Minimal K-types for flat G-bundles, moduli spaces, and isomonodromy

Daniel Sage–joint work with Christopher Bremer

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Overview

New approach to the local theory of flat G-bundles over curves, i.e. formal flat G-bundles, using methods from representation theory: Systematic study of the "leading terms" of the flat structures with respect to Moy-Prasad filtrations

Two main motivations:

- Moduli spaces and the isomonodromy problem for meromorphic flat G-bundles with nondiagonalizable irregular singularities
- The wild ramification case of the geometric Langlands program

A few words about geometric Langlands

k local field, G connected reductive group over k, ${}^LG$ Langlands dual group
Let W_k be the Weil-Deligne group– defined in terms of $\mathrm{Gal}(\bar{k}/k)$.
The local Langlands conjecture asserts a relationship between "admissible" homos $W_k \to {}^LG$ and smooth irreps of $G(k)$.

Geometric Langlands

Replace k by $F = \mathbb{C}((z))$. $\mathrm{Spec}(F) = \Delta^\times$, formal punctured disk
$\mathrm{Gal}(\overline{F}/F) \cong \hat{\mathbb{Z}} = \pi_1(\Delta^\times)$
Naive local Langlands parameters are $\mathrm{Hom}(\hat{\mathbb{Z}}, {}^LG)$, which correspond to ${}^LG$-local systems; equivalently, a flat ${}^LG$-bundle whose connection has a regular singularity at the origin.
There are not enough of these–wild ramification is missing. One must allow irregular singularities as well.
In the global picture, Langlands parameters are meromorphic flat ${}^LG$-bundles over curves.

Flat G-bundles

$X = \mathbb{P}^1(\mathbb{C})$ (for convenience), $\mathcal{O}$ structure sheaf of $\mathbb{P}^1(\mathbb{C})$, K function field (meromorphic functions)
$\Omega^1_{K/\mathbb{C}}$ meromorphic 1-forms
Recall: A flat GL_n-bundle on $\mathbb{P}^1(\mathbb{C})$ is a rank n trivializable vector bundle with a meromorphic connection, i.e., a $\mathbb{C}$-derivation $\nabla : V \to V \otimes_{\mathcal{O}} \Omega^1_{K/\mathbb{C}}$.
If one fixes a trivialization $\phi : V \to V^{\mathrm{triv}}$, then

$$\nabla = d + [\nabla]_\phi, \text{ where } [\nabla]_\phi \in M_n(\Omega^1_{K/\mathbb{C}}).$$

Definition

A flat G-bundle on X is a trivializable principal G-bundle $E \to X$ with an abstract meromorphic connection ∇; equivalently, it is a compatible family of flat vector bundles $(E \times_G W, \nabla_W)$, W f.d. rep of G, with structure group G.

Here, $\nabla = d + [\nabla]_\phi$ with $[\nabla]_\phi \in \Omega^1_{K/\mathbb{C}}(\mathfrak{g})$.

Localization

(E, ∇) flat G-bundle induces formal flat structures at each $y \in \mathbb{P}^1$
Let z be a parameter at y
$\mathfrak{o} = \mathbb{C}[[z]]$ completion of local ring at y, $F = \mathbb{C}((z))$ fraction field,
$\Delta_y^\times = \mathrm{Spec}(F)$ is a formal punctured disk at y
One obtains an induced formal connection $(\hat{E}_y, \hat{\nabla}_y)$ on $\Delta_y^\times$. Note that $[\hat{\nabla}_y] \in \mathfrak{g}(F)\frac{dz}{z}$.
If the singular points are $y_1, \ldots, y_m$, one gets a localization functor $L : \nabla \mapsto (\hat{\nabla}_{y_i})$.
($\hat{\nabla}_y$ is trivial except at the singularities.)
If $[\hat{\nabla}_{y_i}]_\phi$ has a simple pole for some trivialization ϕ, then y_i is a regular singular point. Otherwise, it is irregular.

Gauge and coadjoint actions

Fix a G-invariant nondegenerate symm bilinear form $(,)$ on $\mathfrak{g}$
eg for GL_n, $(X, Y) = \mathrm{Tr}(XY)$
There are two natural actions of $\hat{G} := G(F)$ on $\Omega^1_{F/\mathbb{C}}(\mathfrak{g})$.
$[\hat{\nabla}]$ may be viewed as an element of $\mathfrak{g}(F)^\vee$ via

$$X \mapsto \mathrm{Res}(X, [\hat{\nabla}]), \text{ where } X \in \hat{\mathfrak{g}} := \mathfrak{g}(F).$$

Hence, the coadjoint action makes sense.
Change of trivialization gives rise to gauge change on the connection matrix; this gives the coadjoint action with an additional factor.

$$g \cdot [\hat{\nabla}] = \mathrm{Ad}^*(g)([\hat{\nabla}]) - (dg)g^{-1}, \text{ where } g \in \hat{\mathfrak{g}}.$$

Fact

If we view $[\hat{\nabla}]$ as an element of $\mathfrak{g}(\mathfrak{o})^\vee$ (via restriction), then the coadjoint and gauge actions of $G(\mathfrak{o})$ coincide.

From now on, we usually view $[\hat{\nabla}]$ as a functional on $\mathfrak{g}(\mathfrak{o})$ or suitable subalgebras.

Two functors on the category of flat G-bundles

$$
\begin{array}{ccc}
\left\{\begin{array}{c}\text{flat } G\text{-bundles with}\\ \text{singularities at } y_1, \ldots, y_m\end{array}\right\} & \xrightarrow{M} & \left\{\begin{array}{c}\text{enhanced monodromy}\\ \text{data}\end{array}\right\} \\
\Big\downarrow{\scriptstyle L=\prod L_i} & & \\
\prod_i \left\{\begin{array}{c}\text{formal flat}\\ G\text{-bundles on } \Delta_{y_i}^{\times}\end{array}\right\} & &
\end{array}
$$

Want to study these categories via the geometry of the moduli spaces. In general, these moduli spaces are stacks; to understand, look for better-behaved subcategories of flat G-bundles.

Some problems

1. Find classes of formal isomorphism types for which $L^{-1}((\hat{\nabla}_i))$ are well-behaved moduli spaces.
2. When are such moduli spaces nonempty (Deligne-Simpson problem)? Reduced to a singleton (a version of rigidity)?
3. Investigate the fibers of the monodromy map restricted to reasonable moduli spaces.

Nonresonant case for GL_n (reg semisimple leading term)

$[\hat{\nabla}_y] = (M_{-r}z^{-r} + M_{1-r}z^{1-r} + \dots)\frac{dz}{z}$, $M_i \in \mathfrak{gl}_n(\mathbb{C})$, $M_{-r} \neq 0$.

If M_{-r} is regular semisimple, then $[\hat{\nabla}_y]$ is gauge equivalent to an element of $\mathcal{A}(r)\frac{dz}{z} = \{D_{-r}z^{-r} + \dots + D_0 \mid D_i \text{ diag}, D_{-r} \text{ reg}\}\frac{dz}{z}$.
($\mathcal{A}(r)$ is the set of "formal types").

Consider only connections with nonresonant singularities.

$$\begin{array}{ccc} \widetilde{\mathcal{M}}^{\mathrm{nr}}(\mathbf{r}) & \xrightarrow{M} & \widetilde{\mathcal{S}}^{\mathrm{nr}}(\mathbf{r}) \\ \Big\downarrow {\scriptstyle L=\prod L_i} & & \\ \prod_i \mathcal{A}(r_i) & & \end{array}$$

Results of Boalch (2001) building on Jimbo-Miwa-Ueno (1981)

- The moduli space $\widetilde{\mathcal{M}}^{\mathrm{nr}}(\mathbf{r})$ is a Poisson manifold; its symplectic leaves are the connected components of the fibers of L.
- The fibers of M form an integrable system (solutions of the isomonodromy equations).
- These two foliations are "orthogonal".

Understanding nonresonant connections is not enough. One only gets formal connections with integral slope. The p-adic case suggests that the fractional slope connections will be particularly interesting–eg for GL_n, connections with slope $1/n$ should correspond to supercuspidal representations of loop groups.

Example (Nilpotent leading term)

$$d + \begin{pmatrix} 0 & z^{-(s+1)} \\ z^{-s} & 0 \end{pmatrix} \frac{dz}{z} = d + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z^{-(s+1)} \frac{dz}{z} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} z^{-s} \frac{dz}{z}$$

Slope $s + \frac{1}{2}$. Classical techniques don't work. How to proceed?

Observations on the nonresonant case

- *Look at* $[\hat{\nabla}]$ *with respect to the filtration of* $\mathfrak{gl}_n(F)$ *by* $z^{-r}\,\mathfrak{gl}_n(\mathfrak{o})$.
- *The image of the leading term in the associated graded is non-nilpotent.*
- *The centralizer* $S = Z(M_r z^{-r}) \subset \mathrm{GL}_n(F)$ *is a split maximal torus; the filtration on* $\mathfrak{s}$ *is induced by the filtration on* $\mathfrak{gl}_n(F)$.

Moy-Prasad filtrations

$T \subset B \subset G$, T maximal torus, B Borel subgroup, W Weyl group
$\mathfrak{B}$ the Bruhat-Tits building for G, $\mathcal{A} = \mathcal{A}(T) = X_*(T) \otimes \mathbb{R}$ apartment associated to $T(F)$
V a representation of G, $\hat{V} := V_F = V \otimes_{\mathbb{C}} F$, enough to define filtration for $x \in \mathcal{A}$
For $\lambda \in X^*(T)$, let V_λ be the corresponding weight space
Set $\hat{V}_x(r) = \bigoplus_{\lambda(x)+m=r} V_\lambda z^m$. This gives an $\mathbb{R}$-grading $V \otimes \mathbb{C}[z, z^{-1}] = \bigoplus_{r \in \mathbb{R}} \hat{V}_x(r)$.
For any $r \in \mathbb{R}$, define $\mathfrak{o}$-lattices

$$\hat{V}_{x,r} = \prod_{s \geq r} \hat{V}_x(s) \subset \hat{V}; \quad \hat{V}_{x,r+} = \prod_{s > r} \hat{V}_x(s) \subset \hat{V}.$$

$\{\hat{V}_{x,r} \mid r \in \mathbb{R}\}$ determines the Moy-Prasad filtration on $\hat{V}$;
$\hat{V}_{x,r} \supset \hat{V}_{x,s}$ whenever $s > r$.
Set of critical numbers $\{r \in \mathbb{R} \mid \hat{V}_{x,r} \neq \hat{V}_{x,r+}\}$ is discrete and 1-periodic.

Most important examples: the adjoint and coadjoint reps $\mathfrak{g}$ and $\mathfrak{g}^\vee$.

Moy-Prasad filtrations (cont.)

$\pi : G(\mathfrak{o}) \to G$, $z \mapsto 0$

Definition

An Iwahori subgroup is a $G(F)$-conjugate of $\pi^{-1}(B)$. A parahoric subgroup (G semisimple) is a subgroup containing an Iwahori subgroup. Iwahori (parahoric) subalgebras defined similarly.

Facets in $\mathfrak{B}$ correspond to parahorics. If $\hat{\mathfrak{g}}_x$ is the parahoric for $x \in \mathfrak{B}$, then $\hat{\mathfrak{g}}_{x,0} = \hat{\mathfrak{g}}_x$. One can also define an $\mathbb{R}_{\geq 0}$-filtration on the parahoric subgroup $\hat{G}_x = \hat{G}_{x,0}$; $\hat{G}_{x,0+}$ is the prounipotent radical.

The pairing $\langle X, Y \rangle = \operatorname{Res}(X, Y) \frac{dz}{z}$ induces

$$(\hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+})^\vee \cong \hat{\mathfrak{g}}_{x,-r}/\hat{\mathfrak{g}}_{-x,-r+}.$$

Also, for $r > 0$, there is a natural isomorphism

$$\hat{G}_{x,r}/\hat{G}_{x,r+} \cong \hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+}.$$

Lattice chain filtrations, $G = GL_n$

Assume $G = GL_n$.

Definition

A lattice chain $\mathcal{L}$ in F^n is a "periodic" (with period e), decreasing chain of $\mathfrak{o}$-lattices $(L^i)_{i\in\mathbb{Z}}$: $L^i \supsetneq L^{i+1}$, and $L^{i+e} = zL^i$.

$P = \mathrm{Stab}_{\mathrm{GL}_n(F)}(\mathcal{L})$ is a parahoric subgroup.
$\mathfrak{P} := \mathrm{Lie}(P) = \{x \in \mathfrak{gl}_n(F) \mid x(L_i) \subset L_i \text{ for all } i\}$.

One gets a natural filtration of $\mathfrak{gl}_n(F)$ (resp. P) by congruence subalgebras (resp. subgroups).
Congruent subalgebras: $\mathfrak{P}^k = \{x \in \mathfrak{gl}_n(F) \mid x(L_i) \subset L_{i+k}\, \forall i\}$.
Congruent subgroups: $P^k = \mathrm{Id} + \mathfrak{P}^k$ for $k \geq 1$.

Let x be the barycenter of the simplex in the reduced building corresponding to P.
Then $\mathfrak{P}^k = \hat{\mathfrak{g}}_{x,k/e}$ and $P^k = \hat{G}_{x,k/e}$; in particular, $\mathfrak{P} = \hat{\mathfrak{g}}_x$.

Fundamental strata

In p-adic representation theory, fundamental strata (or minimal K-types) were introduced by Bushnell and Kutzko (GL_n) and Moy and Prasad.

Definition

- A stratum (x, r, β) consists of $x \in \mathfrak{B}$, a real number $r \geq 0$, and a functional $\beta \in (\hat{\mathfrak{g}}_{x,r}/\hat{\mathfrak{g}}_{x,r+})^\vee$.
- (x, r, β) is fundamental if every representative $\tilde{\beta} \in \hat{\mathfrak{g}}_{x,-r}$ of β is non-nilpotent. If $x \in \mathcal{A}$, enough to check that the unique graded representative is non-nilpotent.
- Two fundamental strata (x, r, β), (x', r', β') are associate if $r = r'$ and (for $r > 0$) the $G(F)$-orbits of $\tilde{\beta} + \hat{\mathfrak{g}}_{x,-r+}$ and $\tilde{\beta}' + \hat{\mathfrak{g}}_{x',-r'+}$ intersect.

For lattice chain filtrations of $GL_n(F)$, write (P, r, β).
Moy-Prasad: Every irreducible admissible representation W of a p-adic group contains a minimal K-type. Any such has the same depth, allowing one to define the depth of W.

Fundamental strata give the correct notion of the leading term of a formal flat G-bundle.

Definition

The formal flat G-bundle $\hat{\nabla}$ contains the stratum (x, r, β) (for $r > 0$) if $\mathrm{Res}([\hat{\nabla}], \mathfrak{g}_{x,r+}) = 0$ and $[\hat{\nabla}]$ induces the same functional as β on $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$.

For $G = \mathrm{GL}_n$, $r > 0$, $\hat{\nabla}$ contains (P, r, β) iff $\iota_{z\partial_z}[\hat{\nabla}]$ and $\tilde{\beta}$ induce the same endomorphism of the associated graded space $\bigoplus L^j/L^{j+1}$.

Examples

- $[\hat{\nabla}] = \left(z^{-r}M_{-r} + z^{-r+1}M_{1-r} + \mathrm{h.o.t.}\right)\frac{dz}{z}$ with $M_i \in \mathfrak{g}$.
  $\hat{\nabla}$ contains the G-stratum (x, r, β), where $x \in \mathfrak{B}$ is the vertex corresponding to $G(\mathfrak{o})$, $\beta \in (z^r\mathfrak{g}(\mathfrak{o})/z^{r+1}\mathfrak{g}(\mathfrak{o}))^\vee$ is induced by $z^{-r}M_{-r}\frac{dz}{z}$, fundamental if M_{-r} is non-nilpotent.
- $\hat{V} = F^2$, $[\hat{\nabla}] = \begin{pmatrix} 0 & z^{-(s+1)} \\ z^{-s} & 0 \end{pmatrix}\frac{dz}{z}$.
  Here, $(\hat{V}, \hat{\nabla})$ contains the fundamental GL_2-stratum $(I, s + \frac{1}{2}, \beta)$, where $I \subset \mathrm{GL}_2(\mathfrak{o})$ is the standard Iwahori subgroup, $\beta \in (\mathfrak{I}^{2s+1}/\mathfrak{I}^{2s})^\vee$.

Theorem (Bremer-S. 2013b, 2014)

Every formal flat G-bundle $\hat{\nabla}$ contains a fundamental stratum (x, r, β) with x an optimal point (so $r \in \mathbb{Q}$); the depth r is positive iff $\hat{\nabla}$ is irregular singular. Moreover,

- *If $\hat{\nabla}$ contains a stratum (x', r', β'), then $r' \geq r$.*
- *If $r > 0$, (x', r', β') is fundamental if and only if $r' = r$.*
- *Any two fundamental strata contained in $\hat{\nabla}$ are associate.*

We can now define the slope of $\hat{\nabla}$ as this minimal depth.

Theorem (Bremer-S, 2013b)

The slope of the formal flat G-bundle $(\hat{E}, \hat{\nabla})$ is a nonnegative rational number. It is positive if and only if $(\hat{E}, \hat{\nabla})$ is irregular singular. The slope may also be characterized as

1. *the maximum slope of the associated flat connections; or*
2. *the maximum slope of the flat connections associated to the adjoint representations and the characters.*

Other defs of slope by Frenkel-Gross and Chen-Kamgarpour.

Regular strata

Need stronger condition on strata to get nice moduli spaces.
Let $S \subset G(F)$ be a (possibly non-split) maximal torus. There is a unique Moy-Prasad filtration $\{\mathfrak{s}_r\}$ on $\mathfrak{s} = \mathrm{Lie}(S)$.

Definition

A point $x \in \mathfrak{B}$ is compatible with $\mathfrak{s}$ if $\mathfrak{s}_r = \hat{\mathfrak{g}}_{x,r} \cap \mathfrak{s}$ for all r.

Definition

A fundamental stratum (x, r, β) is a regular stratum centralized by S if x is compatible with $\mathfrak{s}$ and for any representative $\tilde{\beta} \in \mathfrak{g}_{x,-r}$ of β, $\mathrm{Stab}_{G(F)}(\tilde{\beta})$ is a $\hat{G}_{x,0+}$-conjugate of S (for $r > 0$).

$$\left\{\begin{matrix}\text{conj classes maximal} \\ \text{tori in } G(F)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{conj classes} \\ \text{in } W\end{matrix}\right\}$$

Proposition

A torus centralizes a regular stratum (x, r, β) if and only if its class corresponds to a regular conjugacy class in W. In this case, $e^{2\pi i r}$ is a regular eigenvalue of this class.

Regular strata (cont.)

For $G = \mathrm{GL}_n$, S is regular if it is uniform i.e., $S = (E^\times)^k$ for some field extension E/F, or if it is of the form $S' \times \mathbb{C}^*$ where S' is uniform for GL_{n-1}.

Examples

- Take $M_{-r} \in \mathfrak{g}$ regular semisimple, let $T = Z_G(M_{-r})$, and let $x \in \mathfrak{B}$ be the vertex corresponding to $G(\mathfrak{o})$. Then $(x, r, z^{-r} M_{-r} \frac{dz}{z})$ is a regular stratum centralized by $T(F)$.

- The Frenkel-Gross rigid flat G-bundle is S-regular of slope $1/h$, where h is the Coxeter number and S corresponds to the Coxeter element in W.
  Explicitly for $G = \mathrm{GL}_2$: Let $\omega = \left(\begin{smallmatrix} 0 & 1 \\ z & 0 \end{smallmatrix}\right)$, so $S = \mathbb{C}((\omega))^*$ is a non-split maximal torus in $\mathrm{GL}_2(F)$. Then, $(I, \frac{1}{2}, \left(\begin{smallmatrix} 0 & z^{-1} \\ 1 & 0 \end{smallmatrix}\right) \frac{dz}{z})$ is a regular stratum centralized by S (as are the previous GL_2 examples).

Formal types

The set $\mathcal{A}(S, r)$ of S-formal types of depth r is an open subset of the affine space $(\mathfrak{s}_0/\mathfrak{s}_{r+})^\vee \cong \mathfrak{s}_{-r}/\mathfrak{s}_{0+}$. (It can also be interpreted as a subset of $\mathfrak{s}_{-r}$.)

Examples

- $T \subset G$ split,
  $\mathcal{A}(T(F), r) = \{D_{-r}z^{-r} + \cdots + D_0 \mid D_i \in \mathfrak{t}, D_{-r} \text{ reg}\}$.
- $S = \mathbb{C}((\omega)) \subset \mathrm{GL}_2(F)$
  $\mathcal{A}(S, s + 1/2) = \{\text{deg } 2s + 1 \text{ polys in } \omega^{-1}\}$.

Theorem (Bremer-S. 2013c)

If $\hat{\nabla}$ contains the regular stratum (x, r, β) centralized by S, then $[\hat{\nabla}]$ is $\hat{G}_{x,0+}$-gauge equivalent to a unique elt of $\mathcal{A}(S, r)\frac{dz}{z}$ with "leading term" β.

We call the element of $\mathcal{A}(S, r)$ a formal type for $\hat{\nabla}$.
The formal type determines the formal isomorphism class.

Formal types vs formal isomorphism classes

$W_S = N(S)/S$, $W_S^{\mathrm{aff}} = N(S)/S_0 \cong W_S \ltimes S/S_0$ relative Weyl and affine Weyl groups

The gauge action of $N(S)$ induces a natural action of W_S^{aff} on $\mathcal{A}(S,r)$.

Let $\mathcal{C}(S,r)$ be the full subcategory of rank n formal connections $(\hat{V},\hat{\nabla})$ containing a regular stratum with formal type in $\mathcal{A}(S,r)$.

One can construct a "framed" version $\mathcal{C}^{\mathrm{fr}}(S,r)$ of this category, together with a forgetful "deframing" functor $\mathcal{C}^{\mathrm{fr}}(S,r) \to \mathcal{C}(S,r)$.

Theorem (Bremer-S 2013c)

This functor induces the quotient map $\mathcal{A}(S,r) \to \mathcal{A}(S,r)/W_S^{\mathrm{aff}}$ on moduli spaces.

Framable connections ($G = \mathrm{GL}_n$)

∇ global flat G-bundle; fix a trivialization ϕ.
Assume $\hat{\nabla}_y$ has formal type A_y.

Definition

$g \in G$ is a compatible framing for ∇ at y if $g \cdot [\hat{\nabla}_y]$ has the same leading term as $A_y \frac{dz}{z}$. If such a g exists, ∇ is framable at y.

$g \circ \phi$ is a global trivialization which makes the leading term of $[\hat{\nabla}_y]$ match the leading term of $A_y \frac{dz}{z}$.

Example

$P = G(\mathfrak{o})$, $A_y = D_{-r} z^{-r} + \cdots + D_0$
$g \cdot [\hat{\nabla}_y] = (D_{-r} z^{-r} + M_{1-r} z^{1-r} + \mathrm{h.o.t.}) \frac{dz}{z}$.

Moduli spaces ($G = \mathrm{GL}_n$)

Starting data

- $\{y_i\}$ *irregular singular points*
- $\mathbf{A} = (A_i)$ *collection of S_i-formal types at y_i (which determine regular strata (P_i, r_i, β_i) at each y_i), each S_i uniform.*

Let $\mathcal{C}(\mathbf{A})$ be the category of framable connections (V, ∇) with formal types $\mathbf{A}$:

- V is a trivializable rank n vector bundle on $\mathbb{P}^1$;
- ∇ is a mero. connection on V with sing. points only at $\{y_i\}$;
- ∇ is framable and has formal type A_i at y_i.

The morphisms are vector bundle maps compatible with the connections.

$\mathcal{M}(\mathbf{A})$ is the corresponding moduli space.

Note that if two framable connections are isomorphic as meromorphic connections (i.e. as D-modules), then they are isomorphic as framable connections. Thus, $\mathcal{M}(\mathbf{A})$ is a subspace of the moduli stack of meromorphic connections.

Variants

There are also moduli spaces $\widetilde{\mathcal{M}}(\mathbf{A})$ (resp. $\widetilde{\mathcal{M}}(\mathbf{S}, \mathbf{r})$) of framed connections with fixed formal types (resp. fixed regular combinatorics), which include data of compatible framings.

One can also allow additional regular singular points $\{q_j\}$; formal isomorphism classes are given by coadjoint orbit of the residue $\mathrm{res}_{q_j}([\hat{\nabla}]) := [\hat{\nabla}]|_{\mathfrak{gl}_n(\mathbb{C})}$.
If $\mathbf{B} = (\mathcal{O}^j)$ collection of nonresonant coadjoint orbits in $\mathfrak{gl}_n(\mathbb{C})^\vee$, can construct $\mathcal{M}(\mathbf{A}, \mathbf{B})$ etc; here, ∇ has residue at q_j in $\mathcal{O}^j$.

Symplectic and Poisson reduction

We will construct these moduli spaces via symplectic (or Poisson) reduction of a symplectic (Poisson) manifold which is a direct product of local pieces. This is a result of Boalch (2001) in the case of regular diagonalizable leading terms.

Setup

- *X symplectic mfld with Hamiltonian action of the group G*
- *$\mu : X \to \mathfrak{g}^\vee$ the moment map*
- *$\alpha \in \mathfrak{g}^\vee$ is a singleton coadjoint orbit.*

Definition

The symplectic reduction $X /\!\!/_\alpha G$ is defined to be the quotient $\mu^{-1}(\alpha)/G$.

Fact

If $\mu^{-1}(\alpha)/G$ is smooth, then the symplectic structure on X descends to $X /\!\!/_\alpha G$.

Poisson reduction is analogous.

Local pieces

A a formal type with parahoric P. A can be viewed as an elt of $\mathfrak{P}^\vee$; let $\mathcal{O}_A$ be the P-coadjoint orbit.
Associated parabolic to P: $P/z\,\mathrm{GL}_n(\mathfrak{o}) \cong Q \subset \mathrm{GL}_n(\mathbb{C})$

Let $\mathcal{M}(A) \subset (Q\backslash \mathrm{GL}_n(\mathbb{C})) \times \mathfrak{gl}_n(\mathfrak{o})^\vee$ be the subvariety

$$\mathcal{M}(A) = \{(Qg, \alpha) \mid (\mathrm{Ad}^*(g)(\alpha))|_{\mathfrak{P}} \in \mathcal{O}_A)\}.$$

$GL_n(\mathbb{C})$ acts on $\mathcal{M}(A)$ via $h(Qg, \alpha) = (Qgh^{-1}, Ad^*(h)\alpha)$.

Proposition

$\mathcal{M}(A)$ is a symplectic manifold, and the $\mathrm{GL}_n(\mathbb{C})$-action is Hamiltonian with moment map $(Qg, \alpha) \mapsto \mathrm{res}(\alpha) := \alpha|_{\mathfrak{gl}_n(\mathbb{C})}$.

$\mathcal{M}(A_i)$ encodes the local data of $\nabla \in \mathcal{M}(\mathbf{A})$ at y_i.
There are similar local manifolds $\widetilde{\mathcal{M}}(A)$ and $\widetilde{\mathcal{M}}(P, r)$ (symplectic and Poisson respectively) corresponding to the other moduli spaces.

Structure of the moduli spaces

Theorem (Bremer-S. 2013a)

1. *The moduli space* $\widetilde{\mathcal{M}}(\mathbf{A},\mathbf{B})$ *is a symplectic manifold obtained as a symplectic reduction of the product of local data:*

$$\widetilde{\mathcal{M}}(\mathbf{A},\mathbf{B}) \cong \left[\left(\prod_i \widetilde{\mathcal{M}}(A_i)\right) \times \left(\prod_j \mathcal{O}^j\right)\right] /\!\!/_0 \, \mathrm{GL}_n(\mathbb{C}).$$

2. *The moduli space* $\mathcal{M}(\mathbf{A},\mathbf{B})$ *may be constructed in a similar way. Moreover, it is the symplectic reduction of* $\widetilde{\mathcal{M}}(\mathbf{A},\mathbf{B})$ *via a torus action.*

The condition that the moment map take value 0 just says that the sum of the residues over all singular points is 0.

These results and those on the next slide are due to Boalch (2001) in the case where all irregular formal types have regular semisimple leading term.

Theorem (Bremer-S. 2012)

1. *The space* $\widetilde{\mathcal{M}}(\mathbf{P}, \mathbf{r})$ *is a Poisson manifold obtained by Poisson reduction of the product of local pieces.*
2. *The fibers of the localization map* L *are the* $\widetilde{\mathcal{M}}(\mathbf{A})$*.*
3. *The symplectic leaves are the connected components of the* $\widetilde{\mathcal{M}}(\mathbf{A})$*'s.*

Theorem (Bremer-S. 2012)

There is an explicitly defined, Frobenius integrable Pfaffian system $\mathcal{I}$ *on* $\widetilde{\mathcal{M}}(\mathbf{P}, \mathbf{r})$ *such that the solution leaves of* $\mathcal{I}$ *correspond to the fibers of the monodromy map* M*. The independent variables of this system are the coefficients of the formal types.*

Some rigid connections

Let $x=0$, $y=\infty$. Let the formal type at 0 be the simplest possible Iwahori type $A=\omega^{-1}$. Let $\mathcal{O}$ be any nonresonant adjoint orbit at ∞.

Proposition (Bremer-S)

$\mathcal{M}(A,\mathcal{O})$ is a singleton when $\mathcal{O}$ is regular and empty otherwise. Thus, one obtains a family of rigid connections including the Frenkel-Gross example.

Idea of proof when $\mathcal{O}$ irregular ($n=3$)

- Let $X=\{\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & y & 0 \end{pmatrix}+b \mid x,y\in\mathbb{C}^*, b\in\mathfrak{b}\cap\mathfrak{sl}_3(\mathbb{C})\}$.
- The moment map conditions imply that $\mathcal{M}(A,\mathcal{O})$ is the set of B orbits in the set $X\cap\mathcal{O}$.
- All elements of X are regular, so if $\mathcal{O}$ is not regular, the moduli space is empty.

References

- C. Bremer and D. S. Sage, *Moduli spaces of irregular singular connections*, *Int. Math. Res. Not. IMRN* **2013** (2013), 1800–1872.
- C. Bremer and D. S. Sage, *Isomonodromic deformations of connections with singularities of parahoric formal type*, *Comm. Math. Phys.* **313** (2012), 175–208.
- C. Bremer and D. S. Sage, *A theory of minimal K-types for flat G-bundles*, arXiv:1306.3176[math.AG].
- C. Bremer and D. S. Sage, *Flat G-bundles and regular strata for reductive groups*, arXiv:1309.6060[math.AG].