GENERALIZED SERRE CONDITIONS AND PERVERSE COHERENT SHEAVES

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ABSTRACT. In algebraic geometry, one often encounters the following problem: given a scheme X, find a proper birational morphism $Y \to X$ where the geometry of Y is "nicer" than that of X. One version of this problem, first studied by Faltings, requires Y to be Cohen-Macaulay; in this case $Y \to X$ is called a Macaulayfication of X. In another variant, one requires Y to satisfy the Serre condition S_r. In this paper, the authors introduce generalized Serre conditions–these are local cohomology conditions which include S_r and the Cohen-Macaulay condition as special cases. To any generalized Serre condition S_ρ, there exists an associated perverse t-structure on the derived category of coherent sheaves on a suitable scheme X. Under appropriate hypotheses, the authors characterize those schemes for which a canonical finite S_ρ-ification exists in terms of the intermediate extension functor for the associated perversity. Similar results, including a universal property, are obtained for a more general morphism extension problem called S_ρ-extension.

1. INTRODUCTION

In algebraic geometry, one often encounters the following problem: given a scheme X, find a proper birational morphism $Y \to X$ where the geometry of Y is "nicer" than that of X. The strongest version of this problem is the resolution of singularities. On the other hand, there are many weaker variations expressed in terms of local cohomology. For example, one might require Y to satisfy Serre's condition S_2. In another version, introduced by Faltings [4], Y is required to be Cohen-Macaulay; $Y \to X$ is then called a Macaulayfication of X. Kawasaki has shown that Macaulayfications exist for a broad class of schemes [7], and they can often be constructed in contexts where desingularizations do not exist.

In general, Macaulayfications are not canonical. However, it was shown in [1] that there may exist a finite Macaulayfication, restricting to an isomorphism over the Cohen-Macaulay locus, that satisfies an appropriate universal property. This is a special case of a more general morphism extension problem.

Consider the following diagram, where U is an open dense subscheme of X and ζ_1 is a finite dominant morphism.

$$
\begin{array}{ccc}
\tilde{U} & \lhook\joinrel\dashrightarrow & \tilde{X} \\
{\scriptstyle \zeta_1}\big\downarrow & & \big\downarrow{\scriptstyle \zeta} \\
U & \stackrel{j}{\lhook\joinrel\longrightarrow} & X
\end{array}
$$

In [1], Achar and Sage investigated the problem of constructing an "S_2-extension" of (X, ζ_1): this is a scheme $\tilde{X}$ together with a finite morphism $\zeta : \tilde{X} \to X$ such that $\tilde{X}$ contains $\tilde{U}$ as an open subscheme, ζ extends ζ_1, $\tilde{X}$ is S_2 off of $\tilde{U}$, and $(\tilde{X}, \zeta)$ satisfies an appropriate universal property. If a pair satisfies all conditions except for the universal property, it is called a weak S_2-extension. They applied the theory of perverse coherent sheaves to show that the S_2-extension exists under suitable hypotheses (for example, the "componentwise codimension" of the complement of U must be at least 2) . When ζ_1 is the identity, S_2-extension gives a canonical S_2-ification, which restricts to an isomorphism over U. Achar and Sage used similar techniques to show that a canonical finite Macaulayfication exists when a certain perverse coherent sheaf is defined and a sheaf (i.e., concentrated in degree 0). Moreover, in this case, the finite Macaulayfication coincides with the S_2-ification. This last fact was first observed in a local ring context by Schenzel [8].

In this paper, we strengthen and generalize these results. A summary of the theory of perverse coherent sheaves appears in Section 2. In Section 3, we introduce "generalized Serre conditions"–these are local cohomology conditions which include the Serre conditions S_r and the Cohen-Macaulay condition as special cases. A generalized Serre condition is defined in terms of a function $\rho : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ which has slope at most one and satisfies $\rho(0) = 0$. These conditions have a close relationship to perversities in the theory of coherent sheaves, and we show how to associate an S_ρ-perversity to the pair (X, U) in Definition 3.6. We also provide an example of an S_ρ variety that is "strictly S_ρ through codimension n". We then investigate the S_ρ-extension problem in Section 4 and show that under appropriate assumptions, an S_ρ-extension exists if and only if a certain intermediate extension sheaf (with respect to the S_ρ-perversity) exists and is a sheaf (Theorem 4.4). If it exists, it is the only weak S_ρ-extension; moreover, it coincides with the S_2-extension. This result is applied to the finite S_ρ-ification problem in Theorem 4.13. We obtain similar results when the codimension condition is relaxed, although here stronger hypotheses are required.

2. Serre conditions and perverse coherent sheaves

Throughout the paper, X will be a semi-separated scheme of finite type over a Noetherian base scheme S admitting a dualizing complex. We will further assume that X is equidimensional. For any $x \in X$, we will write $\bar{x} \subset X$ for the subscheme corresponding to the closure of $\{x\}$. We will write codim x for codim $\bar{x}$. If Z is a closed subscheme of X, we will let c-codim(Z) denote the "componentwise codimension" (or "c-codimension") of Z in X; if the X_i's are the irreducible components of X, then

$$
\text{c-codim}(Z) = \min_{Z \cap X_i \neq \varnothing} \text{codim}_{X_i} Z \cap X_i.
$$

Let $\mathfrak{Coh}(X)$ be the category of coherent sheaves on X, and let $\mathcal{D}(X)$ be the bounded derived category of $\mathfrak{Coh}(X)$. Suppose that $i_x : \{x\} \to X$ is the inclusion of a point. If $\mathcal{F} \in \mathfrak{Coh}(X)$, we define $i_x^*(\mathcal{F})$ to be the stalk of $\mathcal{F}$ at x. Since i_x^* is exact, it induces an exact functor from $\mathcal{D}(X)$ to the bounded derived category of $\mathcal{O}_{X,x}$-modules (written $\mathcal{D}(\mathcal{O}_{X,x})$). We define $\Gamma_x(\mathcal{F})$ to be the submodule of $i_x^*(\mathcal{F})$ consisting of sections with support on $\overline{\{x\}}$. This is a left exact functor, and we write $i_x^! : \mathcal{D}(X) \to \mathcal{D}(\mathcal{O}_{X,x})$ for the corresponding derived functor.

Recall that the *depth* of a coherent sheaf $\mathcal{F}$ at y is defined to be r if $H^r(i_x^! \mathcal{F})$ is the first non-vanishing local cohomology sheaf; in other words, $H^r(i_x^! \mathcal{F}) \neq 0$ and $H^k(i_x^! \mathcal{F}) = 0$ for $k < r$. We will denote the depth of $\mathcal{F}$ at y by $\mathrm{depth}_y(\mathcal{F})$.

Definition 2.1. Let r be a positive integer. We say that a coherent sheaf $\mathcal{F} \in \mathfrak{Coh}(X)$ is S_r if $\mathrm{depth}_x(\mathcal{F}) = \dim_x(\mathcal{F})$ for all $x \in X$ satisfying $\dim_x(\mathcal{F}) \leq r$. The sheaf $\mathcal{F}$ is Cohen-Macaulay if $\mathrm{depth}_x(\mathcal{F}) = \dim_x(\mathcal{F})$ for all $x \in X$.

Now, we recall the Deligne-Bezrukavnikov theory of perverse coherent sheaves and its extension by Achar and Sage [2, 1].

Definition 2.2. A *perversity* is a function $p : X \to \mathbb{Z}$ satisfying

$$p(y) \geq p(x) \qquad \text{and}$$
$$\mathrm{codim}(y) - p(y) \geq \mathrm{codim}(x) - p(x) \qquad \text{whenever } \mathrm{codim}(y) \geq \mathrm{codim}(x).$$

(In particular, $p(x)$ only depends on $\mathrm{codim}(x)$.) Given a perversity p, we define the dual perversity $\bar{p}$ by $\bar{p}(x) = \mathrm{codim}(\bar{x}) - p(x)$.

By [2, Theorem 3.10], a perversity determines two full subcategories, $^{\mathrm{P}}\mathcal{D}(X)^{\leq 0}$ and $^{\mathrm{P}}\mathcal{D}(X)^{\geq 0}$, such that $(^{\mathrm{P}}\mathcal{D}(X)^{\leq 0}, {}^{\mathrm{P}}\mathcal{D}(X)^{\geq 0})$ is a t-structure on $\mathcal{D}(X)$. Specifically,

$$^{\mathrm{P}}\mathcal{D}(X)^{\leq 0} = \{\mathcal{F} \in \mathcal{D}(X) \mid \forall x \in X, H^k(i_x^*(\mathcal{F})) = \{0\} \text{ whenever } k > p(x)\}$$
$$^{\mathrm{P}}\mathcal{D}(X)^{\geq 0} = \{\mathcal{F} \in \mathcal{D}(X) \mid \forall x \in X, H^k(i_x^!(\mathcal{F})) = \{0\} \text{ whenever } k < p(x)\}.$$

We call this t-structure the *perverse t-structure* with respect to the perversity p. There are associated truncation functors $\tau_{\leq 0}^{\mathrm{P}} : \mathcal{D}(X) \to {}^{\mathrm{P}}\mathcal{D}(X)^{\leq 0}$ and $\tau_{\geq 0}^{\mathrm{P}} : \mathcal{D}(X) \to {}^{\mathrm{P}}\mathcal{D}(X)^{\geq 0}$. The heart of the t structure, denoted $\mathcal{M}^p(X)$, is the category of *perverse coherent sheaves* with respect to p. For example, if p is the trivial perversity defined by $p(x) = 0$ for all $x \in X$, then one easily sees that $\mathcal{M}^p(X)$ is the usual category of coherent sheaves on X.

One of the powerful tools in the theory of perverse coherent sheaves is the intermediate extension functor. Suppose that $U \subset X$ is an open dense subscheme of X, and let $Z = X \setminus U$. In certain cases, there exists an intermediate extension functor $\mathcal{IC}^p$ which defines an equivalence between a subcategory of $\mathcal{M}^p(U)$ and a subcategory of $\mathcal{M}^p(X)$. The definition of $\mathcal{IC}^p$ requires the construction of two new perversities associated to p, denoted p^+ and p^-, which depend on the pair (X, U) (although this dependence will be suppressed in the notation). In order to ensure that the domain of $\mathcal{IC}^p$ is non-empty, we will eventually need to impose conditions on the perversity p as well as on c-codim(Z).

Definition 2.3. Fix a pair (X, U) as above. Let p be a perversity on X, and let z be any point in X such that $\mathrm{codim}(z) = \text{c-codim}(Z)$. We define

$$p^-(x) = \begin{cases} p(x) - 1 & \text{if } p(x) \geq p(z), \\ p(x) & \text{if } p(x) < p(z), \end{cases}$$

and

$$p^+(x) = \begin{cases} p(x) + 1 & \text{if} \quad \text{codim}(x) - p(x) \geq \text{codim}(z) - p(z), \\ p(x) & \text{if} \quad \text{codim}(x) - p(x) < \text{codim}(z) - p(z). \end{cases}$$

Additionally, we define $\mathcal{M}_{\pm}^{\mathrm{p}}(U)$ and $\mathcal{M}_{\pm}^{\mathrm{p}}(X)$ to be the full subcategories

$$\mathcal{M}_{\pm}^{\mathrm{p}}(U) = {}^{p^-}\mathcal{D}(U)^{\leq 0} \cap {}^{p^+}\mathcal{D}(U)^{\geq 0} \subset \mathcal{M}^p(U),$$

$$\mathcal{M}_{\pm}^{\mathrm{p}}(X) = {}^{p^-}\mathcal{D}(X)^{\leq 0} \cap {}^{p^+}\mathcal{D}(X)^{\geq 0} \subset \mathcal{M}^p(X).$$

Proposition 2.4 ([1, Proposition 2.3])**.** *Let* $j : U \to X$ *be the inclusion map. Then,* $j^* : \mathcal{M}_{\pm}^{\mathrm{p}}(X) \to \mathcal{M}_{\pm}^{\mathrm{p}}(U)$ *is an equivalence of categories.*

Definition 2.5. The intermediate extension functor $\mathcal{IC}^p(X, \cdot) : \mathcal{M}_{\pm}^{\mathrm{p}}(U) \to \mathcal{M}_{\pm}^{\mathrm{p}}(X)$ is defined as the inverse equivalence to that of Proposition 2.4.

Remark 2.6. By [1, Remark 2.7], if U is irreducible, then the category $\mathcal{M}_{\pm}^{\mathrm{p}}(U)$ reduces to the zero object whenever $\text{codim}(Z) \leq \text{codim}(U) + 1$.

From now on, unless otherwise mentioned, we will assume that U is an open dense subset of X and that c-codim $Z \geq 2$. These conditions are needed for many of the results we use from [1]. Moreover, we will primarily consider *standard* perversities on X.

Definition 2.7. We say that a perversity p on (X, U) is *standard* if

$$p(x) = p^-(x) = p^+(x) = 0 \quad \text{if} \quad \text{codim}(x) = 0.$$

There are unique maximal and minimal standard perversities on X defined by:

$$s(x) = \begin{cases} 0, \\ 1, \end{cases} \quad c(x) = \begin{cases} \text{codim}(x) & \text{if codim}(x) < \text{c-codim}(Z), \\ \text{codim}(x) - 1 & \text{if codim}(x) \geq \text{c-codim}(Z). \end{cases}$$

Lemma 2.8 ([1, Lemma 3.3])**.** *Every standard perversity* p *satisfies* $s(x) \leq p(x) \leq c(x)$ *for all* $x \in X$.

Remark 2.9. If p is any standard perversity, then $p^- \geq 0$. It follows that any coherent sheaf $\mathcal{F}$ is contained in ${}^{p^-}\mathcal{D}(X)^{\leq 0}$.

Remark 2.10. The complex $\mathcal{IC}^s(X, \mathcal{F})$ is automatically a sheaf if it is defined. Indeed, it is just $j_* \mathcal{F}$ by [1, Proposition 3.7].

3. Generalized Serre Conditions

3.1. **The conditions** S_ρ**.** In this section, we introduce a class of local cohomology conditions which generalize Serre's conditions S_r and the Cohen-Macaulay condition. We also show their connection to perversity functions.

Let W' be the set of weakly increasing functions $\rho : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $\rho(0) = 0$ and $\rho(k+1) - \rho(k) \leq 1$ for all k. Note that W' is a lattice with respect to the usual partial order with the identity (resp. the zero function) as the maximum (resp. minimum). If we set $\rho_r(k) = \min(k, r)$, then $\{\rho_r\}$ is an increasing sequence whose supremum is id. Let $W = \{\rho \in W' \mid \rho \geq \rho_2\}$.

Definition 3.1. Given $\rho \in W'$, we say that $\mathcal{F} \in \mathfrak{Coh}(X)$ is S_ρ at x if $H^k(i_x^!(\mathcal{F})) = \{0\}$ for $k < \rho(\dim_x(\mathcal{F}))$; $\mathcal{F}$ is S_ρ if it is S_ρ at x for all $x \in X$.

Note this says that $\mathcal{F}$ is S_ρ at x if and only if $\operatorname{depth}_x(\mathcal{F}) \geq \rho(\dim_x(\mathcal{F}))$ for all $x \in X$. In particular, S_{ρ_r} is the usual condition S_r while a sheaf is S_{id} if and only if it is Cohen-Macaulay. Elements of W correspond to local cohomology conditions which are at least as strong as S_2.

To see the relationship between these conditions and perversities, we need a numerical version of perversity functions.

Definition 3.2. For $n \geq 2$, a *standard numerical perversity of level n* is a function $\pi : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $\pi(0) = 0$, $0 < \pi(n) < n$, and both π and its dual $\bar{\pi} \overset{\text{def}}{=} \mathrm{id} - \pi$ are nondecreasing. The set of all such functions is denoted by P_n; these satisfy $P_n \subset P_m$ if $n \leq m$. A standard numerical perversity is an element of $P = \cup_{n \geq 2} P_n$.

Note that $0 \leq \pi(k+1) - \pi(k) \leq 1$ for any $\pi \in P$.

Given (X, U), any element $\pi \in P_{\text{c-codim } Z}$ induces a standard perversity $p_\pi = \pi \circ \operatorname{codim} : X \to \mathbb{Z}_{\geq 0}$. Conversely, any perversity comes from a (non-unique) element of $P_{\text{c-codim } Z}$.

For $\pi \in P_n$, we set

$$\pi_n^+(k) = \begin{cases} p(k) + 1 & \text{if } k - p(k) \geq n - p(n), \\ p(k) & \text{if } k - p(k) < n - p(n). \end{cases}$$

We will suppress n from the notation when it is unambiguous. In particular, if we are considering a pair (X, U), then we will always take $n = \text{c-codim}(Z)$. With this convention, we see that $p_{\pi^+} = (p_\pi)^+$.

Given $\rho : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $r(0) = 0$ and $n \geq 2$, let $P_n(\rho)$ be the set of standard numerical perversities $\pi \in P_n$ such that $\pi^+(k) \leq \rho(k)$ for all x with equality if $k \geq n$. Given $\rho \in W$ and $n \geq 2$, let m be the largest index such that $\rho(m) < \rho(n)$. We then define $\hat{\pi}_{\rho,n}, \pi_{\rho,n} \in P_n$ via

$$\hat{\pi}_{\rho,n}(k) = \begin{cases} \rho(k), & \text{if } k \leq m, \\ \rho(k) - 1, & \text{if } k \geq m + 1, \end{cases} \qquad \pi_{\rho,n}(k) = \begin{cases} \max(k - (n - \rho(n) + 1), 0), & \text{if } k < n, \\ \rho(k) - 1, & \text{if } k \geq n. \end{cases}$$

It is easy to check that these are indeed standard numerical perversities in $P_n(\rho)$. We also define a function $\phi : P_2 \to W$:

$$\phi(\pi)(k) = \begin{cases} k, & \text{if } k \leq 1, \\ \pi(k) + 1, & \text{if } k \geq 2. \end{cases}$$

Proposition 3.3.

(1) *Let $\rho : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be a function such that $\rho(k) = k$ for $k \leq 1$. The set $P_2(\rho)$ is nonempty if and only if $\rho \in W$.*

(2) *For any $\rho \in W$ and $n \geq 2$, $\pi_{\rho,n}$ (resp. $\hat{\pi}_{\rho,n}$) is the unique minimum (resp. maximum) element of $P_n(\rho)$.*

(3) *There exists $\pi \in P_n(\rho)$ such that $\pi_n^+ = \rho$ if and only if $\rho(n-1) < \rho(n)$.*

(4) *The function $\phi : P_2 \to W$ is surjective and two-to-one with $\phi^{-1}(\rho) = P_2(\rho) = \{\pi_{\rho,2}, \hat{\pi}_{\rho,2}\}$. Moreover, the duality map on numerical perversities induces a duality map $\rho \mapsto \bar{\rho}$ on W such that $\phi(\pi_{\rho,2}) = \hat{\pi}_{\bar{\rho},2}$ and $\phi(\hat{\pi}_{\rho,2}) = \pi_{\bar{\rho},2}$.*

Proof. If $\pi \in P_2(\rho)$, then $\pi(k) = \rho(k) - 1$ for $k \geq 2$. Also, $\pi(2) = 1$, so $\rho(2) = 2$. This implies that ρ is nondecreasing and $\rho \geq \rho_2$. Moreover, since $\pi(k+1) - \pi(k) \leq 1$,

the same holds for ρ. Thus, $\rho \in W$. The converse holds, since $\pi_{\rho,2} \in P_2(\rho)$ for $\rho \in W$.

For the second part, note that if $\pi \in P_n(\rho)$, then $\pi(n) = \rho(n) - 1$. If $\pi(k) < k - (n - \rho(n) + 1)$ for $n - \rho(n) + 1 \leq k < n$, then $\pi(n) < \rho(n) - 1$, a contradiction, so $\pi \geq \pi_{\rho,n}$. On the other hand, $\pi \leq \rho$, so if $\pi \not\leq \hat{\pi}_{\rho,n}$, there exists k such that $m < k < n$ such that $\pi(k) = \rho(k) = \rho(n)$. Since π is nondecreasing, we obtain $\pi(n) \geq r(n)$, which is again a contradiction.

Note that $\hat{\pi}_{\rho,n}^+(k) = \rho(k)$ except for $m < k < n$, so $\hat{\pi}_{\rho,n}^+(k) = \rho$ precisely when $\rho(n-1) < \rho(n)$. Since $\pi \leq \pi'$ implies $\pi^+ \leq \pi'^+$, the third statement follows.

For $\pi \in P_2$, it is immediate that $\pi \in P_2(\rho)$ if and only if $\phi(\pi) = \rho$. In addition, $\pi \in P_2(\rho)$ is uniquely determined by $\pi(1) \in \{0, 1\}$, so $\pi_{\rho,2}$ and $\hat{\pi}_{\rho,2}$ are the only elements of $\phi^{-1}(\rho) = P_2(\rho)$. The final statement about duality is obvious. $\qquad\square$

Definition 3.4. Given $\rho \in W$, we say that $S_{\bar{\rho}}$ is the generalized Serre condition dual to S_ρ.

Example 3.1.1
The Serre condition dual to S_r is given by $S_{\bar{\rho}_r}$, where

$$\bar{\rho}_r(k) = \begin{cases} k, & \text{if } k \leq 1; \\ 2, & \text{if } 2 \leq k \leq r; \\ k - (r - 2), & \text{if } k > r. \end{cases}$$

In particular, S_2 and Cohen-Macaulay condition are dual to each other.

Proposition 3.5. *Let $\pi \in P_{\text{c-codim}\,Z}(\rho)$, and suppose that $\mathcal{F} \in \mathfrak{Coh}(U)$.*
   (1) *The complex $\mathcal{IC}^{p_\pi}(X, \mathcal{F})$ is defined if and only if $\mathcal{F} \in {}^{p_\pi^+}\mathcal{D}(U)^{\geq 0}$. In particular, this is true if $\mathrm{Supp}(\mathcal{F}) = U$ and $\mathcal{F}$ is S_ρ.*
   (2) *If $\mathcal{IC}^{p_\pi}(X, \mathcal{F})$ is a sheaf, it is S_ρ at all points not in U.*

Proof. Since $\mathcal{F}$ is a sheaf and p_π is standard, $\mathcal{F}$ lies in ${}^{p_\pi^-}\mathcal{D}(U)^{\leq 0}$ by Remark 2.9. Hence, $\mathcal{IC}^{p_\pi}(X, \mathcal{F})$ is defined if and only if $\mathcal{F} \in {}^{p_\pi^+}\mathcal{D}(U)^{\geq 0}$. If $\mathrm{Supp}(\mathcal{F}) = U$, then $\dim_u \mathcal{F} = \mathrm{codim}\,u$ for all $u \in U$, so $\mathcal{F} \in {}^{p_\pi^+}\mathcal{D}(U)^{\geq 0}$. Finally, if $x \in Z$, then $H^k(i_x^!(\mathcal{IC}^{p_\pi}(X, \mathcal{F}))) = 0$ for all $k < p_\pi^+(x) = \rho(\mathrm{codim}(x))$. If $\mathcal{IC}^{p_\pi}(X, \mathcal{F})$ is a sheaf, then it is S_ρ at x because the dimension of a coherent sheaf at x is at most $\mathrm{codim}(x)$. $\qquad\square$

Definition 3.6. The S_ρ-perversity on (X, U) is the standard perversity $p_\rho(x) = \pi_{\rho,\text{c-codim}\,Z}(\mathrm{codim}(x))$.

Note that this perversity gives the least restrictive conditions on $\mathcal{F} \in \mathfrak{Coh}(U)$ guaranteeing that $\mathcal{IC}(X, \mathcal{F})$ is S_ρ for all points in Z (if it is a sheaf). In our previous notation, s is the S_2-perversity. However, c is not the S_{id} (i.e., the Cohen-Macaulay) perversity, since it corresponds to $\hat{\pi}_{\text{id},\text{c-codim}\,Z}$.

We end this section with an explicit description of what it means for a sheaf to be in ${}^{p_\rho^+}\mathcal{D}(X)^{\geq 0}$.

Proposition 3.7. *Let $n = \text{c-codim}\,Z$. A sheaf $\mathcal{F}$ is in ${}^{p_\rho^+}\mathcal{D}(X)^{\geq 0}$ if and only if*

$$(3.1) \quad \mathrm{depth}_x \mathcal{F} \geq \begin{cases} \rho(\mathrm{codim}(x)), & \text{if } \mathrm{codim}(x) \geq n; \\ \rho(n) - (n - \mathrm{codim}(x)), & \text{if } n - \rho(n) + 1 \leq \mathrm{codim}(x) < n; \\ 0, & \text{if } \mathrm{codim}(x) \leq n - \rho(n). \end{cases}$$

The proof is a simple calculation using the definition of p_ρ^+. Note that the first two cases correspond to points for which $p_\rho^+(x) = p_\rho^+(x) + 1$.

3.2. **Example.** In this section, we will construct an S_ρ scheme X that is "strictly S_ρ through codimension n". This example is adapted from [5, Section 4].

Let K be an algebraically closed field of characteristic zero, and let $K[X_1, \ldots, X_a, Z]$ be a polynomial ring in $a + 1$ variables. We define a graded ring T^a by $T^a = K[X_1, \ldots, X_a, Z]/(f)$, where (f) is the ideal generated by $f = Z^{a+1} - \sum_{i=1}^{a} X_i^{a+1}$. Let A^b be the polynomial ring $K[Y_0, Y_1, \ldots, Y_b]$. We let $X_{a,b} = \mathrm{Spec}(T^a \times_K A^b)$, where $T^a \times_K A^b$ denotes the Segre product. (Recall that this is the subalgebra of $T^a \otimes_K A^b$ generated by elements $r \otimes s$ with r and s homogeneous of the same degree.) Griffith showed in [5, Theorem 4.5] that this variety has the following properties:

Lemma 3.8.
  (1) The variety $X_{a,b}$ is S_a, but not S_{a+1}.
  (2) The non-smooth locus of $X_{a,b}$ is the singleton point corresponding to the irrelevant maximal ideal.

Let $\rho \in W$ be a weakly increasing function as in Section 3.1, so in particular $\rho \geq \rho_2$. We define the nth inclination of ρ by

$$t_n\rho(k) = \begin{cases} \rho(k) & k \leq n \\ \rho(n) + k - n & k > n. \end{cases}$$

Note that $t_n\rho$ is the maximum element of W that agrees with ρ on $[0, n]$. It is trivial that $t_m\rho \geq t_n\rho$ whenever $m \leq n$.

Fix $\rho \in W$ and $n \geq 3$, and assume that $\rho|_{[0,n]} \neq \mathrm{id}_{[0,n]}$. This data determines a (nonempty) increasing sequence $(d_1, d_2, ..., d_s)$, consisting of those indices $m \leq n$ satisfying $t_n\rho(m + 1) > t_n\rho(m) = t_n\rho(m - 1)$. Set $e_i = d_i - \rho(d_i)$ and $r_i = \rho(d_i)$. We define a variety $\overline{X}_{\rho,n}$ by

$$\overline{X}_{\rho,n} = \prod_{i=1}^{s} X_{r_i,e_i}.$$

Let x_{r_i,e_i} be the closed point of X_{r_i,e_i} corresponding to the irrelevant ideal of $T^{r_i} \times_K A^{e_i}$. Note that x_{r_i,e_i} has codimension d_i. Define a subvariety $X'_{\rho,n} \subset \overline{X}_{\rho,n}$ by

$$X'_{\rho,n} = \bigcup_{1 \leq i \leq j \leq s} (\{x_{r_i,e_i}\} \times \{x_{r_j,e_j}\} \times \prod_{\substack{1 \leq \ell \leq s \\ \ell \neq i,j}} X_{r_\ell,e_\ell}).$$

Finally, we write $X_{\rho,n} = \overline{X}_{\rho,n} \setminus X'_{\rho,n}$.

Proposition 3.9. The variety $X_{\rho,n}$ satisfies the generalized Serre condition S_ρ. Moreover, if $\rho' \in W$ is a function such that $t_n\rho' > t_n\rho$, then $X_{\rho,n}$ does not satisfy $S_{\rho'}$.

Proof. Let Y_i denote the subvariety $\{x_{r_i,e_i}\} \times_{X_{r_i,e_i}} X_{\rho,n}$. By assumption, $Y_i \cap Y_j = \varnothing$. Note that $Y_i \cong \prod_{j \neq i} X_{r_j,e_j} \setminus \{x_{r_j,e_j}\}$, so, by Lemma 3.8, it is smooth.

First, we calculate the depth at any point $x \in X_{\rho,n}$. Since $\cap_i Y_i^c$ is contained in the smooth locus, $\mathrm{depth}_x(\mathcal{O}_{X_{\rho,n}}) = \mathrm{codim}(x)$ if $x \notin \cup_i Y_i$. Now, suppose that $x \in Y_i$. Let $R = \mathcal{O}_{X_{r_i,e_i},x_{r_i,e_i}}$ and $S = \mathcal{O}_{X_{\rho,n},x}$, with corresponding maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$. Since the projection map $X_{\rho,n} \to X_{r_i,e_i}$ is flat, $\mathrm{depth}_{\mathfrak{n}}(S) =$

$\text{depth}_{\mathfrak{m}}(R) + \text{depth}_{\mathfrak{n}}(S/\mathfrak{m}S)$. There is a similar equation for codimension. Since the fiber over x_{r_i, e_i} of this projection is isomorphic to Y_i, $S/\mathfrak{m}S \cong \mathcal{O}_{Y_i, x}$. An application of Lemma 3.8 gives $\text{depth}_{\mathfrak{m}}(R) = r_i$ and $\text{depth}_{\mathfrak{n}}(S/\mathfrak{m}S) = \text{codim}_{Y_i}(x)$. We deduce that

$$\text{codim}(x) = d_i + \text{codim}_{Y_i}(x)$$

(3.2)

$$\text{depth}_x(\mathcal{O}_{X_{\rho,n}, x}) = r_i + \text{codim}_{Y_i}(x).$$

whenever $x \in Y_i$. (Since $r_i < d_i$, it follows that the Cohen-Macaulay locus (and the smooth locus) is precisely $\cap_i Y_i^c$.)

To show that $X_{\rho,n}$ is $S_{t_n \rho}$, we need only consider $x \in Y_i$. Equation (3.2) shows that the generalized Serre condition corresponding to $t_{d_i} \rho_{r_i}$ is satisfied at such an x. The function $t_{d_i} \rho_{r_i}$ is strictly increasing until $k = r_i$, nonincreasing on the interval $[r_i, d_i]$, and then strictly increasing afterwards. It is easily checked that $t_{d_i} \rho_{r_i} \geq t_n \rho$.

Finally, we suppose that $\rho' \in W$ is a function such that $t_n \rho' > t_n \rho$. In particular, there exists a smallest integer k such that $2 < k \leq d_s$ and $\rho'(k) > \rho(k)$. Note that one cannot have $k \leq d_1$ unless $\rho(k) = \rho(d_1)$. It is clear that $X_{\rho,n}$ can not be $S_{\rho'}$ if $k \leq d_i$ and $\rho(k) = \rho(d_i)$, since the generic point of Y_i is a codimension d_i point that has depth $\rho(d_i)$. Suppose now that $d_i < k < d_{i+1}$ and $\rho(k) < \rho(d_{i+1})$. Then, ρ is strictly increasing on the interval $[d_i, k]$. It follows that $\rho'(k) > r_i + k - d_i$. Choose any point $x \in Y_i$ with codimension k in $X_{\rho,n}$. By equation (3.2), $\text{depth}_x(\mathcal{O}_{X_{\rho,n}, x}) = r_i + (k - d_i)$. Therefore, $X_{\rho,n}$ does not satisfy the condition $S_{\rho'}$, since $r_i + (k - d_i) < \rho'(k)$.

$\square$

4. S_ρ-EXTENSION AND FINITE S_ρ-IFICATION

In this section, we investigate the "S_ρ-extension problem" for any $\rho \in W$ and its relationship to S_2-extension. In particular, we apply our results to the finite S_ρ-ification problem.

4.1. S_ρ-extension.

Definition 4.1. Let $U \subset X$ be an open dense subscheme with complement Z. A finite morphism $f : Y \to X$ is S_ρ relative to U if $f_* \mathcal{O}_Y$ satisfies the depth conditions in (3.1).

If c-codim$(Z) \geq 1$, this is equivalent to the statement $f_* \mathcal{O}_Y \in {}^{p_\rho^+} \mathcal{D}(X)^{\geq 0}$. If c-codim $Z = 1$, this simply means that $f_* \mathcal{O}_Y$ is S_ρ.

The initial data for S_ρ-extension consists of an open dense subscheme U of the scheme X and a finite dominant morphism $\zeta_1 : \tilde{U} \to U$ that maps generic points to generic points and satisfies $\zeta_{1*} \mathcal{O}_{\tilde{U}} \in {}^{p_\rho^+} \mathcal{D}(U)^{\geq 0}$. We will let $j : U \to X$ denote the inclusion.

Definition 4.2. We say that a scheme $\tilde{X}$ together with a morphism $\zeta : \tilde{X} \to X$ is an S_ρ-extension of (X, ζ_1) if it satisfies the following conditions:

(1) $\tilde{X}$ contains $\tilde{U}$ as an open dense subscheme;
(2) ζ extends ζ_1 and is finite;
(3) ζ is S_ρ relative to U; and

(4) $(\tilde{X}, \zeta)$ is universal among finite dominant morphisms $f : Y \to X$ which send generic points to generic points, which are S_ρ relative to U, and whose restriction $f|_{f^{-1}(U)}$ factors through ζ_1. (In other words, there exists a unique $g : Y \to \tilde{X}$ such that $f = \zeta \circ \gamma$.)

We say that $(\tilde{X}, \zeta)$ is a *weak S_ρ-extension* if it satisfies conditions (1) and (2) and $\tilde{X}$ is S_ρ off of $\tilde{U}$.

Note that ζ is automatically dominant and takes generic point to generic points.

Remark 4.3. Let $f : Y \to X$ be a finite dominant map extending ζ_1 and taking generic points to generic points. Under these conditions, c-codim$_Y(Y - f^{-1}(U)) =$ c-codim$_X Z$ and $\operatorname{codim}(y) = \operatorname{codim}(f(y))$. (Note that we are using the equidimensionality of X.) This means that the perversities corresponding to ρ on X and Y are related by $p_\rho^Y = p_\rho^X \circ f$, so there is no ambiguity in denoting both simply by p_ρ. Moreover, the argument given in [1, Proposition 3.5] shows that $f_* \mathcal{O}_Y \in {}^{p_\rho^+}\mathcal{D}(X)^{\geq 0}$ implies that $\mathcal{O}_Y \in {}^{p_\rho^+}\mathcal{D}(Y)^{\geq 0}$ in the case c-codim $Z \geq 2$. In particular, if f is S_ρ-relative to U, then Y is S_ρ at all points y with $\operatorname{codim}(y) \geq$ c-codim Z.

If c-codim $Z \geq 2$, (X, ζ_1) has an S_2-extension if and only if $\mathcal{IC}^s(X, \zeta_{1*}\mathcal{O}_{\tilde{U}})$ is defined; moreover, it is given by $\mathbf{Spec}(\mathcal{IC}^s(X, \zeta_{1*}\mathcal{O}_{\tilde{U}})) \to X$ [1, Theorem 4.5]. (By Remark 2.10, $\mathcal{IC}^s(X, \zeta_{1*}\mathcal{O}_{\tilde{U}}) = j_* \zeta_{1*}\mathcal{O}_{\tilde{U}})$ is automatically a sheaf of $\mathcal{O}_X$-algebras if it is defined. Thus, the global Spec makes sense.) We now give the corresponding result for S_ρ-extension.

Theorem 4.4. *Suppose that* c-codim $Z \geq 2$. *Then, the pair (X, ζ_1) has an S_ρ-extension if and only if $\mathcal{IC}^{p_\rho}(X, \zeta_{1*}\mathcal{O}_{\tilde{U}})$ is defined and is a sheaf. If it exists, it is given by $\mathbf{Spec}(\mathcal{IC}^{p_\rho}(X, \zeta_{1*}\mathcal{O}_{\tilde{U}})) \to X$ and coincides with the S_2-extension.*

Proof. First, suppose that $(\tilde{X}, \zeta)$ is an S_ρ-extension. By property (3), $\zeta_* \mathcal{O}_{\tilde{X}} \in \mathcal{M}_\pm^{p_\rho}(X)$; restricting to U, we see that $\zeta_{1*}\mathcal{O}_{\tilde{U}} \in \mathcal{M}_\pm^{p_\rho}(U)$. This means that $\mathcal{IC}^{p_\rho}(X, \zeta_{1*}\mathcal{O}_{\tilde{U}})$ is defined. By [1, Lemma 3.4], if a coherent sheaf on X extending $\zeta_{1*}\mathcal{O}_{\tilde{U}}$ is contained in ${}^{p_\rho^+}\mathcal{D}(X)^{\geq 0}$, then it is isomorphic to $\mathcal{IC}^{p_\rho}(X, \zeta_{1*}\mathcal{O}_{\tilde{U}})$. Thus, $\mathcal{IC}^{p_\rho}(X, \zeta_{1*}\mathcal{O}_{\tilde{U}}) \cong \zeta_* \mathcal{O}_{\tilde{X}}$ is a sheaf.

Conversely, suppose $\mathcal{IC}^{p_\rho}(X, \zeta_{1*}\mathcal{O}_{\tilde{U}})$ is defined and is a sheaf. Since $p_\rho \geq s$, $\mathcal{IC}^s(X, \zeta_{1*}\mathcal{O}_{\tilde{U}})$ is defined, and the same argument given in the previous paragraph implies that $\mathcal{IC}^{p_\rho}(X, \zeta_{1*}\mathcal{O}_{\tilde{U}}) = \mathcal{IC}^s(X, \zeta_{1*}\mathcal{O}_{\tilde{U}})$. Accordingly, $\tilde{X} = \mathbf{Spec}(\mathcal{IC}^{p_\rho}(X, \zeta_{1*}\mathcal{O}_{\tilde{U}}))$ is the S_2-extension of (X, ζ_1) and a fortiori satisfies the S_ρ-extension universal property. Finally, ζ is S_ρ relative to U because $\zeta_* \mathcal{O}_{\tilde{X}} = \mathcal{IC}^{p_\rho}(X, \zeta_{1*}\mathcal{O}_{\tilde{U}}) \in {}^{p_\rho^+}\mathcal{D}(X)^{\geq 0}$. $\square$

We will see later that under the conditions of the theorem, a weak S_ρ-extension is automatically an S_ρ-extension. We first prove this for S_2-extension.

Proposition 4.5. *Suppose that* c-codim $Z \geq 2$ *and that (X, ζ_1) has an S_2-extension. Then a weak S_2-extension $(\hat{X}, \hat{\zeta})$ coincides with the S_2-extension.*

Proof. Let $\hat{j} : \tilde{U} \to \hat{X}$ be the inclusion. By Remark 4.3, $\tilde{U}$ is S_2 at all points with codimension at least c-codim Z, so $\mathcal{IC}^s(\hat{X}, \mathcal{O}_{\tilde{U}})$ is defined. Since $\hat{X}$ is S_2 off of $\tilde{U}$, we obtain $\mathcal{O}_{\hat{X}} \cong \mathcal{IC}^s(\hat{X}, \mathcal{O}_{\tilde{U}}) \cong \hat{j}_* \mathcal{O}_{\tilde{U}}$, where the last isomorphism holds by Remark 2.10. Applying $\hat{\zeta}_*$ gives $\hat{\rho}_* \mathcal{O}_{\hat{X}} \cong j_* \zeta_{1*}\mathcal{O}_{\tilde{U}} \cong \mathcal{IC}^s(X, \zeta_{1*}\mathcal{O}_{\tilde{U}})$. Thus, $\hat{X}$ is isomorphic to the S_2 extension $\mathbf{Spec}(\mathcal{IC}^s(X, \zeta_{1*}\mathcal{O}_{\tilde{U}}))$. $\square$

Let $f : Y \to X$ be a finite map, and let $j' : f^{-1}(U) \hookrightarrow Y$ be the inclusion. Write $f_1 = f|_{f^{-1}(U)} : f^{-1}(U) \to U$. We say that Y is the integral closure of X relative to $f^{-1}(U)$ (resp. Y is integrally closed relative to $f^{-1}(U)$) if $Y = \mathbf{Spec}(\mathcal{F})$, where $\mathcal{F}$ is the integral closure of $\mathcal{O}_X$ in $j_* f_{1*}(\mathcal{O}_{f^{-1}(U)})$ (resp. $\mathcal{O}_Y$ is integrally closed in $j'_* \mathcal{O}_{f^{-1}(U)}$). (See [6, Proposition 6.3.4]). If we relax the condition that c-codim $Z \geq 2$, we can prove a weaker version of S_2-extension as long as the integral closure of X relative to $\tilde{U}$ is finite over X.

Proposition 4.6. *Suppose that the integral closure of X relative to $\tilde{U}$ is finite over X. Let $\hat{X}$ be the associated reduced scheme. The natural morphism $\zeta : \hat{X} \to X$ is universal with respect to finite morphisms $f : Y \to X$ satisfying the following properties:*

(1) *$f^{-1}(U)$ is dense in Y;*
(2) *$f|_{f^{-1}(U)}$ factors through ζ_1; and*
(3) *Y is reduced and integrally closed relative to $f^{-1}(U)$.*

Remark 4.7. In this proposition, we do not need to assume that ζ_1 is dominant or that it takes generic points to generic points.

Remark 4.8. We note that condition (3) is equivalent to the following by Serre's criterion [3, Theorem 11.5].

(3') *Y is reduced, and Y satisfies S_2 and R_1 away from $f^{-1}(U)$.*

Remark 4.9. If the base S is a Nagata scheme, then the condition that $\tilde{X}$ is finite over X is automatically satisfied.

The pair $(\hat{X}, \zeta)$ given in the proposition is a particular weak S_2-extension of (X, ζ_1). If c-codim $Z \geq 2$, then it is the S_2-extension by Proposition 4.5.

Proof. There is a natural morphism of quasi-coherent sheaves of $\mathcal{O}_X$-algebras

$$(4.1) \qquad j_* \zeta_{1*}(\mathcal{O}_{\tilde{U}}) \to j_* f_{1*}(\mathcal{O}_{f^{-1}(U)})$$

defined as follows. Since f_1 factors finitely through ζ_1, we may write $f_1 = \zeta_1 \circ f'_1$ where $f'_1 : f^{-1}(U) \to \tilde{U}$ is a finite map. To obtain (4.1), we simply apply the functor $j_* \zeta_{1*}$ to the adjunction map $\mathcal{O}_{\tilde{U}} \to (f'_1)_* (f'_1)^* (\mathcal{O}_{\tilde{U}}) \cong (f'_1)_*(\mathcal{O}_{f^{-1}(U)})$.

By assumption, $f_* \mathcal{O}_Y$ is integrally closed in $j_*(f_{1*})(\mathcal{O}_{f^{-1}(U)})$. Moreover, the map $f_* \mathcal{O}_Y \to j_*(f_{1*})(\mathcal{O}_{f^{-1}(U)})$ is injective; indeed, since $f^{-1}(U)$ is dense in Y and Y is reduced, the morphism $\mathcal{O}_Y \to (j')_* \mathcal{O}_{f^{-1}(U)}$ is injective. Finally, [6, 6.3.5] shows that there is a canonical morphism $Y \to \hat{X}$. $\qquad \square$

Lemma 4.10. *Suppose that $f : Y \to X$ is a finite dominant morphism of reduced Noetherian schemes that takes generic points to generic points. Suppose that U is a dense open subscheme of X such that f induces an isomorphism between $f^{-1}(U)$ and U. Let $x \in X \setminus U$ be a codimension one point that is regular. Then, there exists a unique point $y \in Y$ (necessarily of codimension one) lying above x. Moreover, y is regular, and the map $f_y^* : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is an isomorphism.*

Proof. The hypotheses imply that f induces a bijection between the irreducible components of Y and X. Since x is regular, it lies in a single component of X. This means that $f^{-1}(y)$ can only intersect the corresponding component of Y. We

may accordingly assume without loss of generality that X and Y are irreducible, hence integral, and f induces an isomorphism of function fields $K(X) \cong K(Y)$.

Since x is regular, $f^{-1}(x)$ contains a single point y by the valuative criterion of properness. Moreover, the local ring $\mathcal{O}_{Y,y}$ dominates the valuation ring $f_y^* \mathcal{O}_{X,x}$ in $K(Y)$, so they are equal. Hence, $\mathcal{O}_{Y,y}$ is also a discrete valuation ring, and $f_y^* : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is an isomorphism.

$\square$

Proposition 4.11. *Let X, Y, and $f : Y \to X$ satisfy the conditions in Proposition 4.6. Suppose that f is dominant and takes generic points to generic points and that the given factorization map $g_1 : f^{-1}(U) \to \tilde{U}$ is an isomorphism. If we further assume that Y (and hence $\tilde{U}$) is S_2, then the map $g : Y \to \hat{X}$ constructed in Proposition 4.6 is an isomorphism.*

Proof. Suppose that we can find an open subset $\tilde{U} \subset V \subset \hat{X}$ such that $g|_{g^{-1}(V)} : g^{-1}(V) \to V$ is an isomorphism and c-codim$(\hat{X} \setminus V) \geq 2$. Since g is a weak S_2-extension of $g|_{g^{-1}(V)} : g^{-1}(V) \to V$, by Proposition 4.5, g is the S_2-extension. Moreover, the identity map $V \to V$ has an S_2-extension, namely, $\hat{X} \to \hat{X}$. It follows that g is an isomorphism.

Let V be the complement of the support of the coherent $\mathcal{O}_{\hat{X}}$-module $f_* \mathcal{O}_Y / \mathcal{O}_{\hat{X}}$. This is an open set containing $\tilde{U}$. Note that $g|_{g^{-1}(V)} : g^{-1}(V) \to V$ is a continuous bijection; finiteness implies that it is closed, hence a homeomorphism. Since the induced map of sheaves is obviously an isomorphism, we obtain a scheme isomorphism. By Lemma 4.10, V contains all codimension one points not in $\tilde{U}$. It follows that codim $\hat{X} \setminus V \geq 2$. The same holds for the c-codimension. Indeed, the equidimensionality hypothesis implies that a point x has the same codimension in any irreducible component containing it. $\square$

Putting together Propositions 4.5 and 4.11 and using the fact that a weak S_ρ-extension is a weak S_2-extension, we obtain the following result above weak S_ρ-extensions.

Theorem 4.12.

(1) *Suppose* c-codim $Z \geq 2$ *and* $\mathcal{IC}^s(X, \zeta_{1*} \mathcal{O}_{\tilde{U}})$ *is defined. Then any weak S_ρ-extension of (X, ζ_1) coincides with the S_2-extension.*

(2) *Suppose that $\tilde{U}$ is reduced and the integral closure of X relative to $\tilde{U}$ is finite over X. Then any reduced weak S_ρ-extension (Y, f) of (X, ζ_1) that is R_1 outside of $f^{-1}(U)$ coincides with the weak S_2-extension $(\hat{X}, \zeta)$ constructed in Proposition 4.6.*

4.2. **Finite S_ρ-ification.** We now apply our results to the finite S_ρ-ification problem. Recall that a finite S_ρ-ification of a scheme X is an S_ρ scheme $\hat{X}$ together with a finite birational map $\hat{\zeta} : \hat{X} \to X$. If we let U be an open dense subset of X on which $\hat{\zeta}$ is an isomorphism, we see that a finite S_ρ-ification may be viewed as a weak S_ρ-extension of the identity map $U \to U$. (Observe, however, that a weak S_ρ-extension of the identity can be defined without assuming U is S_ρ.)

Theorem 4.13.

(1) *Assume that the S_ρ locus of X contains an open dense set whose complement has c-codimension at least 2. Then,*

(a) *If the complex $\mathfrak{IC}^{p_\rho}(X, \mathcal{O}_U)$ is a sheaf for such an open set U, then $\mathfrak{IC}^{p_\rho}(X, \mathcal{O}_V)$ is a sheaf for any such V, and they are all isomorphic.*

(b) *The scheme X has a finite S_ρ-ification which is an isomorphism off of a closed set of c-codimension at least 2 if and only if $\mathfrak{IC}^{p_\rho}(X, \mathcal{O}_U)$ is a sheaf for any such open set U.*

(c) *If such an S_ρ-ification exists, it is unique and coincides with the unique finite S_2-ification which is an isomorphism on the S_2 locus W. In particular, the S_2 and S_ρ loci coincide, and this finite S_ρ-ification can be given explicitly as $\mathbf{Spec}(\mathfrak{IC}^s(X, \mathcal{O}_W)) \to X$.*

(2) *Assume that the S_ρ locus of X contains a reduced open dense set U such that the integral closure $\hat{X}$ of X relative to U is finite over X. Then, X has a finite S_ρ-ification which is R_1 off of U if and only if $\hat{X}$ is S_ρ. If such a finite S_ρ-ification exist, it coincides with the unique S_2-ification which is R_1 off of U.*

Proof. The second part follows immediately from Proposition 4.6 and Theorem 4.12. Note that since U is reduced, the integral closure of X relative to U is automatically reduced. Thus, it is unnecessary to pass to the associated reduced scheme as in Proposition 4.6.

For the first part, assume that $U \hookrightarrow X$ is open, dense, and S_ρ with c-codim$(X \setminus U) \geq 2$. Since a finite S_ρ-ification that is an isomorphism over U is the same thing as a weak S_ρ-extension of id $: U \to U$, Theorem 4.4 implies that this exists if and only if $\mathfrak{IC}^{p_\rho}(X, \mathcal{O}_U)$ is a sheaf, in which case it coincides with $\mathfrak{IC}^s(X, \mathcal{O}_U) \cong j_*(\mathcal{O}_U)$ (where the last isomorphism uses Remark 2.10).

Suppose that this is the case and that $V \overset{i}{\hookrightarrow} X$ is another open, dense, S_ρ subscheme with the c-codimension of its complement at least 2. Both $j_*(\mathcal{O}_U)$ and $i_*(\mathcal{O}_V) \cong \mathfrak{IC}^s(X, \mathcal{O}_V)$ are S_2 coherent sheaves extending $\mathcal{O}_{U \cap V}$, so they are isomorphic; they are both isomorphic to $\mathfrak{IC}^s(X, \mathcal{O}_{U \cap V})$. (Since c-codim$(X \setminus (U \cap V)) \geq 2$, the IC sheaf is defined.) In particular, $i_*(\mathcal{O}_V) \in {}^{p_\rho}\mathcal{D}(X)^{\geq 0}$, so it equals $\mathfrak{IC}^{p_\rho}(X, \mathcal{O}_V)$. Thus, this complex is a sheaf and coincides with $\mathfrak{IC}^{p_\rho}(X, \mathcal{O}_U)$.

Part (1b) now follows from Theorems 4.4 and 4.12 because such a finite S_ρ-ification is the same thing as a weak S_ρ extension of id $: U \to U$ (with U as above). Part (1a) also shows that if this S_ρ-ification exists, it is unique. Finally, observe that $U \subset W$, so $\mathfrak{IC}^s(X, \mathcal{O}_U) = \mathfrak{IC}^s(X, \mathcal{O}_W)$. This implies that $\mathbf{Spec}(\mathfrak{IC}^s(X, \mathcal{O}_U)|_W)| \cong W$, so W is S_ρ.

$\square$

Remark 4.14. Even in the context of part (1) of the theorem, there can be other finite S_ρ-ifications which are not isomorphisms over an open, dense set whose complement has c-codimension at least 2. Indeed, take any variety which is S_2, but not R_1. Then, the identity map and the normalization are non-isomorphic finite S_2-ifications.

Corollary 4.15. *If the non-S_2 locus of X has c-codimension at least 2, then there is a unique finite S_2-ification which is an isomorphism on a dense, open set whose complement has c-codimension at least 2. Moreover, it is an isomorphism over the S_2 locus.*

Proof. This follows from the theorem because $\mathfrak{IC}^s$ takes sheaves to sheaves. $\square$

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