathfrak{s}$ of type γ which is graded compatible with some $x \in \mathcal{A}_0$ may also be derived from the construction in [12, Section 5.3].

Theorem 3.13. *Suppose that $\mathfrak{s}$ is a Cartan subalgebra of type γ graded compatible with $x \in \mathcal{A}_0$. Choose a class representative $w \in W$ for γ. Then, there exists a finite extension E of F and $g \in G(E)$ satisfying the following properties:*

(1) *g is a w-diagonalizer of $\mathfrak{s}$;*
(2) *$g^{-1}\sigma(g) \in N$ is a finite order coset representative for $w \in N/T$;*
(3) *$-\tau(g)g^{-1} \in (\tilde{x} + \mathfrak{s}(0)) \cap \mathfrak{t}_{\mathbb{Q}}$; and*
(4) *$g = z^{-t}h$, where $t = -\tau(g)g^{-1} \in \mathfrak{t}_{\mathbb{Q}}$, the image y of t in $\mathcal{A}_0$ is a point compatible with $\mathfrak{s}$ (indeed, y is graded compatible with $\mathfrak{s}$), $h \in G$ simultaneously diagonalizes $g^{-1}\sigma(g)$ and $\mathfrak{t}^w$ into $\mathfrak{t}$, and $\mathfrak{s}(0) = \mathrm{Ad}(h)(\mathfrak{t}^w) \subset \mathfrak{t}$.*

As explained in Section 2, the element z^{-t} depends on a choice of uniformizer for E. However, for any such choice $h = z^t g$ will satisfy the properties in part (4).

Remark 3.14. One may think of the element g described above as a generalized Vandermonde matrix. Consider the case where $G = \mathrm{GL}_n$ and γ is the orbit of the Coxeter element. Let ξ be a primitive n^{th} root of unity and choose $u \in E$ to be an n^{th} root of z. Then, it is possible to choose g to be a conjugate to the matrix with entries $\left(\xi^{(i-1)j}u^{i-1}\right)$. Here, t is the diagonal matrix with entries $(0, 1/n, \dots, (n-1)/n)$ corresponding to the barycenter of the fundamental chamber in $\mathcal{A}_0$. An explicit embedding of the Coxeter torus satisfying the properties in the theorem is given by setting $S = F[\varpi]^\times$, where $\varpi = ze_{n1} + \sum_{i=1}^{n-1} e_{i(i+1)}$. More generally, one obtains a good embedding of any maximal torus by taking a block-diagonal embedding of Coxeter tori.

Proof. We note that Lemma 3.9 implies that there exists $g \in G(E)$ satisfying parts (1) and (3) and such that $g^{-1}\sigma(g) \in N(E)$ is a coset representative for $w \in N/T$. Assuming $g = z^{-t}h$ as in part (4), $g^{-1}\sigma(g) = \mathrm{Ad}(h^{-1})e^{-2\pi i t} \in N$ has finite order and is diagonalized by h.

It remains to prove the rest of part (4). Let $t = -(dg)g^{-1} \in \mathfrak{t}(\mathbb{Q})$. We first prove that $g = z^{-t}h$ for some $h \in G$. Indeed,

$$\tau(z^t g)(g^{-1}z^{-t}) = \mathrm{Ad}(z^t)\left(t + \tau(g)g^{-1}\right) = 0.$$

It follows that $h = z^t g \in G$. Since $t = -\tau(g)g^{-1} \in \tilde{x} + \mathfrak{s}(0)$, Lemma 3.8 implies that y is graded compatible with $\mathfrak{s}$.

Finally, suppose that $X \in \mathfrak{t}$. Computing, we get $(\tau + \mathrm{ad}(\tilde{y}))(\mathrm{Ad}(g)(X)) = (\tau + \mathrm{ad}(t))(\mathrm{Ad}(g)(X)) = 0$, so $\mathrm{Ad}(g)(X) \in \mathfrak{s}(E)(0) \subset \mathfrak{t}(E)(0) = \mathfrak{t}$. In particular, since $z^t \in T(E)$, $\mathrm{Ad}(h)(X) = \mathrm{Ad}(g)(X)$. Moreover, $\sigma(\mathrm{Ad}(g)(X)) = \mathrm{Ad}(g)(w \circ \sigma(X)) = \mathrm{Ad}(g)(wX)$, so $\mathrm{Ad}(g)(X) \in \mathfrak{s}(0) = (\mathfrak{s}(E)(0))^\Gamma$ if and only if $X \in \mathfrak{t}^w$. It follows that $\mathfrak{s}(0) = \mathrm{Ad}(g)(\mathfrak{t}^w) = \mathrm{Ad}(h)(\mathfrak{t}^w)$. □

We can now identify the set of points in $\mathcal{A}_0$ which are compatible with some maximal torus of type γ. We denote this set by Π_γ. Note that this set is nonempty by Theorem 3.11. A similar problem for p-adic classical groups is considered in [7].

Corollary 3.15. *Let γ denote a conjugacy class in W, and choose $x \in \mathcal{A}_0$ and $\mathfrak{s}$ of type γ such that x is graded compatible with $\mathfrak{s}$ and $\tilde{x} \in \mathfrak{t}_{\mathbb{Q}}$. Then,* $\Pi_\gamma = \{y \in \mathcal{A}_0 \mid \tilde{y} \in \hat{W}(\tilde{x} + \mathfrak{s}(0))\}$.

Theorem 3.7 shows the existence of x and $\mathfrak{s}$ as in the statement of the theorem. Note that this guarantees that $\tilde{x} \in \mathfrak{t}$ for any base field k.

Remark 3.16. Theorem 3.7 shows that Π_γ can also be characterized as the set of $y \in \mathcal{A}_0$ graded compatible with some conjugate of $\mathfrak{s}$.

Proof. Applying Theorem 3.7, assume without loss of generality that x is graded compatible with S. Lemma 3.8 then implies that any point y with $\tilde{y} \in \tilde{x} + \mathfrak{s}(0)$ is (graded) compatible with S. Now, suppose that $v \in \hat{W}$. We may choose a representative $m = z^t n$ for v, where $t \in \mathfrak{t}_{\mathbb{Z}} \cong X_*(T)$ and $n \in N$. Observe that $\tau(m)m^{-1} \in \mathfrak{t}_{\mathbb{Q}}$, so Lemma 3.10 implies that $\mathrm{Ad}(m)(\mathfrak{g}_x(i)) = \mathfrak{g}_{vx}(i)$. We deduce that vx is graded compatible with $\mathrm{Ad}(m)(S)$. This shows that $\hat{W}(\tilde{x} + \mathfrak{s}(0)) \subset \Pi_\gamma$.

Now, choose $x' \in \Pi_\gamma$, and let $\mathfrak{s}'$ be a Cartan subalgebra of type γ graded compatible with x'. Choose a representative $w \in W$ for γ. By Theorem 3.13(4), there exists a w-diagonalizer $g = z^{-t}h$ (resp. $g' = z^{-t'}h'$) for S (resp. S') such that h simultaneously diagonalizes $g^{-1}\sigma(g)$ and $\mathfrak{t}^w$ (resp. h', $(g')^{-1}\sigma(g')$). We shall write $\zeta = z^t\sigma(z^{-t}) = e^{-2\pi i t} \in T$ and $\zeta' = z^{t'}\sigma(z^{-t'})$. We also set $n = h^{-1}\zeta h = g^{-1}\sigma(g) \in N$ and similarly for n'. They are both representatives for w, so we can take $\tilde{\zeta} \in T$ such that $n = n'\tilde{\zeta}$.

Let $S^\flat \subset T$ be the subgroup generated by rational cocharacters of S. One may deduce from Theorem 3.13(4) that $hT^wh^{-1} = S^\flat$. We now show that $\zeta \in (W \cdot \zeta')S^\flat$. Applying Lemma 3.17, write $\tilde{\zeta} = \delta_w(\zeta_1)\zeta_2$, with $\zeta_2 \in T^w$. In particular, $n = (\zeta_1 n' \zeta_1^{-1})\zeta_2$. Moreover,

$$\begin{aligned}(h\zeta_1)n'(h\zeta_1)^{-1} &= (h\zeta_1)n'\zeta_2(\zeta_2)^{-1}(h\zeta_1)^{-1}\\ &= (h\zeta_1)n'\zeta_2(h\zeta_1)^{-1}h\zeta_2^{-1}h^{-1} \in \zeta S^\flat.\end{aligned}$$

The last line follows from the observation above that $s = h\zeta_2^{-1}h^{-1} \in hT^wh^{-1} = S^\flat$. Thus, $q = h\zeta_1 h'^{-1}$ conjugates ζ' to ζs. We further note that $qS'^\flat q^{-1} = h\zeta_1 T^w \zeta_1^{-1}h^{-1} = hT^wh^{-1} = S^\flat$. This implies that $q\zeta' S'^\flat q^{-1} = \zeta S^\flat$. The connected centralizer $Z(\zeta' S'^\flat)^0$ is reductive. Moreover, it contains the maximal tori T and $q^{-1}Tq$, so there exists $p \in Z(\zeta' S'^\flat)^0$ such that $pq^{-1}Tqp^{-1} = T$. It follows that $qp^{-1} \in N$ and that conjugation by qp^{-1} and q coincide on $\zeta' S'^\flat$. If we let $u \in W$ be the image of q, we see that $u(\zeta') = \zeta s$ and $u(S'^\flat) = S^\flat$.

Finally, it is evident that $s = e^{2\pi i(u(t')-t)}$ and therefore $t \in u(t') + X_*(T) + \mathfrak{s}(0)$. Since $\tilde{x} \in t + \mathfrak{s}(0)$, $\tilde{x}' \in t' + \mathfrak{s}'(0)$, and $u(\mathfrak{s}'(0)) = \mathfrak{s}(0)$, we conclude that $\tilde{x}'$ lies in $\hat{W}(\tilde{x} + \mathfrak{s}(0))$. ☐

Lemma 3.17. *Given $w \in W$, let $\delta_w : T \to T$ be the homomorphism $\delta_w(t) = w^{-1}(t)t^{-1}$. Then, T has the direct product decomposition $T = \delta_w(T)T^w$.*

Proof. This follows immediately from the obvious fact that $\mathrm{Lie}(\delta_w(T)) = \{w^{-1}(X) - X \mid X \in \mathfrak{t}\}$ and $\mathfrak{t}^w$ are complementary subspaces of $\mathfrak{t}$. ☐

4. Regular Strata

In this section, we introduce a class of strata with regular semisimple "leading term". We will first do so only for strata coming from the standard apartment.

If $\tilde{\beta} \in \hat{\mathfrak{g}}^\vee$, we let $Z(\tilde{\beta})$ (resp. $Z^0(\tilde{\beta})$) denote the stabilizer (res. connected stabilizer) of $\tilde{\beta}$ in $\hat{G}$ under the coadjoint action. The corresponding Lie algebra $\mathfrak{z}(\tilde{\beta})$ is the stabilizer of the Lie algebra action.

Definition 4.1. A stratum (x, r, β) with $x \in \mathcal{A}_0$ is *graded regular* if $Z^0(\tilde{\beta}_0)$ is a maximal torus which is compatible with x. We call $Z^0(\tilde{\beta}_0)$ the connected centralizer of the stratum.

It is of course equivalent to check that the Lie centralizer $\mathfrak{z}(\tilde{\beta}_0)$ of the stratum is a Cartan subalgebra compatible with x. It is obvious that a graded regular stratum is fundamental.

Given a torus S, we write $\rho_{\mathfrak{s}} : \hat{\mathfrak{g}}^\vee \to \mathfrak{s}^\vee$ for the restriction map. If $\mathfrak{s}$ is compatible with $x \in \mathcal{B}$, $\rho_{\mathfrak{s}}(\hat{\mathfrak{g}}^\vee_{x,r})) \subset \mathfrak{s}^\vee_r$ and $\rho_{\mathfrak{s}}(\hat{\mathfrak{g}}^\vee_{x,r+})) \subset \mathfrak{s}^\vee_{r+}$ for all r. Moreover, if $\mathfrak{s}$ is graded compatible with $x \in \mathcal{A}_0$, then $\rho_{\mathfrak{s}}(\hat{\mathfrak{g}}^\vee_x(r)) \subset \mathfrak{s}^\vee(r)$.

Lemma 4.2 (Tame Corestriction)**.** *Take $x \in \mathcal{A}_0$, and let (x, r, β) be a graded regular stratum with connected centralizer S. Then, there is a morphism of $\mathfrak{s}$-modules $\pi_{\mathfrak{s}} : \hat{\mathfrak{g}} \to \mathfrak{s}$ satisfying the following properties:*

1. *$\pi_{\mathfrak{s}}$ restricts to the identity on $\mathfrak{s}$;*
2. *$\pi_{\mathfrak{s}}(\hat{\mathfrak{g}}_{x,\ell}) = \mathfrak{s}_\ell$ and $\pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_\ell) \subset \hat{\mathfrak{g}}^\vee_{x,\ell}$;*
3. *the kernel of the restriction map*
$$\bar{\rho}_{\mathfrak{s},\ell} : (\pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee) + \hat{\mathfrak{g}}^\vee_{x,\ell-r})/\hat{\mathfrak{g}}^\vee_{x,(\ell-r)+} \to \mathfrak{s}^\vee/\mathfrak{s}^\vee_{(\ell-r)+}$$
is given by the image of $\mathrm{ad}^*(\hat{\mathfrak{g}}_{x,\ell})(\tilde{\beta})$ modulo $\hat{\mathfrak{g}}^\vee_{x,(\ell-r)+}$, where $\tilde{\beta} \in \hat{\mathfrak{g}}^\vee_{x,-r}$ is any representative of β;
4. *if $Z \in \mathfrak{s}$ and $X \in \hat{\mathfrak{g}}$, then $\langle Z, X\rangle_{\frac{dz}{z}} = \langle Z, \pi_{\mathfrak{s}}(X)\rangle_{\frac{dz}{z}}$;*
5. *$\pi_{\mathfrak{s}}$ (resp. $\pi^*_{\mathfrak{s}}$) commutes with the adjoint action of the normalizer $N(S)$ of S; and*
6. *the image $\pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_\ell)$ consists of those elements in $\hat{\mathfrak{g}}^\vee_{x,\ell}$ stabilized by S.*

Remark 4.3. The proof will actually show that $\mathrm{ad}^*(\hat{\mathfrak{g}}_{x,\ell})(X)$ modulo $\hat{\mathfrak{g}}^\vee_{x,(\ell-r)+}$ is in the kernel of $\bar{\rho}_{\mathfrak{s},\ell}$ for any $X \in \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_{-r}) + \hat{\mathfrak{g}}^\vee_{x,-r+}$.

Remark 4.4. We will usually omit the subscript ℓ on $\bar{\rho}_{\mathfrak{s},\ell}$ when it is clear from context.

Proof. Recall that $\pi_{\mathfrak{s}}$ is the orthogonal projection of $\hat{\mathfrak{g}}$ onto $\mathfrak{s}$ with respect to the F-bilinear invariant pairing $\langle,\rangle$ on $\hat{\mathfrak{g}}$ obtained by extending scalars. The invariance of the form immediately shows that $\pi_{\mathfrak{s}}$ is an $\mathfrak{s}$-module map that is equivariant with respect to the adjoint action of $N(S)$. The first and fourth statements are trivial.

Recall that F-duals and smooth k-duals of F-vector spaces are identified using the map κ described before Proposition 2.2. We will use this identification and the results of this proposition in the rest of the proof without comment.

Since x is compatible with $\mathfrak{s}$, we have $\mathfrak{s}_\ell \subset \hat{\mathfrak{g}}_{x,\ell}$, so $\pi_{\mathfrak{s}}(\hat{\mathfrak{g}}_{x,\ell}) \supset \mathfrak{s}_\ell$. Now, suppose that $Z \in \mathfrak{s}_{-\ell+}$ and $X \in \hat{\mathfrak{g}}_{x,\ell}$. In order to show that $\pi_{\mathfrak{s}}(X) \in \mathfrak{s}_\ell$, it suffices to show that $\langle X, Z\rangle_{\frac{dz}{z}} = 0$. This follows immediately, since $Z \in \hat{\mathfrak{g}}_{x,-\ell+} = \hat{\mathfrak{g}}^\perp_{x,\ell}$. Also, if $\alpha \in \mathfrak{s}^\vee_\ell$, then $\pi^*_{\mathfrak{s}}(\alpha)(\hat{\mathfrak{g}}_{x,\ell+}) = \alpha(\pi(\hat{\mathfrak{g}}_{x,\ell+})) = \alpha(\mathfrak{s}^\vee_{\ell+}) = 0$, so $\pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_\ell) \subset \hat{\mathfrak{g}}^\vee_{x,\ell}$. This proves the second statement.

Suppose that $\alpha \in \mathfrak{s}^\vee$ and $s \in S$. Then, if $X \in \hat{\mathfrak{g}}$, $\mathrm{Ad}^*(s)(\pi^*_{\mathfrak{s}}(\alpha))(X) = \pi^*_{\mathfrak{s}}(\alpha)(\mathrm{Ad}(s^{-1})(X)) = \alpha(\pi_{\mathfrak{s}}(\mathrm{Ad}(s^{-1})(X)))$. The right hand side is equal to $\pi^*_{\mathfrak{s}}(\alpha)(X)$ by part (5). Thus, $\pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee)$ is stabilized by S. On the other hand, if $\alpha \in \hat{\mathfrak{g}}^\vee$ is stabilized by S, then, writing $\alpha = \langle Y, \cdot\rangle$, one sees that $\mathrm{Ad}(s)(Y) = Y$ for all $s \in S$, i.e., $Y \in \mathfrak{s}$. By part (4), $\alpha(X) = \alpha(\pi_{\mathfrak{s}}(X))$, so $\alpha \in \pi^*_{\mathfrak{s}}(\mathfrak{s})$. This proves (6).

Finally, we consider the third statement. The image of $\mathrm{ad}^*(\hat{\mathfrak{g}}_{x,j})(\tilde{\beta})$ modulo $\hat{\mathfrak{g}}^\vee_{x,(j-r)+}$ is independent of the choice of representative, so we can take $\tilde{\beta} = \tilde{\beta}_0 \in \hat{\mathfrak{g}}^\vee_x(-r)$. Since S stabilizes $\tilde{\beta}_0$, $\mathrm{ad}^*(Z)(\tilde{\beta}_0) = 0$ for $Z \in \mathfrak{s}$, and it follows that for any $X \in \hat{\mathfrak{g}}$, $\mathrm{ad}^*(X)(\tilde{\beta}_0)(Z) = -\mathrm{ad}^*(Z)(\tilde{\beta}_0)(X) = 0$, i.e., $\rho_{\mathfrak{s}}(\mathrm{ad}^*(\hat{\mathfrak{g}})(\tilde{\beta}_0)) = 0$. Therefore,

$(\mathrm{ad}^*(\hat{\mathfrak{g}}_{x,j})(\tilde{\beta}_0) + \hat{\mathfrak{g}}^\vee_{x,(j-r)+})/\hat{\mathfrak{g}}^\vee_{x,(j-r)+}) \subset \ker(\bar{\rho}_{\mathfrak{s},j})$. For convenience, we denote the former space by C_j.

We will show that $\dim C_j = \dim\ker(\bar{\rho}_{\mathfrak{s},j})$. Observe that $\rho_{\mathfrak{s}} \circ \pi^*_{\mathfrak{s}}$ is the identity on $\mathfrak{s}^\vee$, so $\ker(\bar{\rho}_{\mathfrak{s},j}) = \ker(\hat{\mathfrak{g}}^\vee_{x,j-r}/\hat{\mathfrak{g}}^\vee_{x,(j-r)+} \to \mathfrak{s}^\vee_{j-r}/\mathfrak{s}^\vee_{(j-r)+})$. This second map is surjective, so $\dim\ker(\bar{\rho}_{\mathfrak{s},j}) = \dim\hat{\mathfrak{g}}^\vee_x(j-r) - \dim\mathfrak{s}^\vee(j-r)$. Next, suppose that $X \in \hat{\mathfrak{g}}_{x,j}$ and $\mathrm{ad}^*(X)(\tilde{\beta}_0) \in \hat{\mathfrak{g}}_{x,(j-r)+}$. Since this is always the case if $X \in \hat{\mathfrak{g}}_{x,j+}$, we can assume that $X \in \hat{\mathfrak{g}}_x(j)$. By Proposition 2.1, $\mathrm{ad}^*(X)(\tilde{\beta}_0) \in \hat{\mathfrak{g}}^\vee_x(j-r) \cap \hat{\mathfrak{g}}^\vee_{x,(j-r)+} = \{0\}$, so $X \in \mathfrak{s} \cap \hat{\mathfrak{g}}_x(j) = \mathfrak{s}(j)$. We thus obtain $\dim(C_j) = \dim\hat{\mathfrak{g}}_x(j) - \dim\mathfrak{s}(j)$.

We now sum these two dimensions over j in the full period $\ell \leq j < \ell+1$. Note that summing j over the full period $\ell \leq j < \ell+1$ gives $\sum_{\ell\leq j<\ell+1}\dim\hat{\mathfrak{g}}_x(j) = \dim\mathfrak{g} = \dim\mathfrak{g}^\vee = \sum_{\ell\leq j<\ell+1}\dim\hat{\mathfrak{g}}^\vee_x(j-r)$ and similarly, $\sum_{\ell\leq j<\ell+1}\dim\mathfrak{s}_x(j) = \sum_{\ell\leq j<\ell+1}\dim\mathfrak{s}^\vee_x(j)$. (Of course, both sides are zero if $\hat{\mathfrak{g}}^\vee_x(j-r) = \{0\}$.) Thus, $\sum_{\ell\leq j<\ell+1}\dim C_j = \sum_{\ell\leq j<\ell+1}\dim\ker(\bar{\rho}_{\mathfrak{s},j})$. Since each term on the right is greater or equal to the corresponding term on the left, we get $\dim C_j = \dim\ker(\bar{\rho}_{\mathfrak{s},j})$ for all j. In particular, $C_\ell = \ker(\bar{\rho}_{\mathfrak{s},\ell})$. □

Proposition 4.5. *If $(x,0,\beta)$ is a graded regular stratum with $x \in \mathcal{A}_0$, then there exists $m \in \hat{G}_x$ such that $m\cdot(x,0,\beta) = (x,0,\beta')$ is regular with connected centralizer $\hat{T}$. The stratum $(x,0,\beta')$ is uniquely determined by $(x,0,\beta)$ up to the action of $(\hat{N}\cap\hat{G}_x)/\hat{T}_x \cong (N\cap H_x)/T$.*

Proof. Write $S = Z^0(\tilde{\beta}_0)$, so $\mathfrak{t}' = (d\theta_x)^{-1}(\mathfrak{s}_0/\mathfrak{s}_{0+})$ is a Cartan subalgebra in $\mathfrak{h}_x$. Choose $h \in H_x$ such that $\mathrm{Ad}(h)\mathfrak{t}' = \mathfrak{t}$, and let $m \in \hat{G}_x$ be a lift of $\theta_x(h)$. Define a new stratum $(x,0,\beta') = m\cdot(x,0,\beta)$ which has representative $\tilde{\beta}'_0 \in (\mathrm{Ad}^*(m)(\tilde{\beta}_0) + \hat{\mathfrak{g}}^\vee_{x+}) \cap \hat{\mathfrak{g}}^\vee_x(0)$. It suffices to show that $Z^0(\tilde{\beta}'_0) = \hat{T}$.

Let $Z_0(\tilde{\beta}'_0) = Z^0(\tilde{\beta}'_0)\cap\hat{G}_x$ and $Z_+(\tilde{\beta}'_0) = Z^0(\tilde{\beta}'_0)\cap\hat{G}_{x+}$. Observe that $\theta_x^{-1}(Z_0(\tilde{\beta}'_0)/Z_+(\tilde{\beta}'_0)) = T$, so by the T-equivariance of θ_x, if $t \in T \subset \hat{G}_x$, $\mathrm{Ad}^*(t)(\tilde{\beta}'_0) - \tilde{\beta}'_0 \in \hat{\mathfrak{g}}^\vee_{x+}\cap\hat{\mathfrak{g}}^\vee_x(0) = \{0\}$. We deduce that $\hat{T} \subset Z^0(\tilde{\beta}'_0)$.

On the other hand, the T-equivariance of θ_x also implies that $Z_0(\tilde{\beta}'_0) \subset T\hat{G}_{x,+}$. Suppose $g \in Z_0(\tilde{\beta}'_0)\cap\hat{G}_{x,\ell}$ for $\ell > 0$. Write $g \in \exp(X)\hat{G}_{x,\ell+}$ for some $X \in \hat{\mathfrak{g}}_x(\ell)$. Then, $\tilde{\beta}'_0 = \mathrm{Ad}^*(g)(\tilde{\beta}'_0) \in \tilde{\beta}'_0 + \mathrm{ad}^*(X)(\tilde{\beta}'_0) + \hat{\mathfrak{g}}_{x,\ell+}$. It follows that $\mathrm{ad}^*(X)(\tilde{\beta}'_0) = 0$. Therefore, $\mathrm{ad}^*(\mathrm{Ad}(m^{-1})(X))(\tilde{\beta}_0) \in \hat{\mathfrak{g}}^\vee_{x,\ell+}$. Finally, this implies that $\mathrm{Ad}(m^{-1})(X) \in \mathfrak{s}_\ell + \hat{\mathfrak{g}}_{x,\ell+}$ and $X \in \hat{\mathfrak{t}}_\ell + \hat{\mathfrak{g}}_{x,\ell+}$. Thus, $Z_0(\tilde{\beta}'_0)\cap\hat{G}_{x,\ell} \subset \hat{T}_\ell\hat{G}_{x,\ell+}$. A standard limit argument shows that $Z_0(\tilde{\beta}'_0) \subset \hat{T}_0$ and thus $Z^0(\tilde{\beta}'_0) \subset \hat{T}$.

Now suppose that $(x,0,\beta)$ is a graded regular stratum with connected centralizer by $\hat{T}$, and there exists $m \in \hat{G}_x$ such that the same holds for $m\cdot(x,0,\beta)$. It suffices to show that there exists $n \in \hat{N}\cap\hat{G}_x$ such that $n\cdot(x,0,\beta) = m\cdot(x,0,\beta)$. Recall that $m'\cdot(x,0,\beta) = m\cdot(x,0,\beta)$ whenever $m' \in m\hat{G}_{x+}$. Let $\bar{m}$ be the image of m in H_x under the composition $\hat{G}_x \to \hat{G}_x/\hat{G}_{x,+} \xrightarrow{\theta_x^{-1}} H_x$. Then, $\theta'_x(\bar{m}) \in m\hat{G}_{x,+}$ and $\mathrm{Ad}^*(\theta'_x(\bar{m}))(\tilde{\beta}_0) \in \hat{\mathfrak{g}}^\vee_x(0)$. Since $\tilde{\beta}_0$ is a regular element of $\hat{\mathfrak{g}}_x(0)$ stabilized by $\hat{T}$, and the same is true for $\mathrm{Ad}^*(\theta'_x(\bar{m}))(\tilde{\beta}_0)$, we deduce that $\theta'_x(\bar{m}) \in \hat{N}\cap\hat{G}_x$.

Finally, it is clear that the action of $\hat{T}_x$ fixes $(x,0,\beta)$. Moreover, the correspondence $m \mapsto \bar{m}$ above determines a homomorphism from $\hat{N}\cap\hat{G}_x \to N\cap H_x$. It is easily checked that this homomorphism induces an isomorphism $(\hat{N}\cap\hat{G}_x)/\hat{T}_x \cong (N\cap H_x)/T$. □

If $(x,0,\beta)$ is graded regular with centralizer $\hat{T}$, then, since x is graded compatible with $\hat{T}$, $\tilde{\beta}_0 \in \mathfrak{t}\frac{dz}{z}$ and $\operatorname{Res}(\tilde{\beta}_0)$ is a regular element of $\mathfrak{t}$.

Proposition 4.6. *Suppose that $(x,0,\beta)$ is a graded regular stratum with connected centralizer $\hat{T}$, and let $m \in \hat{N} \cap \hat{G}_x$ determine an element $w \in W$. Then, for any root α, $\alpha(\operatorname{Res}(\tilde{\beta}_0)) + \alpha(\tilde{x}) \in \mathbb{Z}_{<\alpha(\tilde{x})}$ if and only if $w(\alpha)(\operatorname{Res}(\operatorname{Ad}^*(m)(\tilde{\beta}_0)) + w(\alpha)(\tilde{x}) \in \mathbb{Z}_{<w(\alpha)(\tilde{x})}$.*

Proof. It is evident that $w(\alpha)(\operatorname{Res}(\operatorname{Ad}^*(m)(\tilde{\beta}_0)) = w(\alpha)(w\operatorname{Res}(\tilde{\beta}_0)) = \alpha(\operatorname{Res}(\tilde{\beta}_0))$. It is now obvious that $\alpha(\operatorname{Res}(\tilde{\beta}_0))+\alpha(\tilde{x}) < \alpha(\tilde{x})$ if and only if $w(\alpha)(\operatorname{Res}(\operatorname{Ad}^*(m)(\tilde{\beta}_0))+w(\alpha)(\tilde{x}) < w(\alpha)(\tilde{x})$, so it remains to show that if one of the expressions on the left is an integer, the other is as well. In particular, it suffices to show that $\alpha(\tilde{x}) - w(\alpha)(\tilde{x}) \in \mathbb{Z}$ for all α, or equivalently (replacing α by $w^{-1}\alpha$), $\alpha(w\tilde{x} - \tilde{x}) \in \mathbb{Z}$.

Exponentiating, it is enough to show that $\operatorname{Ad}(\exp(2\pi i(w\tilde{x} - \tilde{x})))(X) = X$. However, by Proposition 4.5, there is an element $h \in H_x \cap N$ such that $w\tilde{x} = \operatorname{Ad}(h)(\tilde{x})$. Therefore, $\exp(2\pi i(w\tilde{x} - \tilde{x})) = h\exp(2\pi i\tilde{x})h^{-1}\exp(2\pi i\tilde{x})^{-1}$. Since h commutes with $\exp(2\pi i\tilde{x})$, this is the identity. □

Definition 4.7. Let $(x,0,\beta)$ be a graded regular stratum with $x \in \mathcal{A}_0$. Without loss of generality, assume that $\hat{T} = Z^0(\tilde{\beta}_0)$. We say that $(x,0,\beta)$ is *resonant* if for some root α of $\hat{T}$,

$$\alpha(\operatorname{Res}(\tilde{\beta}_0)) + \alpha(\tilde{x}) \in \mathbb{Z}_{<\alpha(\tilde{x})}.$$

By Proposition 4.5, one can always chose $m \in \hat{G}_x$ such that $m \cdot (x,0,\beta) = (x,0,\beta')$ has connected centralizer $\hat{T}$. The only ambiguity in the choice of m lies in the fact that, for any $n \in \hat{N} \cap \hat{G}_x$, nm will also "diagonalize" $(x,0,\beta)$. However, Proposition 4.6 implies that the condition in the definition above holds for $\tilde{\beta}_0$ if and only if it holds for $\operatorname{Ad}^*(n)(\tilde{\beta}_0)$.

Definition 4.8. A stratum (x,r,β) is called *regular* if the following conditions are satisfied: all subgroups $Z^0(\tilde{\beta})$ for any representative $\tilde{\beta}$ are $\hat{G}_{x+}$-conjugate maximal tori compatible with x, and additionally the stratum must be nonresonant when it has depth 0. If Σ is the $\hat{G}_{x+}$-conjugacy class of maximal tori determined by these stabilizers, we say the stratum is Σ-*regular* (or even S-regular, if $S \in \Sigma$).

Note that any $\hat{G}_{x+}$-conjugate of a representative $\tilde{\beta}$ is also a representative, so the set of connected centralizers of a regular stratum is a full $\hat{G}_{x+}$-orbit. Also, the compatibility with x need only be checked on a single $S \in \Sigma$. The following proposition shows that (x,r,β) is regular if and only if it is conjugate to a graded regular stratum coming from $x' \in \mathcal{A}_0$.

Proposition 4.9.

1. *If $x \in \mathcal{A}_0$, then the stratum (x,r,β) is graded regular (and nonresonant if $r = 0$) if and only if it is regular.*
2. *Let (x,r,β) be an S-regular stratum. If $\tilde{\beta} \in \pi_{\mathfrak{s}}^*(\mathfrak{s}_{-r}^\vee) + \hat{\mathfrak{g}}_{x,\ell-r}^\vee$ for $\ell > 0$ is any representative, then there exists $p \in \hat{G}_{x,\ell}$ such that $\operatorname{Ad}^*(p)(\tilde{\beta}) \in \pi_{\mathfrak{s}}^*(\mathfrak{s}_{-r}^\vee)$.*

Proof. It is trivial that regular implies graded regular. Now, assume that (x,r,β) is graded regular, and set $S = Z^0(\tilde{\beta}_0)$. First, suppose that $\tilde{\beta} \in \pi_{\mathfrak{s}}^*(\mathfrak{s}^\vee)$ is a representative. We show that $\mathfrak{z}(\tilde{\beta}) = \mathfrak{s}$. By Lemma 4.2(6), $\mathfrak{s} \subset \mathfrak{z}(\tilde{\beta})$. Conversely, suppose this inclusion is strict. Then, there exists an element of $\mathfrak{z}(\tilde{\beta})$ of the form $Y + Y'$

with $Y \in \hat{\mathfrak{g}}_x(\ell) \setminus \mathfrak{s}(\ell)$ and $Y' \in \hat{\mathfrak{g}}_{x,\ell+} \setminus \hat{\mathfrak{g}}_{x,\ell}$. Since $\tilde{\beta} = \tilde{\beta}_0 + \alpha$ with $\alpha \in \hat{\mathfrak{g}}^\vee_{x,-r+}$, $\mathrm{ad}^*(Y+Y')(\tilde{\beta})$ equals $\mathrm{ad}^*(Y)(\tilde{\beta}_0) \in \hat{\mathfrak{g}}^\vee(\ell - r)$ plus higher order terms. It follows that $\mathrm{ad}^*(Y)(\tilde{\beta}_0) = 0$, contradicting that fact that $Y \notin \mathfrak{s}$.

Next, let $\tilde{\beta}$ be any representative. It suffices to find $p \in \hat{G}_{x,0+}$ such that $\mathrm{Ad}^*(p)(\tilde{\beta}) \in \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee)$, as the previous argument will then imply that $\mathfrak{z}(\tilde{\beta}) = \mathrm{Ad}(p^{-1})(\mathfrak{s})$. Note that $\tilde{\beta} \in \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_{-r}) + \hat{\mathfrak{g}}^\vee_{x,(\ell-r)+}$ for some $\ell > 0$. Since $\tilde{\beta} - \pi^*_{\mathfrak{s}}(\rho_{\mathfrak{s}}(\tilde{\beta})) + \hat{\mathfrak{g}}^\vee_{x,(\ell-r)+} \in \ker(\bar{\rho}_{\mathfrak{s}})$, Lemma 4.2(3) states that there exists $X \in \hat{\mathfrak{g}}_{x,\ell}$ such that $\mathrm{ad}^*(X)(\tilde{\beta}) \in \tilde{\beta} - \pi^*_{\mathfrak{s}}(\rho_{\mathfrak{s}}(\tilde{\beta})) + \hat{\mathfrak{g}}^\vee_{x,(\ell-r)+}$. Take $p_\ell = \exp(-X) \in \hat{G}_{x,\ell}$. Then, $\mathrm{Ad}^*(p_\ell)(\tilde{\beta}) \in \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_{-r}) + \hat{\mathfrak{g}}^\vee_{x,(\ell-r)+}$. Recursively applying this argument, we obtain an increasing sequence ℓ_i (with $\ell_1 = \ell$) and elements $p_{\ell_i} \in \hat{G}_{x,\ell_i}$ such that $\mathrm{Ad}^*(p_{\ell_m}) \cdots \mathrm{Ad}^*(p_{\ell_1})(\tilde{\beta}) \in \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_{-r}) + \hat{\mathfrak{g}}^\vee_{x,(\ell_m-r)+}$ for all m. Setting $p = \lim(p_{\ell_m} \cdots p_{\ell_1}) \in \hat{G}_{x,\ell}$, we have $\mathrm{Ad}^*(p)(\tilde{\beta}) \in \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_{-r})$ as desired. This also proves the second statement for $x \in \mathcal{A}_0$.

Finally, the general case of part (2) follows by conjugating a regular stratum (x, r, β) to (x', r, β') with $x' \in \mathcal{A}_0$. One need only observe that conjugation preserves Moy-Prasad filtrations while if $S' = gSg^{-1}$, then $\pi_{\mathfrak{s}'}\mathrm{Ad}(g) = \mathrm{Ad}(g)\pi_{\mathfrak{s}}$. □

Not every maximal torus can be the connected centralizer of a regular stratum. Recall that a Weyl group element w is called regular if it has an eigenvector in the reflection representation whose stabilizer is trivial [17]. Equivalently, it has a regular semisimple eigenvector in $\mathfrak{t}$. Note that the eigenvalue of such a regular eigenvector can equal one only for the identity element of W. We say that a maximal torus (or Cartan subalgebra) has *regular type* if it has type γ, with γ a regular conjugacy class in W.

Corollary 4.10. *Let γ be a conjugacy class in W. Then, there exists an S-regular stratum for some maximal torus S of type γ if and only if γ is regular. In this case, the set of $x \in \mathcal{A}_0$ which support an S-regular stratum for some S of type γ is precisely Π_γ, which is described explicitly in Corollary 3.15.*

Proof. It is obvious that the split torus $\hat{T}$, corresponding to $e \in W$, admits regular strata (with $r = 0$), so we assume that S is nonsplit. Let E/F be a degree e extension over which S splits, and let $g \in G(E)$ be a w-diagonalizer of S. It follows that $\alpha = \mathrm{Ad}^*(g^{-1})(\tilde{\beta}) \in \hat{\mathfrak{t}}^\vee_E(-er)$ has connected centralizer $\hat{T}$. One can then find $v \in z^{-r}\mathfrak{t}$ regular semisimple such that $\alpha = \langle v, \cdot \rangle$. With σ our fixed generator for $\mathrm{Gal}(E/F)$ and ξ the e-th root of unity defined by $\sigma(z^{1/e}) = \xi z^{1/e}$, we have $w^{-1}(v) = \sigma(v) = \xi^{-re} v$. We deduce that $z^r v \in \mathfrak{t}$ is a regular eigenvector for w (with eigenvalue ξ^{re}).

If γ is a nonidentity regular conjugacy class, $e^{2\pi i r} (\neq 1)$ is a regular eigenvalue for γ with $r > 0$, and $x \in \Pi$, it is easy to write down an explicit regular stratum based at x of depth r whose connected centralizers are of type γ. A classification of such strata is given in Theorem 5.13. □

Example 4.11. For GL_n, a conjugacy classes in the Weyl group S_n is regular if its cycle decomposition consists of either n/k k-cycles or $(n-1)/k$ k-cycles and an additional 1-cycle. The corresponding maximal tori are the isomorphic to $(F[z^{1/k}]^\times)^{n/k}$ and $(F[z^{1/k}]^\times)^{(n-1)/k} \times F^\times$ respectively. A maximal torus of the former type is called uniform; formal connections contains S-regular strata for S uniform were studied in [3].

5. Isomorphism classes of flat G-bundles

In this section, we give a parameterization for the space of isomorphism classes of formal flat G-bundles that contain regular strata. More precisely, let S be a maximal torus of regular type which is compatible with some point in $\mathcal{A}_0$. If $r \geq 0$, let $\mathcal{C}(S,r)$ be the category of connections of slope r that contain an S-regular stratum. For each $x \in \mathcal{A}_0$ compatible with x, we also define the category $\mathcal{C}^{\mathrm{fr}}_x(S,r)$ of *framed* flat formal G-bundles whose objects are quadruples $\mathcal{F} = (\mathcal{G}, \nabla, (x,r,\beta), \phi)$, where $(\mathcal{G}, \nabla)$ is an object in $\mathcal{C}(S,r)$ containing the regular stratum (x,r,β) with respect to the trivialization ϕ. The morphisms in $\mathrm{Hom}(\mathcal{F}, \mathcal{F}')$ consists of isomorphisms $\psi : (\mathcal{G}, \nabla) \to (\mathcal{G}', \nabla')$ such that $\psi' = \phi' \circ \psi \circ \phi^{-1} \in \hat{G}_x$ and $\psi'^*(\beta') = \beta$. There is a forgetful "deframing" functor $\mathcal{C}^{\mathrm{fr}}_x(S,r) \to \mathcal{C}(S,r)$. We show that the moduli space of $\mathcal{C}^{\mathrm{fr}}_x(S,r)$ is the space of (S,r)-*formal types*–an explicitly-determined open[1] subset of $\mathfrak{s}^\vee_{-r}/\mathfrak{s}^\vee_{0+}$. The space of formal types is endowed with an action of the relative affine Weyl group, and the moduli space of $\mathcal{C}(S,r)$ is the corresponding orbit space.

Throughout this section, S will denote a maximal torus of regular type compatible with some point in $\mathcal{A}_0$. For clarity of exposition, we will assume that $S = \hat{T}$ when $r = 0$. This restriction is unnecessary, but it allows one to avoid the notational complications inherent in discussing the resonance condition for other split maximal tori.

5.1. Framed flat G-bundles and formal types.

In this section, we show that the category $\mathcal{C}^{\mathrm{fr}}_x(S,r)$ is essentially independent of the choice of x for $r > 0$ and compute its moduli space.

Theorem 5.1. *Suppose that $(\mathcal{G}, \nabla)$ contains an S-regular stratum (x,r,β) with respect to the trivialization ϕ.*

(1) *There exists $p \in \hat{G}_{x+}$ and an element $\tilde{A} \in \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_{-r})$ such that $[\nabla]_{p\phi} - \tilde{x}\frac{dz}{z} = \tilde{A} - \tilde{x}\frac{dz}{z}$.*
(2) *The orbit of $\tilde{A} - \tilde{x}\frac{dz}{z}$ under $\hat{G}_{x+}$-gauge transformations contains $\tilde{A} - \tilde{x}\frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{x+}$.*
(3) *If $\tilde{A}^1, \tilde{A}^2 \in \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_{-r})$ both determine the same regular stratum (x,r,β), and there exists $p \in \hat{G}_x$ such that $p \cdot \tilde{A}^1 \in \tilde{A}^2 + \hat{\mathfrak{g}}^\vee_{x+}$, then $\tilde{A}^1 \in \tilde{A}^2 + \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_{0+})$.*

Before proving this theorem, we show how it can be recast in the language of *formal types*. Since $\pi^*_{\mathfrak{s}} : \mathfrak{s}^\vee \to \hat{\mathfrak{g}}^\vee$ is an injection which is compatible with the Moy-Prasad filtration induced by x, the above theorem suggests that one may parameterize flat G-bundles of slope r containing S-regular strata by elements of $\mathfrak{s}^\vee_{-r}/\mathfrak{s}^\vee_{0+}$. In the following, we will adopt the notational convention that whenever $\tilde{A} \in \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_{-r})$, then $A = \rho_{\mathfrak{s}}(\tilde{A}) + \mathfrak{s}^\vee_{0+} \in \mathfrak{s}^\vee_{-r}/\mathfrak{s}^\vee_{0+}$. Similarly if $A \in \mathfrak{s}^\vee_{-r}/\mathfrak{s}^\vee_{0+}$, then $\tilde{A} \in \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_{-r})$, unless already defined, will denote an arbitrary element of $\pi^*_{\mathfrak{s}}(A)$.

Definition 5.2. A functional $A \in (\mathfrak{s}_0)^\vee \cong \mathfrak{s}^\vee/\mathfrak{s}^\vee_{0+}$ is called an *S-formal type of depth r* if

(1) the smallest congruence ideal contained in $A^\perp$ is $\mathfrak{s}_{r+}$;
(2) there exists $x \in \mathcal{A}_0$ compatible with S; and
(3) for some x compatible with S, the corresponding stratum (x,r,A^x) is S-regular, where A^x is the functional induced by $\tilde{A} - \tilde{x}\frac{dz}{z}$.

[1] for $r > 0$

We denote the space of S-formal types of depth r by $\mathbf{A}(S,r)$, which we will view as a subset of $\mathfrak{s}^\vee_{-r}/\mathfrak{s}^\vee_{0+} \cong (\mathfrak{s}_0/\mathfrak{s}_{r+})^\vee$. An S-formal type is any element of $\cup\, \mathbf{A}(S,r)$.

Remark 5.3. When $r>0$, $\mathbf{A}(S,r)$ is an open subset of the affine space $\mathfrak{s}^\vee_{-r}/\mathfrak{s}^\vee_{0+} \cong \bigoplus_{j=-r}^{0} \mathfrak{s}^\vee(j)$ with the summation only including $j \in \mathrm{Crit}(\mathfrak{s})$. Indeed, a coset $\mathfrak{s}^\vee_{-r}/\mathfrak{s}^\vee_{0+}$ corresponds to an element of $\mathbf{A}(S,r)$ if and only if its projection onto $\mathfrak{s}^\vee(-r)$ is regular, which is clearly an open condition. If $r=0$, the proof of Proposition 5.4 below shows that $\mathbf{A}(\hat{T},0) \cong \{X \in \mathfrak{t} \mid \alpha(X) \notin \mathbb{Z} \text{ for all } \alpha \in \Phi\}$. This is not Zariski-open, but if $k=\mathbb{C}$, it is open in the complex topology.

Proposition 5.4. *Suppose that $A \in (\mathfrak{s}_0)^\vee$ is an S-formal type of depth r.*

1. *If $r>0$, (x,r,A^x) is S-regular for all $x \in \mathcal{A}_0$ compatible with S.*
2. *When $r=0$, the stratum (x,r,A^x) is $\hat{T}$-regular if and only if $\alpha(\mathrm{Res}(\tilde{A})) \neq \alpha(\tilde{x})$ for all $\alpha \in \Phi$. In particular, if $H_x = G$, then (x,r,A^x) is $\hat{T}$-regular for all $A \in \mathbf{A}(\hat{T},0)$.*

Proof. Choose $y \in \mathcal{A}_0$ such that (y,r,A^y) is S-regular. When $r>0$, the functional A^x is induced by $\pi_{\mathfrak{s}}^*(A)$, so it is immediate that (x,r,A^x) is S-regular for any $x \in \mathcal{A}_0$ compatible with S.

Now, suppose that $r=0$ and $S=\hat{T}$. The nonresonance condition states that for all $\alpha \in \Phi$, $\alpha(\mathrm{Res}(\tilde{A})) = \alpha(\mathrm{Res}(\tilde{A} - \tilde{y}\frac{dz}{z})) + \alpha(\tilde{y}) \notin \mathbb{Z}_{<\alpha(\tilde{y})}$. Since this condition holds for $-\alpha$, either $\alpha(\mathrm{Res}(\tilde{A}))$ is not an integer or $\alpha(\mathrm{Res}(\tilde{A})) = \alpha(\tilde{y})$. However, if $\alpha(\mathrm{Res}(\tilde{A})) = \alpha(\tilde{y})$, then $\alpha(\mathrm{Res}(\tilde{A} - \tilde{y}\frac{dz}{z})) = 0$ and $(y,0,A^y)$ is not a regular stratum. We deduce that $\alpha(\mathrm{Res}(\tilde{A})) \notin \mathbb{Z}$ for all $\alpha \in \Phi$.

The set of $x \in \mathcal{A}_0$ such that $\tilde{A} - \tilde{x}\frac{dz}{z}$ does not determine a regular stratum thus consists of those points for which there exists $\alpha \in \Phi$ such that $\alpha(\mathrm{Res}(\tilde{A})) = \alpha(\tilde{x})$. This is a finite union of hyperplanes. (Since $\tilde{x} \in \mathfrak{t}_{\mathbb{R}}$, this condition is vacuous for roots α with $\alpha(\mathrm{Res}(\tilde{A})) \notin \mathbb{R}$.) Finally, suppose that $H_x = G$. If the corresponding stratum (x,r,A^x) is not regular, then there exists α such that $\alpha(\mathrm{Res}(\tilde{A})) = \alpha(\tilde{x})$. Since $H_x = G$, $\alpha(\tilde{x}) \in \mathbb{Z}$. This contradicts the fact that $\alpha(\mathrm{Res}(\tilde{A})) \notin \mathbb{Z}$. On the other hand, if $H_x \neq G$, there exists α for which $\alpha(\tilde{x}) \notin \mathbb{Z}$. We can thus choose $A \in \mathbf{A}(\hat{T},0)$ such that $\alpha(\mathrm{Res}(\tilde{A})) = \alpha(\tilde{x})$. □

Corollary 5.5. *If x and y are both compatible with S and $r>0$, then the categories $\mathcal{C}_x^{\mathrm{fr}}(S,r)$ and $\mathcal{C}_y^{\mathrm{fr}}(S,r)$ are canonically isomorphic.*

Proof. If A is the formal type for $\mathcal{F}_x = (\mathcal{G}, \nabla, (x,r,\beta^x), \phi)$, then the proposition shows that the functor $\mathcal{F}_x \mapsto \mathcal{F}_y = (\mathcal{G}, \nabla, (y,r,\beta^y), \phi)$ is the desired isomorphism. □

We can now describe the moduli spaces of these categories of framed flat G-bundles. Let $\mathbf{A}_x(\hat{T},0)$ be the Zariski-open subset of $\mathbf{A}(\hat{T},0)$ consisting of those A satisfying $\alpha(\mathrm{Res}(\tilde{A})) \neq \alpha(\tilde{x})$ for all $\alpha \in \Phi$. It is clear that $\mathbf{A}_x(\hat{T},0) = \mathbf{A}(\hat{T},0)$ if and only if $\alpha(\tilde{x}) \in \mathbb{Z}$ for all $\alpha \in \Phi$; equivalently, this holds precisely when $H_x = G$.

Theorem 5.6.

1. *If $r>0$, $\mathbf{A}(S,r)$ is the moduli space of $\mathcal{C}_x^{\mathrm{fr}}(S,r)$.*
2. *The moduli space of $\mathcal{C}_x^{\mathrm{fr}}(\hat{T},0)$ is a subset of $\mathbf{A}(\hat{T},0)$, with equality if and only if $H_x = G$.*

Proof. If $(\mathcal{G},\nabla)$ contains an S-regular stratum (x,r,β), then Theorem 5.1(1) shows that there is an element $\tilde{A}\in\pi_{\mathfrak{s}}^*(\mathfrak{s}_{-r}^\vee)$ and a trivialization ϕ of ∇ such that $\tilde{A}-\tilde{x}\frac{dz}{z}$ determines β and $[\nabla]_\phi|_{\hat{\mathfrak{g}}_{x,0}}=\tilde{A}$. Thus, any isomorphism class can be represented by a quadruple $\mathcal{F}=(\mathcal{G},\nabla,(x,r,\beta),\phi)$ satisfying these properties. We define a map from the moduli space to $\mathbf{A}(S,r)$, sending the class of $\mathcal{F}$ to the formal type $A=\rho_{\mathfrak{s}}(\tilde{A})+\mathfrak{s}_{0+}^\vee$. Part (3) shows that this map is well-defined. Moreover, the map is injective: if $\mathcal{F}'$ is another such good representative of an isomorphism class for which $\tilde{A}'$ determines the same formal type A, then part (2) shows that $\mathcal{F}'\cong\mathcal{F}$. Thus, the moduli space of $\mathcal{C}_x^{\mathrm{fr}}(S,r)$ is a subset of $\mathbf{A}(S,r)$. One easily sees that the image lies in $\mathbf{A}_x(\hat{T},0)$ when $r=0$.

It remains to compute the image of the formal types map. Let $\mathcal{G}^{\mathrm{triv}}$ be the trivial principal formal G-bundle. If $A\in\mathbf{A}(S,r)$, define a flat structure on $\mathcal{G}^{\mathrm{triv}}$ via $[\nabla_{\tilde{A}}]_{\mathrm{id}}=\tilde{A}$, where id is the identity trivialization. The flat G-bundle $(\mathcal{G}^{\mathrm{triv}},\nabla_{\tilde{A}})$ contains the stratum (x,r,A^x) with respect to id. When $r>0$, the class of $(\mathcal{G}^{\mathrm{triv}},\nabla_{\tilde{A}},(x,r,A^x),\mathrm{id})$ is mapped to A, so the formal types map is surjective. If $r=0$, the elements of $\mathbf{A}_x(\hat{T},0)$ are precisely those formal types for which (x,r,A^x) is a regular stratum, and we see that the image is $\mathbf{A}_x(\hat{T},0)$ in the same way. □

We now turn to the proof of Theorem 5.1. We begin with two lemmas.

Lemma 5.7. *Suppose that $(\mathcal{G},\nabla)$ contains an S-regular stratum (x,r,β) with respect to the trivialization ϕ, and assume that $[\nabla]_\phi\in\pi_{\mathfrak{s}}^*(\mathfrak{s}^\vee)+\hat{\mathfrak{g}}_{x,\ell-r}^\vee$ for some $\ell>0$.*

(1) *There exists $p\in\hat{G}_{x,\ell}$ such that $[\nabla]_{p\phi}\in\pi_{\mathfrak{s}}^*\circ\rho_{\mathfrak{s}}([\nabla]_\phi)+\hat{\mathfrak{g}}_{x,(\ell-r)+}^\vee$.*

(2) *If $q\in\hat{G}_{x,\ell}$ and $r>0$, then*
$$\rho_{\mathfrak{s}}([\nabla]_{q\phi})\in\rho_{\mathfrak{s}}([\nabla]_\phi)+\mathfrak{s}_{(\ell-r)+}^\vee.$$

Proof. First, assume $r>0$, so $[\nabla]_\phi$ is a representative of β. In the notation of Lemma 4.2, $[\nabla]_\phi-\pi_{\mathfrak{s}}^*\circ\rho_{\mathfrak{s}}([\nabla]_\phi)+\hat{\mathfrak{g}}_{x,(\ell-r)+}^\vee\in\ker(\bar{\rho}_{\mathfrak{s}})$. Part (3) of the Lemma implies that there exists $X\in\hat{\mathfrak{g}}_{x,\ell}$ such that $[\nabla]_\phi+\mathrm{ad}^*(X)([\nabla]_\phi)-\pi_{\mathfrak{s}}^*\circ\rho_{\mathfrak{s}}([\nabla]_\phi)\in\hat{\mathfrak{g}}_{x,(\ell-r)+}^\vee$. Set $p=\exp(X)$. By Lemma 2.4(4), $[\nabla]_{p\phi}-\tilde{x}\frac{dz}{z}\in\mathrm{Ad}^*(p)\left([\nabla]_\phi-\tilde{x}\frac{dz}{z}\right)+\hat{\mathfrak{g}}_{x,\ell}^\vee$. Since $\mathrm{Ad}^*(p)([\nabla]_\phi-\tilde{x}\frac{dz}{z})\in[\nabla]_\phi-\tilde{x}\frac{dz}{z}+\mathrm{ad}^*(X)([\nabla]_\phi)+\hat{\mathfrak{g}}_{x,(\ell-r)+}^\vee$ and $\hat{\mathfrak{g}}_{x,\ell}^\vee\subset\hat{\mathfrak{g}}_{x,(\ell-r)+}^\vee$, it follows from the observations above that $[\nabla]_{p\phi}-\tilde{x}\frac{dz}{z}\in\pi_{\mathfrak{s}}^*\circ\rho_{\mathfrak{s}}([\nabla]_\phi)-\tilde{x}\frac{dz}{z}+\hat{\mathfrak{g}}_{x,(\ell-r)+}^\vee$. This proves the first statement in the irregular singular case.

Writing $q=\exp(Y)$ for $Y\in\hat{\mathfrak{g}}_{x,\ell}$, we obtain

$$\begin{aligned}[\nabla]_{q\phi}-\tilde{x}\frac{dz}{z}\in\mathrm{Ad}^*(q)([\nabla]_\phi-\tilde{x}\frac{dz}{z})+\hat{\mathfrak{g}}_{x,(\ell-r)+}^\vee\\ =[\nabla]_\phi+\mathrm{ad}^*(Y)([\nabla]_\phi)-\tilde{x}\frac{dz}{z}+\hat{\mathfrak{g}}_{x,(\ell-r)+}^\vee.\end{aligned}$$

Since $\mathrm{ad}^*(Y)([\nabla]_\phi)+\hat{\mathfrak{g}}_{x,(\ell-r)+}\in\ker(\bar{\rho}_{\mathfrak{s}})$ by Lemma 4.2(3), the second statement follows.

Now, assume that $r=0$ and $S=\hat{T}$, and write $[\nabla]_\phi-\tilde{x}\frac{dz}{z}\in\pi_{\mathfrak{t}}^*\rho_{\mathfrak{t}}([\nabla]_\phi)-\tilde{x}\frac{dz}{z}+\sum_{\psi\in\Phi}Y_\psi\frac{dz}{z}+\hat{\mathfrak{g}}_{x,\ell+}^\vee$, where $Y_\psi\in\hat{\mathfrak{u}}_\psi\cap\hat{\mathfrak{g}}_{x,\ell}$. Choose $\alpha\in\Phi$ such that $\ell-\alpha(\tilde{x})\in\mathbb{Z}_{>-\alpha(\tilde{x})}$; this condition is necessarily satisfied by α if $Y_\alpha\frac{dz}{z}\in\hat{\mathfrak{g}}_{x,\ell}^\vee\backslash\hat{\mathfrak{g}}_{x,\ell+}^\vee$. Recall that the graded representative $\tilde{\beta}_0\in[\nabla]_\phi-\tilde{x}\frac{dz}{z}+\hat{\mathfrak{g}}_{x,0+}^\vee$ satisfies $b=\mathrm{Res}(\tilde{\beta}_0)\in\mathfrak{t}$. The nonresonance condition ensures that $\ell+\alpha(b)\neq0$. Define $X_\alpha=\frac{1}{\ell+\alpha(b)}Y_\alpha$. By

Lemma 2.4(2),

$$\begin{aligned}[\nabla]_{\exp(X_\alpha)\phi} - \tilde{x}\frac{dz}{z} &\in \mathrm{Ad}^*(\exp(X_\alpha))([\nabla]_\phi - \tilde{x}\frac{dz}{z}) - \ell X_\alpha \frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{x,\ell+} \\ &= [\nabla]_\phi - \tilde{x}\frac{dz}{z} - (\alpha(b)+\ell)X_\alpha\frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{x,\ell+} \\ &= \pi^*_{\hat{\mathfrak{t}}}\rho_{\hat{\mathfrak{t}}}([\nabla]_\phi) - \tilde{x}\frac{dz}{z} + \sum_{\substack{\psi\in\Phi \\ \psi\neq\alpha}} Y_\alpha \frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{x,\ell+}\end{aligned}$$

Repeating this process, we can kill off all of the off-diagonal terms $Y_\alpha \frac{dz}{z}$. This completes the proof. □

Lemma 5.8. *Suppose that $x \in \mathcal{A}_0$ is compatible with S. Let $Z\frac{dz}{z} \in \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_\ell)$ for $\ell > 0$; here, $Z \in \mathfrak{s}_\ell$. Then, there exists $s \in S_\ell$ such that $(ds)s^{-1} \in (Z - \mathrm{ad}(\tilde{x})(\frac{1}{\ell}Z))\frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{x,\ell+}$. In particular, $\pi^*_{\mathfrak{s}} \circ \rho_{\mathfrak{s}}((ds)s^{-1}) \in Z\frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{x,\ell+}$.*

Proof. Take $s = \exp(\frac{1}{\ell}Z)$. Then, $(ds)s^{-1} = \frac{1}{\ell}\tau(Z)\frac{dz}{z}$. However, $\frac{1}{\ell}\tau(Z) \in -\mathrm{ad}(\tilde{x})(\frac{1}{\ell}Z) + Z + \hat{\mathfrak{g}}_{x,\ell+}$. Consider $\mathrm{ad}^*(\tilde{x})(\frac{1}{\ell}Z\frac{dz}{z})$. If $X \in \mathfrak{s}$, then $\mathrm{ad}^*(\tilde{x})(\frac{1}{\ell}Z\frac{dz}{z})\,(X) = -\mathrm{ad}^*(X)(\frac{1}{\ell}Z\frac{dz}{z})\,(\tilde{x}) = 0$. It follows that $\rho_{\mathfrak{s}}(\mathrm{ad}^*(\tilde{x})(\frac{1}{\ell}Z\frac{dz}{z})) = 0$ by definition of the restriction map. Therefore, $\rho_{\mathfrak{s}}((ds)s^{-1}) \in \rho_{\mathfrak{s}}(Z\frac{dz}{z}) + \mathfrak{s}^\vee_{\ell+}$. □

Proof of Theorem 5.1. By Theorem 3.7, we may assume that S is graded compatible with x. Suppose that ϕ' is a trivialization for which $[\nabla]_{\phi'} \in \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee) + \hat{\mathfrak{g}}^\vee_{x,\ell-r}$ with $\ell > 0$. Writing $\rho_{\mathfrak{s}}([\nabla]_\phi) = \sum_{i\geq -r} A_i$ with $A_i \in \mathfrak{s}^\vee(i)$, we have $[\nabla]_{\phi'} = \sum_{-r\leq i<\ell-r}\tilde{A}_i + \hat{\mathfrak{g}}^\vee_{x,\ell-r}$. We suppose further that if $\ell - r > 0$, then $A_i = 0$ for $0 < i < \ell - r$. We now show that we can construct p_ℓ, in $\hat{G}_{x,\ell-r}$ if $\ell - r > 0$ and in $\hat{G}_{x,\ell}$ if not, such that $[\nabla]_{p_\ell\phi'} = \sum_{-r\leq i<\ell-r}\tilde{A}_i + \pi^*_{\mathfrak{s}}(\mathfrak{s}(\ell-r)) + \hat{\mathfrak{g}}^\vee_{x,(\ell-r)+}$. If we can do this, then applying this process recursively, starting with ϕ and the smallest ℓ' for which $\hat{\mathfrak{g}}^\vee_{x,(\ell'-r)+} \neq \hat{\mathfrak{g}}^\vee_{x,-r+}$, we obtain a well-defined $p = \prod_{\ell=\ell'}^\infty p_\ell$ satisfying part (1). Moreover, we can set $A = \sum_{i=-r}^0 A_i$.

By Lemma 5.7, there exists $q \in \hat{G}_{x,\ell}$ such that $[\nabla]_{q\phi'} - \tilde{x}\frac{dz}{z} \in \pi^*_{\mathfrak{s}} \circ \rho_{\mathfrak{s}}([\nabla]_{\phi'}) - \tilde{x}\frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{x,(\ell-r)+}$. If $\ell - r \leq 0$, set $p_\ell = q$. On the other hand, if $\ell - r > 0$, Lemma 5.8 shows that there exists $s \in S_{\ell-r}$ such that $\rho_{\mathfrak{s}}((ds)s^{-1}) \in A_{\ell-r} + \mathfrak{s}^\vee_{(\ell-r)+}$. In particular, $\rho_{\mathfrak{s}}([\nabla]_{sq\phi}) \in \sum_{i=-r}^0 A_i + \mathfrak{s}^\vee_{(\ell-r)+}$. Applying Lemma 5.7 once more, we obtain $q' \in \hat{G}_{x,\ell}$ such that $[\nabla]_{q'sq\phi} - \tilde{x}\frac{dz}{z} \in \sum_{i=-r}^0 \tilde{A}_i - \tilde{x}\frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{x,(\ell-r)+}$. The desired change of trivialization is given by $p_\ell = q'sq$.

Part (2) now follows from the observation that whenever $X \in \hat{\mathfrak{g}}^\vee_{x,0+}$, the preceding algorithm produces an element $p \in \hat{G}_{x,0+}$ such that $p \cdot (\tilde{A} + X) = \tilde{A}$.

It remains to prove part (3). Recall that Lemma 2.4(3) states that $p \cdot \tilde{A}^1 - \tilde{x}\frac{dz}{z} \in \mathrm{Ad}^*(p)(\tilde{A}^1 - \tilde{x}\frac{dz}{z}) + \hat{\mathfrak{g}}^\vee_{x,0+}$. It follows that $\mathrm{Ad}^*(p)(\tilde{A}^1 - \tilde{x}\frac{dz}{z}) \in \tilde{A}^2 - \tilde{x}\frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{x,0+}$. Since $\tilde{A}^1$ and $\tilde{A}^2$ determine the same regular stratum, $\mathrm{Ad}^*(p)(\tilde{A}^1 - \tilde{x}\frac{dz}{z}) \in \tilde{A}^1 - \tilde{x}\frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{x,-r+}$.

We first consider the case $r > 0$. By Proposition 4.9, there exists $q \in \hat{G}_{x,r}$ such that $\mathrm{Ad}^*(qp)(\tilde{A}^1) \in \tilde{A}^2 + \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_{0+})$, so $n = qp \in N(S) \cap \hat{G}_x$. Write $\tilde{A}^1 \in \tilde{A}_{-r} + \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_{-r+})$ where $\tilde{A}_{-r} \in \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee(-r))$; note that $\tilde{A}^2 \in \tilde{A}_{-r} + \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_{-r+})$ by assumption. Fix a w-diagonalizer g for S satisfying Theorem 3.13, and write $n' =$

$g^{-1}ng \in N(E)$. A direct calculation, using the fact that $\tau(n')(n')^{-1} \in \mathfrak{t}(E)$, shows that $\tau(n)n^{-1} \in \tau(g)g^{-1} - \mathrm{Ad}(n)(\tau(g)g^{-1}) + \mathfrak{s} = -\tilde{x} + \mathrm{Ad}(n)(\tilde{x}) + \mathfrak{s}$. Therefore,

$$\begin{aligned}\tau(\mathrm{Ad}^*(n)(\tilde{A}_{-r})) &+ \mathrm{ad}^*(\tilde{x})(\mathrm{Ad}^*(n)(\tilde{A}_{-r}))\\ &= \mathrm{Ad}^*(n)(\tau(\tilde{A}_{-r})) + \mathrm{ad}^*(\tau(n)n^{-1})(\mathrm{Ad}^*(n)(\tilde{A}_{-r})) + \mathrm{ad}^*(\tilde{x})(\mathrm{Ad}^*(n)(\tilde{A}_{-r}))\\ &= \mathrm{Ad}^*(n)(\tau(\tilde{A}_{-r}) + \mathrm{ad}^*(\tilde{x})(\tilde{A}_{-r})) = -r\mathrm{Ad}^*(n)(\tilde{A}_{-r}).\end{aligned}$$

We deduce that $\mathrm{Ad}^*(n)(\tilde{A}_{-r}) = \tilde{A}_{-r}$. Since $\tilde{A}_{-r}$ is regular semisimple, $n \in S \cap \hat{G}_x = S_0$. Using the facts that $(dn)n^{-1} \in \mathrm{Ad}^*(n)(\tilde{x}) - \tilde{x} + \mathfrak{s}^\vee_{0+}$ and ρ_s commutes with $\mathrm{Ad}^*(n)$, we see that $\rho_{\mathfrak{s}}(n \cdot \tilde{A}^1) \in \rho_{\mathfrak{s}}(\tilde{A}^1) + \mathfrak{s}^\vee_{0+}$. We now apply Lemma 5.7 (2) to obtain

$$\rho_{\mathfrak{s}}(\tilde{A}^2) \in \rho_{\mathfrak{s}}(q^{-1}n \cdot \tilde{A}^1) + \mathfrak{s}^\vee_{0+} = \rho_{\mathfrak{s}}(n \cdot \tilde{A}^1) + \mathfrak{s}^\vee_{0+} = \rho_{\mathfrak{s}}(\tilde{A}^1) + \mathfrak{s}^\vee_{0+}.$$

This proves part (3) when $r > 0$.

Finally, assume that $r = 0$. Since $\tilde{A}^1$ and $\tilde{A}^2$ both determine the same stratum, $\tilde{A}^1 - \tilde{x}\frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{x+} = \tilde{A}^2 - \tilde{x}\frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{x+}$. We thus have $\tilde{A}^1 - \tilde{A}^2 \in \pi^*_{\hat{\mathfrak{t}}}(\hat{\mathfrak{t}}^\vee_0) \cap \hat{\mathfrak{g}}^\vee_{x,0+} = \pi^*_{\hat{\mathfrak{t}}}(\hat{\mathfrak{t}}^\vee_{0+})$. □

5.2. **Flat G-bundles and orbits of formal types.** Let S be a maximal torus of type γ, which for the time being is not assumed to be regular. Let $W_S = N(S)/S$ and $\hat{W}_S = N(S)/S_0$ be the relative Weyl group and the relative affine Weyl group associated to S. Note that $\hat{W}_S \cong W_S \ltimes S/S_0$. The group $\hat{W}_S$ consists of the Galois fixed points of $N(S(E))/S_0(E) \cong \hat{W}$.

Proposition 5.9. *The group W_S is isomorphic to a subgroup of the centralizer in W of a representative w for γ.*

Proof. In order to prove this, let $n \in N(S)$ and choose a regular element $s \in \mathfrak{s}$. Choose a w-diagonalizer g for S. Then, $s = \mathrm{Ad}(g)(t)$ for some $t \in \mathfrak{t}(E)$ that satisfies $\sigma(t) = w^{-1}t$. Write $n' = g^{-1}ng$. It is clear that $n' \in N(E)$. Moreover, since $N(E) = N \cdot T(E)$, one sees that $\sigma(n't(n')^{-1}) = n'\sigma(t)(n')^{-1}$. Thus, if $u \in W$ is the image of n', $w^{-1}ut = uw^{-1}t$. Since t is regular, $w^{-1}u = uw^{-1}$. It follows that $n \mapsto u$ defines a monomorphism from $N(S)/S$ into the centralizer of w in W. □

We now show that there is an action $\hat{\varrho}$ of $N(S)$ on $\mathfrak{s}^\vee/\mathfrak{s}^\vee_{0+}$ by affine transformations given by the formula

$$\hat{\varrho}(n)(X + \mathfrak{s}^\vee_{0+}) = \mathrm{Ad}^*(n)(X) - \rho_{\mathfrak{s}}((dn)n^{-1}) + \mathfrak{s}^\vee_{0+}.$$

Proposition 5.10. *The map $\hat{\varrho} : N(S) \to \mathrm{Aff}(\mathfrak{s}^\vee/\mathfrak{s}^\vee_{0+})$ defined above is a group homomorpism. The kernel of $\hat{\varrho}$ contains S_0, so $\hat{\varrho}$ induces a group action ϱ of $\hat{W}_S$ on $\mathfrak{s}^\vee/\mathfrak{s}^\vee_{0+}$. Finally, $\mathfrak{s}^\vee_{-r}/\mathfrak{s}^\vee_{0+} \subset \mathfrak{s}^\vee/\mathfrak{s}^\vee_{0+}$ is a finite dimensional submodule for all $r \geq 0$, and the quotient action on $\mathfrak{s}^\vee/\mathfrak{s}^\vee_0$ is comes from the coadjoint action of $N(S)$.*

Proof. Without loss of generality, assume that $\mathfrak{s}$ is graded compatible with $x \in \mathcal{A}_0$. Choose a w-diagonalizer $g \in G(E)$ such that $\mathrm{Ad}(g)(\mathfrak{t}(E)) = \mathfrak{s}(E)$ and $(\tau g)g^{-1} \in \tilde{x} + \mathfrak{s}(0)$ as in Proposition 3.9.

Suppose that $n_1, n_2 \in N(S)$. Then, $\rho_{\mathfrak{s}}(\mathrm{Ad}^*(n_1)((dn_2)n_2^{-1}) = \mathrm{Ad}^*(n_1)\rho_{\mathfrak{s}}((dn_2)n_2^{-1})$. It follows that

$$\mathrm{Ad}^*(n_1n_2)(X) - \rho_{\mathfrak{s}}((d(n_1n_2))(n_1n_2)^{-1}) = \mathrm{Ad}^*(n_1)\left[\mathrm{Ad}^*(n_2)(X) - \rho_{\mathfrak{s}}((dn_2)n_2^{-1})\right] - d(n_1)n_1^{-1},$$

and thus $\hat{\varrho}$ is a homomorphism. Now, take $s \in S_0$. Write $s = ts'$, where $t \in S_0 \cap T$ and $s' \in S_{0+}$. By Lemma 3.10, $-(dt)t^{-1} \in \mathfrak{t}^\vee_{0+}$. Since $s' = \exp(X)$ for some $X \in \mathfrak{s}_{0+}$, $\rho_{\mathfrak{s}}(-(ds')(s')^{-1}) = \rho_{\mathfrak{s}}(-\tau(X)\frac{dz}{z}) \in \mathfrak{s}^\vee_{0+}$. It follows that S_0 lies in the kernel of $\hat{\varrho}$.

We now prove the second half of the proposition. Let $n \in N(S)$. There exists $n' \in N(E)$ such that $g(n')g^{-1} = n$. Then,

$$\begin{aligned}(dn)n^{-1} &= \mathrm{Ad}^*(gn')(-g^{-1}dg) + \mathrm{Ad}^*(g)(dn'(n')^{-1}) + (dg)g^{-1}\\ &= \mathrm{Ad}^*(g)(dn'(n')^{-1}) - \mathrm{Ad}^*(n)((dg)g^{-1}) + (dg)g^{-1}.\end{aligned}$$

Since $(dg)g^{-1} \in \hat{\mathfrak{g}}^\vee_x(0)$, it follows that $\rho_s((dg)g^{-1}) \subset \mathfrak{s}^\vee(0)$. Moreover, $\rho_s(\mathrm{Ad}^*(n)((dg)g^{-1})) = \mathrm{Ad}^*(n)(\rho_s((dg)g^{-1}))$, so the restriction of the second two terms in the expression above lie in $\mathfrak{s}^\vee(0)$. By Lemma 3.10, $(dn')(n')^{-1} \in \left(\mathrm{Ad}(n')(\tilde{x}) - \widetilde{n'x}\right)\frac{dz}{z} + \mathfrak{t}_{0+}(E)\frac{dz}{z} \subset \mathfrak{t}_0(E)\frac{dz}{z}$. Thus, $\mathrm{Ad}^*(g)((dn')(n')^{-1}) \in \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee(E)_0) \cap \hat{\mathfrak{g}}^\vee = \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_0)$. We conclude that $\rho_{\mathfrak{s}}((dn)n^{-1}) \in \mathfrak{s}^\vee_0$. This proves that $\hat{\varrho}(n)(X + \mathfrak{s}^\vee_{0+}) + \mathfrak{s}^\vee_0 = \mathrm{Ad}^*(n)(X) + \mathfrak{s}^\vee_0$ whenever $X \in \mathfrak{s}^\vee$.

Finally, since the action of the Weyl group preserves the Moy-Prasad filtration on a split torus, it follows that $\mathrm{Ad}^*(n)(\mathfrak{s}^\vee(E)_{-r}) = \mathfrak{s}^\vee(E)_{-r}$. Thus, $\mathrm{Ad}^*(n)(\mathfrak{s}^\vee_{-r}) = \mathfrak{s}^\vee_{-r}$. Since $(dn)n^{-1} \in \mathfrak{s}^\vee_0$, it is now clear that $\hat{\varrho}(n)(\mathfrak{s}^\vee_{-r}) \subset \hat{\varrho}(n)(\mathfrak{s}^\vee_{-r})$ whenever $r \geq 0$. □

From now on, we reimpose the conditions on S from the previous section. In particular, γ is a regular conjugacy class. First, we show that the action ϱ restricts to give an action on the space of S-formal types of depth r.

Proposition 5.11. *The subspace $\mathbf{A}(S,r) \subset \mathfrak{s}^\vee_{-r}/\mathfrak{s}^\vee_{0+}$ is stable under the action of $\hat{W}_S$.*

Proof. By Proposition 5.10, $\mathfrak{s}^\vee_{-r}/\mathfrak{s}^\vee_{0+}$ is closed under the action of $\hat{W}_S$. Assume without loss of generality that $x \in \mathcal{A}_0$ is graded compatible with $\mathfrak{s}$, and write $X_{-r} \in \mathfrak{s}^\vee(-r)$ for the leading term of $X \in \mathbf{A}(S,r)$. Suppose that $\hat{w} \in \hat{W}$ with representative $n \in N(S)$. The same proposition shows that $\varrho(\hat{w})(X) \in \mathrm{Ad}^*(n)(X) + \mathfrak{s}^\vee_0$.

First, we assume that $r > 0$. Since $\mathrm{Ad}^*(n)(\mathfrak{s}(r)) = \mathfrak{s}(r)$, it follows that $\varrho(\hat{w})(X)_{-r} = \mathrm{Ad}^*(n)(X_{-r})$. Thus, $Z^0(\varrho(\hat{w})(X)_{-r}) = \mathrm{Ad}^*(n)(Z^0(X_{-r})) = S$. By Proposition 4.9, $\varrho(\hat{w})(X)$ determines a regular stratum. Therefore, $\varrho(\hat{w})(X) \in \mathbf{A}(S,r)$.

Now, we consider the case $r = 0$ and $S = \hat{T}$. Choose a representative $n \in N$ for $\hat{w}$. The proof of Proposition 5.4 shows that $X \in \hat{\mathfrak{t}}^\vee_0/\hat{\mathfrak{t}}^\vee_{0+}$ is in $\mathbf{A}(\hat{T},0)$ if and only if $\alpha(\mathrm{Res}(\tilde{X}_0)) \notin \mathbb{Z}$ for every root α. It is obvious that if X satisfies this condition, then the same holds for $\mathrm{Ad}^*(n)(X)$. Also, Lemma 3.10 shows that $(dn)n^{-1} \in (-\widetilde{nx} + \hat{\mathfrak{t}}_{0+})\frac{dz}{z}$, so that $\mathrm{Res}((dn)n^{-1}) \in \mathfrak{t}(\mathbb{Z})\frac{dz}{z}$. It follows immediately that $\alpha(\varrho(\hat{w})(\mathrm{Res}(\tilde{X}_0))) \notin \mathbb{Z}$ for all α, i.e., $\varrho(\hat{w})(X) \in \mathbf{A}(\hat{T},0)$. □

We now show that isomorphisms classes of flat G-bundles in $\mathcal{C}(S,r)$ can be identified with orbits in $\mathbf{A}(S,r)$.

Lemma 5.12. *Suppose that x is compatible with $\mathfrak{s}$ and $X \in \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_{-r}) + \hat{\mathfrak{g}}^\vee_x$. If $g \in \hat{G}_{x,r}$ with $r > 0$, $\rho_{\mathfrak{s}}(\mathrm{Ad}^*(g)(X) - (dg)g^{-1}) \in \rho_{\mathfrak{s}}(X) + \mathfrak{s}^\vee_{0+}$.*

Proof. Lemma 2.4(3) implies that $\mathrm{Ad}^*(g)(X) - (dg)g^{-1} \in \mathrm{Ad}^*(g)(X) - \mathrm{Ad}^*(g)(\tilde{x}\frac{dz}{z}) + \tilde{x}\frac{dz}{z} + \hat{\mathfrak{g}}^\vee_{x+}$. Since $g \in \hat{G}_{x+}$, it is clear that $-\mathrm{Ad}^*(g)(\tilde{x}\frac{dz}{z}) + \tilde{x}\frac{dz}{z} \in \hat{\mathfrak{g}}^\vee_{x+}$ It thus suffices to show that $\rho_{\mathfrak{s}}(\mathrm{Ad}^*(g)(X)) \in \rho_{\mathfrak{s}}(X) + \mathfrak{s}^\vee_{0+}$. Write $g = \exp(Y)$, for $Y \in \hat{\mathfrak{g}}_{x,r}$. Then,

$\mathrm{Ad}^*(g)(X) \in X + \mathrm{ad}^*(Y)(X) + \hat{\mathfrak{g}}^\vee_{x+}$. By Remark 4.3, $\rho_{\mathfrak{s}}(X + \mathrm{ad}^*(Y)(X) + \hat{\mathfrak{g}}^\vee_{x+}) \in \rho_{\mathfrak{s}}(X) + \mathfrak{s}^\vee_{0+}$. □

Theorem 5.13. *There is a bijection between the set of isomorphism classes of formal flat G-bundles that contain an S-regular stratum of depth r and the set $\mathbf{A}(S,r)/\hat{W}_S$.*

Proof. Theorem 5.1 shows that whenever $(\mathcal{G},\nabla)$ contains a regular stratum (x,r,β), there is a trivialization ϕ such that $[\nabla]_\phi \in \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_{-r})$. Moreover, the isomorphism class of $(\mathcal{G},\nabla)$ depends only on the restriction of $[\nabla]_\phi$ to $\mathfrak{s}^\vee_{-r}/\mathfrak{s}^\vee_{0+}$, giving rise to an element $A \in \mathbf{A}(S,r)$. One further sees that every formal type in the $\hat{W}_S$-orbit of A can be obtained from $(\mathcal{G},\nabla)$ by changing the trivialization by elements of $\hat{N}$. It remains to show the converse.

With ϕ a trivialization for $\mathcal{G}$ as above, take $g \in \hat{G}$ such that $(\mathcal{G},\nabla)$ contains the regular strata (y,r,β') with respect to the trivialization $g\phi$. Note that the depths are necessarily the same by Theorem 2.3. We may assume that $[\nabla]_{g\phi} \in \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_{-r})$ by Theorem 5.1(1), so that $\beta' = \rho_{\mathfrak{s}}([\nabla]_{g\phi})^y$. If $r > 0$, Proposition 5.4 implies that $(x,r,\rho_{\mathfrak{s}}([\nabla]_{g\phi})^x)$ is a regular stratum contained in $(\mathcal{G},\nabla)$ with respect to the trivialization $g\phi$. On the other hand, if $r = 0$, the same proposition shows that for any x' in a nonempty open subset of the fundamental alcove, $(\mathcal{G},\nabla)$ contains the regular strata $(x',0,\rho_{\hat{\mathfrak{t}}}([\nabla]_\phi)^{x'})$ and $(x',0,\rho_{\hat{\mathfrak{t}}}([\nabla]_{g\phi})^{x'})$ with respect to ϕ and $g\phi$ respectively. Thus, we may assume without loss of generality that $x = y$ and further, that $\hat{G}_x$ is the standard Iwahori subgroup I when $r = 0$.

First, assume that $r > 0$. Write $g = p_1 n p_2$ using the affine Bruhat decomposition, where $p_1, p_2 \in \hat{G}_x$ and $n \in \hat{N}$. By Lemma 2.4(3), $[\nabla]_{p_2\phi} - \tilde{x}\frac{dz}{z} \in \mathrm{Ad}^*(p_2)([\nabla]_\phi - \tilde{x}\frac{dz}{z}) + \hat{\mathfrak{g}}^\vee_{x+}$. Applying $\mathrm{Ad}^*(p_2^{-1})$ to both sides,

$$[\nabla]_\phi - p_2^{-1}(dp_2) \in [\nabla]_\phi - \tilde{x}\frac{dz}{z} + \mathrm{Ad}^*(p_2^{-1})(\tilde{x}\frac{dz}{z}) + \hat{\mathfrak{g}}^\vee_{x+} \subset [\nabla]_\phi + \hat{\mathfrak{g}}^\vee_x.$$

By Lemma 4.9(2), we see that there exists $q_2 \in \hat{G}_{x,r}$ such that

$$Z_1 := \mathrm{Ad}^*(q_2^{-1})\left([\nabla]_\phi - p_2^{-1}dp_2\right) \in [\nabla]_\phi + \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_0). \tag{6}$$

Using the fact that $(dn)n^{-1} \in \hat{\mathfrak{g}}^\vee_x$, a similar argument shows that there exists $q_1 \in \hat{G}_{x,r}$ such that

$$Z_2 := \mathrm{Ad}^*(q_1)\left([\nabla]_{g\phi} + (dp_1)p_1^{-1} + \mathrm{Ad}^*(p_1)((dn)n^{-1})\right) \in [\nabla]_{g\phi} + \pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee_0). \tag{7}$$

Note that Z_1 and Z_2 are regular semisimple elements of $\pi^*_{\mathfrak{s}}(\mathfrak{s}^\vee)$ since they both determine regular strata. Setting $h = q_1 g q_2$, the following calculation shows that $\mathrm{Ad}^*(h)(Z_1) = Z_2$ and thus $h \in N(S)$:

$$\begin{aligned}\mathrm{Ad}^*(h)(Z_1) &= \mathrm{Ad}^*(q_1p_1n)\left([\nabla]_{p_2\phi}\right)\\ &= \mathrm{Ad}^*(q_1p_1)\left([\nabla]_{np_2\phi} + (dn)n^{-1}\right)\\ &= \mathrm{Ad}^*(q_1)\left([\nabla]_{g\phi} + \mathrm{Ad}^*(p_1)\left((dn)n^{-1}\right) + (dp_1)p_1^{-1}\right) = Z_2.\end{aligned}$$

Now, let $X = \rho_{\mathfrak{s}}([\nabla]_\phi)+\mathfrak{s}^\vee_{0+}$ and $Y = \rho_{\mathfrak{s}}([\nabla]_{g\phi})+\mathfrak{s}^\vee_{0+}$, so that $X, Y \in \mathbf{A}(S,r)$. We will show that $Y = \hat{\varrho}(h)(X)$. Applying Lemma 5.7(2), we see that $\rho_{\mathfrak{s}}([\nabla]_{q_1g\phi}) \in \rho_{\mathfrak{s}}([\nabla]_{g\phi}) + \mathfrak{s}^\vee_{0+}$ and $\rho_{\mathfrak{s}}([\nabla]_{q_2^{-1}\phi}) \in \rho_{\mathfrak{s}}([\nabla]_\phi) + \mathfrak{s}^\vee_{0+}$. Since $\rho_{\mathfrak{s}}$ is an $N(S)$-map by

Lemma 4.2(5), it follows that

$$\begin{aligned}\rho_{\mathfrak{s}}([\nabla]_{hq_2^{-1}\phi}) &= \rho_{\mathfrak{s}}(\mathrm{Ad}^*(h)([\nabla]_{q_2^{-1}\phi})) - (dh)h^{-1}) \\ &= \mathrm{Ad}^*(h)(\rho_{\mathfrak{s}}([\nabla]_{q_2^{-1}\phi})) - \rho_{\mathfrak{s}}((dh)h^{-1}) \in \rho_{\mathfrak{s}}([\nabla]_{h\phi}) + \mathfrak{s}_{0+}^\vee.\end{aligned}$$

Finally,

$$\begin{aligned}Y = \rho_{\mathfrak{s}}([\nabla]_{g\phi}) + \mathfrak{s}_{0+}^\vee &= \rho_{\mathfrak{s}}([\nabla]_{q_1 g\phi}) + \mathfrak{s}_{0+}^\vee \\ &= \rho_{\mathfrak{s}}([\nabla]_{hq_2^{-1}\phi}) + \mathfrak{s}_{0+}^\vee = \rho_{\mathfrak{s}}([\nabla]_{h\phi}) + \mathfrak{s}_{0+}^\vee = \hat{\varrho}(h)(X).\end{aligned}$$

Now, take $r = 0$ with $\hat{G}_x = I$. Again, write $g = p_1 n p_2$ with $p_i \in I$ and $n \in \hat{N}$. Furthermore, since $I = T \ltimes I_+$, we may assume that $p_1, p_2 \in \hat{G}_{x+} = I_+$.

Since $p_i \in I_+$, $\rho_{\hat{\mathfrak{t}}}([\nabla]_{p_2\phi}) \in \rho_{\hat{\mathfrak{t}}}([\nabla]_{\phi}) + \hat{\mathfrak{t}}_{0+}^\vee$ and $\rho_{\hat{\mathfrak{t}}}([\nabla]_{p_1^{-1}g\phi}) \in \rho_{\hat{\mathfrak{t}}}([\nabla]_{g\phi}) + \hat{\mathfrak{t}}_{0+}^\vee$ by Lemma 2.4(4). Finally,

$$\begin{aligned}\rho_{\hat{\mathfrak{t}}}([\nabla]_{g\phi}) &\in \rho_{\hat{\mathfrak{t}}}([\nabla]_{np_2\phi}) + \hat{\mathfrak{t}}_{0+}^\vee = \mathrm{Ad}^*(n)(\rho_{\hat{\mathfrak{t}}}([\nabla]_{p_2\phi})) - (dn)n^{-1} + \hat{\mathfrak{t}}_{0+}^\vee \\ &= \mathrm{Ad}^*(n)(\rho_{\hat{\mathfrak{t}}}([\nabla]_{\phi})) - (dn)n^{-1} + \hat{\mathfrak{t}}_{0+}^\vee = \hat{\varrho}(n)(\rho_{\hat{\mathfrak{t}}}([\nabla]_{\phi}) + \hat{\mathfrak{t}}_{0+}^\vee). \quad \square\end{aligned}$$

Corollary 5.14. *The moduli space of $\mathcal{C}(S,r)$ is $\mathbf{A}(S,r)/\hat{W}_S$. Moreover, the deframing functor $\mathcal{C}_x^{\mathrm{fr}}(S,r) \to \mathcal{C}(S,r)$ corresponds to the quotient map $\mathbf{A}(S,r) \to \mathbf{A}(S,r)/\hat{W}_S$ on moduli spaces when $r > 0$ or $r = 0$ and $H_x = G$.*

Thus, the category $\mathcal{C}_x^{\mathrm{fr}}(S,r)$ (with $H_x = G$ if $r = 0$) may be viewed as a "resolution" of $\mathcal{C}(S,r)$.

References

[1] P. Boalch, "Symplectic manifolds and isomonodromic deformations," Adv. Math. **163** (2001), 137–205.

[2] P. Boalch, "G-bundles, isomonodromy, and quantum Weyl groups," Int. Math. Res. Not. **22** (2002), 1129–1166.

[3] C. Bremer and D. S. Sage, "Moduli spaces of irregular singular connections," Int. Math. Res. Not. IMRN **2013** (2013), 1800–1872.

[4] C. Bremer and D. S. Sage, "Isomonodromic deformation of connections with singularities of parahoric formal type," Comm. Math. Phys., **313** (2012), 175–208.

[5] C. Bremer and D. S. Sage, "A theory of minimal K-types for flat G-bundles," arXiv:1306.3176[math.AG].

[6] T. Bridgeland and V. Toledano Laredo, "Stability conditions and Stokes factors," Invent. Math. **187** (2012), 61–98.

[7] P. Broussous and S. Stevens, "Buildings of classical groups and centralizers of Lie algebra elements," J. Lie Theory **19** (2009), 55–78.

[8] C. J. Bushnell and P. Kutzko, *The admissible dual of* GL_N *via compact open subgroups,* Annals of Mathematics Studies, **129**, Princeton University Press, Princeton, NJ, 1993.

[9] C. J. Bushnell and P. Kutzko, "Semisimple types in GL_n," Compositio Math. **119** (1999), 53–97.

[10] R. Fedorov, "Algebraic and Hamiltonian approaches to isostokes deformations," Transform. Groups **11** (2006), 137–160.

[11] E. Frenkel and B. Gross, "A rigid irregular connection on the projective line," Ann. of Math. (2) **170** (2009), 1469–1512.

[12] M. Goresky, R. Kottwitz, and R. Macpherson, "Purity of equivalued affine Springer fibers," Represent. Theory **10** (2006), 130–146.

[13] M. Jimbo, T. Miwa, and K. Ueno, "Monodromy preserving deformations of linear ordinary differential equations with rational coefficients I," Physica D **2** (1981), 306–352.

[14] D. Kazhdan and G. Lusztig, "Fixed point varieties on affine flag manifolds," Israel Journal of Mathematics **62** (1988), 129–168.

[15] A. Moy and G. Prasad, "Unrefined minimal K-types for p-adic groups," Invent. Math. **116** (1994), 393–408.
[16] A. Moy and G. Prasad, "Jacquet functors and unrefined minimal K-types," Comment. Math. Helv. **71** (1996), no. 1, 98–121.
[17] T. A. Springer, "Regular elements of finite reflection groups," Invent. Math. **25** (1974), 159–198.
[18] S. Stevens, "Semisimple strata for p-adic classical groups," Ann. Scient. Éc. Norm. Sup., **35** (2002), 423–435.
[19] J. Tits, "Normalisateurs de tores I. Groupes de Coxeters étendus," J. Algebra. **4** (1966), 96-116.

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803
E-mail address: cbremer@math.lsu.edu

E-mail address: sage@math.lsu.edu