PERVERSE COHERENT SHEAVES AND THE GEOMETRY OF SPECIAL PIECES IN THE UNIPOTENT VARIETY

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Abstract. Let X be a scheme of finite type over a Noetherian base scheme S admitting a dualizing complex, and let $U \subset X$ be an open set whose complement has codimension at least 2. We extend the Deligne-Bezrukavnikov theory of perverse coherent sheaves by showing that a coherent intermediate extension (or intersection cohomology) functor from perverse sheaves on U to perverse sheaves on X may be defined for a much broader class of perversities than has previously been known. We also introduce a derived category version of the coherent intermediate extension functor.

Under suitable hypotheses, we introduce a construction (called "S_2-extension") in terms of perverse coherent sheaves of algebras on X that takes a finite morphism to U and extends it in a canonical way to a finite morphism to X. In particular, this construction gives a canonical "S_2-ification" of appropriate X. The construction also has applications to the "Macaulayfication" problem, and it is particularly well-behaved when X is Gorenstein.

Our main goal, however, is to address a conjecture of Lusztig on the geometry of special pieces (certain subvarieties of the unipotent variety of a reductive algebraic group). The conjecture asserts in part that each special piece is the quotient of some variety (previously unknown for the exceptional groups and in positive characteristic) by the action of a certain finite group. We use S_2-extension to give a uniform construction of the desired variety.

1. Introduction

Let X be a scheme of finite type over a Noetherian base scheme S that admits a dualizing complex, and let $U \subset X$ be an open set whose complement has codimension at least 2. Let $\tilde{U}$ be another scheme, equipped with a finite morphism $\rho_1 : \tilde{U} \to U$. Consider the problem of completing the following diagram in a canonical way:

$$\begin{tikzcd} \tilde{U} \arrow[r, hook, dotted] \arrow[d, "\rho_1"'] & \tilde{X} \arrow[d, dotted, "\rho"] \\ U \arrow[r, hook] & X \end{tikzcd}$$

In other words: "Construct a canonical new scheme $\tilde{X}$ that contains $\tilde{U}$ as an open subscheme, together with a finite morphism $\rho : \tilde{X} \to X$ that extends ρ_1." One may want to impose additional conditions, such as requiring $\tilde{X}$ to obey a regularity condition or requiring the fibers of ρ to have a specified form. Moreover, the pair

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$(\tilde{X}, \rho)$ should satisfy an appropriate universal property. If a group G acts on X with U a G-subscheme and ρ_1 G-equivariant, one would like the entire constructed diagram to be equivariant. The present paper is motivated by a specific instance of this problem, arising in a conjecture of Lusztig on the geometry of special pieces (see below for the definition) in reductive algebraic groups.

In this paper, we give a general construction (called "S_2-extension") of such a scheme $\tilde{X}$ and morphism $\rho : \tilde{X} \to X$, using Deligne's theory of perverse coherent sheaves on X (following Bezrukavnikov's exposition [6]), assuming that the category of coherent sheaves on X has enough locally free objects. (This includes, for example, quasiprojective schemes over S.) This theory parallels the theory of constructible perverse sheaves with the major exception that the intermediate extension (or intersection cohomology) functor is not always defined. Indeed, in [6], this functor is only defined in an equivariant setting with strong restrictions on the group action. In this paper, we first show that the intermediate extension functor may be defined for a much broader class of perversities. In particular, we study two dual perversities, called the "S_2" and "Cohen-Macaulay" perversities.

Next, we construct $\tilde{X}$ as the global Spec of a certain intersection cohomology sheaf with respect to the S_2 perversity. It will be defined whenever $\rho_{1*}\mathcal{O}_{\tilde{U}}$ satisfies certain homological conditions that are weaker than satisfying Serre's condition S_2. The scheme $\tilde{X}$ is locally S_2 outside of $\tilde{U}$; moreover, ρ satisfies a universal property related to this condition, and in that sense $\tilde{X}$ and ρ are canonical. In the particular case of $\tilde{U} = U$ and ρ_1 the identity, we obtain a canonical "S_2-ification" of U. This construction also has applications to the "Macaulayfication" problem. Indeed, we give necessary and sufficient conditions for X to have a universal finite Macaulayfication (i.e., universal among appropriate finite morphisms from Cohen-Macaulay schemes).

Third, we introduce a derived category version of the coherent intermediate extension functor (from a suitable subcategory of the derived category of coherent sheaves on U to the derived category of coherent sheaves on X), and we show that this functor induces an equivalence of categories with its essential image. One corollary of this theorem is that when X is Gorenstein, the coherent intermediate extension functor restricted to Cohen-Macaulay sheaves on U is independent of perversity. Using this, we show that with suitable assumptions on $\tilde{U}$ and X, the scheme $\tilde{X}$ produced by S_2-extension is in fact Cohen–Macaulay or Gorenstein.

Our main goal, however, is to apply these results to the aforementioned conjecture of Lusztig, which we now recall. Let G be a reductive algebraic group over the algebraically closed field k, and assume that the characteristic of k is good for G. Let C_1 be a special unipotent class of G in the sense of [20]. The *special piece* containing C_1 is defined by

$$P = \bigcup C \qquad \begin{array}{c}\text{where } C \text{ ranges over unipotent classes such that } C \subset \overline{C_1} \\ \text{but } C \not\subset \overline{C'} \text{ for any special } C' \subset \overline{C_1} \text{ with } C' \neq C_1.\end{array}$$

Each special piece is a locally closed subvariety of G, and according to a result of Spaltenstein [33], every unipotent class in G is contained in exactly one special piece.

In 1981, Lusztig conjectured that every special piece is rationally smooth [21]. This conjecture can be verified in any particular group by explicit calculation of Green functions, and indeed, the conjecture was quickly verified for all the exceptional groups following work of Shoji [30] and Benyon–Spaltenstein [4]. In the classical groups, however, new techniques were required. In 1989, Kraft and Procesi, relying on their own prior work on singularities of closures of unipotent classes, proved a stronger statement: they showed that every special piece in the classical groups is a quotient of a certain smooth variety by a certain finite group F [19]. In particular, this implies that special pieces are rationally smooth.

A natural question, then, is whether this stronger statement holds in general. The work of Kraft–Procesi makes extensive use of the combinatorics available in the classical groups, so it is not at all obvious how to generalize their construction to all groups. However, in 1997, Lusztig succeeded in characterizing the finite group F in a type-independent manner [23, Theorem 0.4]; he identified F as a certain subgroup of $\bar{A}(C_1)$. (For any unipotent class C, $\bar{A}(C)$ denotes Lusztig's "canonical quotient" of the component group $G^x/(G^x)^\circ$ of the G-stabilizer of a point $x \in C$ [22].) In fact, F is naturally a direct factor of $\bar{A}(C_1)$ (see [2, §3.1]) and inherits from $\bar{A}(C_1)$ the structure of a Coxeter group (see [1]). There is a one-to-one correspondence between parabolic subgroups of F and unipotent classes in P. Given a parabolic subgroup $H \subset F$, we denote the corresponding class C_H. (The trivial subgroup corresponds to the special class, so this notation is consistent with the earlier notation C_1.)

Conjecture 1.1 (Lusztig). *There is a smooth variety $\tilde{P}$ with an action of F such that $P \simeq \tilde{P}/F$ and such that C_H is precisely the image of those points in $\tilde{P}$ whose F-stabilizer is conjugate to H.*

Note that it suffices to examine special pieces for the simple root systems because the unipotent variety of a reductive group is the product of the unipotent varieties of its simple factors.

Before addressing this conjecture further, we remark that it is quite easy to produce candidate varieties that ought to be the preimages of the various C_H's in $\tilde{P}$ by using the results of [2] (a paper to which the present paper might be regarded as a sort of sequel). Fix $x \in C_H$. By [2, Theorem 2.1], we have $\bar{A}(C_H) \simeq N_{\bar{A}(C_1)}(H)/H$ (where $N_J(K)$ denotes the normalizer of K in J) and hence a natural map $G^x \to N_{\bar{A}(C_1)}(H)/H$. Let G^x_F be the kernel of the composed map $G^x \to N_{\bar{A}(C_1)}(H)/H \to N_F(H)/H$, and let $(\tilde{C}_H)^\circ = G/G^x_F$. Clearly, this is a connected variety with a free action of $N_F(H)/H$, and the quotient by that action is C_H. Finally, let

$$\tilde{C}_H = (\tilde{C}_H)^\circ \times_{N_F(H)} F. \tag{1}$$

An element $a \in F$ acts on this variety by $a \cdot (y, f) = (y, fa^{-1})$. The F-stabilizer of any point is conjugate to H, and the natural surjective map $\rho_H : \tilde{C}_H \to C_H$ is the quotient of $\tilde{C}_H$ by the action of F.

Before stating our main result, we observe that, since quotients of smooth varieties are normal, inherent in Lusztig's conjecture is the subconjecture:

Conjecture 1.2. *Every special piece P is normal.*

In characteristic zero, we show how this conjecture can be obtained from known results on unipotent conjugacy classes in the classical types, G_2, F_4, and E_6. For E_7 and E_8, there is a conjectural list of all nonnormal unipotent conjugacy class closures due to Broer, Panyushev, and Sommers [10]. Assuming this is true, then there would remain 5 special pieces (1 in E_7 and 4 in E_8) for which normality is not known. In positive characteristic, much less is known.

In this paper, we will actually construct a variety $\tilde{P}$ whose algebraic quotient by F is the normalization $\bar{P}$ of P. However, we will also show that special pieces are unibranch, i.e., the normalization map $\nu : \bar{P} \to P$ is a bijection and in fact a homeomorphism. This means that P is the topological quotient of $\tilde{P}$. In particular, setting $\bar{C}_H = \nu^{-1}(C_H)$, we see that $\bar{C}_H \simeq C_H$ and that $\bar{P}$ is again stratified by the unipotent orbits corresponding to parabolic subgroups of F.

The main result of the paper is the following.

Theorem 1.3. (1) *There is a canonical normal irreducible G-scheme $\tilde{P}$ together with a finite equivariant morphism $\rho : \tilde{P} \to P$ which extends $\rho_1 : \tilde{C}_1 \to C_1$; the pair $(\tilde{P}, \rho)$ is universal with respect to finite morphisms $f : Y \to P$ that are S_2 relative to C_1 and whose restriction $f|_{f^{-1}(C_1)}$ factors through ρ_1.*

(2) *The variety $\tilde{P}$ is rationally smooth. Moreover, if $\operatorname{char} k = 0$, then $\tilde{P}$ is Gorenstein.*

(3) *The variety $\tilde{P}$ is endowed with a natural F-action commuting with the G-action. The map ρ is the topological quotient by this action while $\bar{\rho} : \tilde{P} \to \bar{P}$ is the algebraic quotient.*

(4) *For each class $C_H \subset P$, the preimage $\rho^{-1}(C_H) = \bar{\rho}^{-1}(\bar{C}_H)$ is isomorphic to $\tilde{C}_H$ and contains exactly those closed points whose F-stabilizer is conjugate to H.*

The first part of this theorem is simply an invocation of the S_2-extension construction. The proof of the Gorenstein property is established by using a theorem of Hinich and Panyushev [18, 25] and the aforementioned results on the derived intermediate extension functor. We remark that the formalism of the S_2-extension construction does not yield a concrete description of the resulting scheme in general, but in our setting, the results of [2] (as noted above) allow us to find an explicit stratification (1) for $\tilde{P}$.

Although we do not prove that $\tilde{P}$ is smooth, we show that if $\hat{P}$ is a smooth variety containing a dense open set isomorphic to $\tilde{C}_1$ and $\hat{\rho} : \hat{P} \to P$ is a finite morphism extending ρ_1, then $\hat{P}$ is isomorphic to $\tilde{P}$. Thus, if Lusztig's conjecture is true, then our $\tilde{P}$ is the desired smooth variety. In particular, for the classical groups, the $\tilde{P}$ constructed here coincides with the Kraft–Procesi variety of [19].

2. Perverse Coherent Sheaves

The theory of perverse coherent sheaves, following Deligne and Bezrukavnikov [6], closely parallels the much better-known theory of constructible perverse sheaves, but one striking difference is that in the coherent setting, the intermediate extension functor does not always exist. Indeed, in loc.cit., it was only constructed in an equivariant setting with strong assumptions on the group action.

In this section, we review the Deligne–Bezrukavnikov theory, and we prove a generalization of [6, Theorem 2] that allows us to use the intermediate extension functor in a much broader class of examples, including many nonequivariant cases.

We begin with the same setting and assumptions as [6]. Let X be a scheme of finite type over a Noetherian base scheme S admitting a dualizing complex, and let G be an affine group scheme acting on X that is flat, of finite type, and Gorenstein over S. (For example, the base scheme could be $S = \operatorname{Spec} k$ with k a field.) By [16, Corollary V.7.2], a scheme X satisfying these assumptions necessarily has finite Krull dimension. Let $\mathfrak{Coh}(X)$ be the category of G-equivariant coherent sheaves on X, and let $\mathcal{D}(X)$ be the bounded derived category of $\mathfrak{Coh}(X)$. We further assume that $\mathfrak{Coh}(X)$ has enough locally free objects. Let $X^{G\text{-gen}}$ be the topological space consisting of generic points of G-invariant subschemes of X, with the subspace topology induced by the underlying topological space of X. We adopt the convention that for any (not necessarily irreducible) G-invariant locally closed subscheme $Y \subset X$, the *codimension* of Y is given by

$$\operatorname{codim} Y = \min_{y \in Y^{G\text{-gen}}} \dim \mathcal{O}_{y,X}.$$

For any point $x \in X$, let $i_x : \{x\} \to X$ denote the inclusion map. (This is merely a topological map, not a morphism of schemes.) For brevity, we will write $\bar{x}$ for the closed subspace $\overline{\{x\}}$ of X. By [6, Proposition 1], X admits an equivariant dualizing complex ω_X. By shifting if necessary, we may assume, as in [6, §3], that for each point $x \in X$, $i_x^! \omega_X$ is concentrated in degree $\operatorname{codim} \bar{x}$.

Although we will always work in this equivariant setting, one can of course obtain nonequivariant versions of our results simply by taking $G = 1$. Occasionally, we will explicitly pass from an equivariant category to a nonequivariant one, and make use of the fact that all the usual functors on sheaves commute with this forgetful functor.

Notational Convention. Throughout this paper, unless otherwise specified, all geometric objects will belong to the appropriate category for the equivariant setting without further mention. Thus, schemes will be G-schemes, morphisms will be G-morphisms, and sheaves will be G-equivariant.

Definition 2.1. A *perversity* is a function $p : X^{G\text{-gen}} \to \mathbb{Z}$ satisfying

$$\begin{aligned} p(y) &\geq p(x) && \text{and} \\ \operatorname{codim} \bar{y} - p(y) &\geq \operatorname{codim} \bar{x} - p(x) && \text{whenever } \operatorname{codim} \bar{y} \geq \operatorname{codim} \bar{x}. \end{aligned} \tag{2}$$

(In particular, $p(x)$ depends only on $\operatorname{codim} \bar{x}$.) For any perversity p, the function $\bar{p} : X^{G\text{-gen}} \to \mathbb{Z}$ defined by $\bar{p}(x) = \operatorname{codim} \bar{x} - p(x)$ is also a perversity, called the *dual perversity* to p.

A slightly more general theory could be obtained by imposing the inequalities (2) only when $y \in \bar{x}$, as is done in [6] (see also Remark 2.8). For the purposes of this paper, however, there would be no practical benefit to defining perversities in this way, and various technical details would become rather more complicated, so we will confine ourselves to perversities as defined above.

Given a perversity p, we define two full subcategories of $\mathcal{D}(X)$ as follows:

$$\begin{aligned} {}^p\mathcal{D}(X)^{\leq 0} &= \{\mathcal{F} \in \mathcal{D}(X) \mid \text{for all } x \in X^{G\text{-gen}},\ H^k(i_x^* \mathcal{F}) = 0 \text{ for all } k > p(x)\} \\ {}^p\mathcal{D}(X)^{\geq 0} &= \{\mathcal{F} \in \mathcal{D}(X) \mid \text{for all } x \in X^{G\text{-gen}},\ H^k(i_x^! \mathcal{F}) = 0 \text{ for all } k < p(x)\} \end{aligned}$$

By [6, Theorem 1], $({}^p\mathcal{D}(X)^{\leq 0}, {}^p\mathcal{D}(X)^{\geq 0})$ is a t-structure on $\mathcal{D}(X)$.

Definition 2.2. The above t-structure is called the *perverse t-structure* (with respect to the perversity p) on $\mathcal{D}(X)$. Its heart, denoted $\mathcal{M}^p(X)$ or simply $\mathcal{M}(X)$, is the category of (G-equivariant) *perverse coherent sheaves* on X with respect to p. The truncation functors for this t-structure will be denoted $\tau^p_{\leq 0} : \mathcal{D}(X) \to {}^p\mathcal{D}(X)^{\leq 0}$ and $\tau^p_{\geq 0} : \mathcal{D}(X) \to {}^p\mathcal{D}(X)^{\geq 0}$.

We denote the standard t-structure on $\mathcal{D}(X)$ by $({}^{\mathrm{std}}\mathcal{D}(X)^{\leq 0}, {}^{\mathrm{std}}\mathcal{D}(X)^{\geq 0})$, and the associated truncation functors by $\tau^{\mathrm{std}}_{\leq 0}$ and $\tau^{\mathrm{std}}_{\geq 0}$. The perverse t-structure associated to the constant perversity $p = 0$ coincides with the standard t-structure.

Now, let U be a locally closed G-invariant subscheme of X, and let $Z = \overline{U} \smallsetminus U$. Let $U^{G\text{-gen}}$ and $Z^{G\text{-gen}}$ be the corresponding subspaces of $X^{G\text{-gen}}$. Given a perverse coherent sheaf on U, we wish to find a canonical way to associate to it a perverse coherent sheaf on $\overline{U}$, analogous to the intermediate extension operation on ordinary (constructible) perverse sheaves. This is not always possible, but [6, Theorem 2] gives one set of conditions under which it can be done. In fact, the conditions of that theorem can be weakened significantly, at the expense of having intermediate extension defined only on some subcategory of $\mathcal{M}(U)$ (see Remark 2.7).

The following proposition provides a general framework for defining intermediate extension on a subcategory of $\mathcal{M}(U)$. Later, we will determine the largest possible subcategory to which the proposition can be applied.

Define a partial order on perversities by pointwise comparison: we say that $p \leq q$ if $p(x) \leq q(x)$ for all $x \in X^{G\text{-gen}}$.

Proposition 2.3. *Suppose q, p, and r are perversities with the following properties: $q \leq p \leq r$, $r(x) - q(x) \leq 2$ for all x, and*

$$q(x) = p(x) - 1 \qquad \textit{and} \qquad r(x) = p(x) + 1 \qquad \textit{for all } x \in Z^{G\text{-gen}}.$$

Define two full subcategories by

$$\mathcal{M}^{q,r}(U) = {}^{q}\mathcal{D}(U)^{\leq 0} \cap {}^{r}\mathcal{D}(U)^{\geq 0} \subset \mathcal{M}^{p}(U),$$
$$\mathcal{M}^{q,r}(\overline{U}) = {}^{q}\mathcal{D}(\overline{U})^{\leq 0} \cap {}^{r}\mathcal{D}(\overline{U})^{\geq 0} \subset \mathcal{M}^{p}(\overline{U}),$$

and let $j : U \hookrightarrow \overline{U}$ be the inclusion map. Then $j^* : \mathcal{M}^{q,r}(\overline{U}) \to \mathcal{M}^{q,r}(U)$ is an equivalence of categories.

Definition 2.4. The inverse equivalence to that of Proposition 2.3, which is denoted $\mathcal{IC}^p(\overline{U}, \cdot) : \mathcal{M}^{q,r}(U) \to \mathcal{M}^{q,r}(\overline{U})$, or simply $\mathcal{IC}(\overline{U}, \cdot) : \mathcal{M}^{q,r}(U) \to \mathcal{M}^{q,r}(\overline{U})$, is called the *intermediate extension functor*.

Proof. Our proof is essentially identical to that of [6, Theorem 2]. Let $J_{!*} : \mathcal{D}(\overline{U}) \to \mathcal{D}(\overline{U})$ be the functor $\tau^q_{\leq 0} \circ \tau^r_{\geq 0}$. We claim that $J_{!*}$ actually takes values in $\mathcal{M}^{q,r}(\overline{U})$. Given $\mathcal{F} \in \mathcal{D}(\overline{U})$, let $\mathcal{F}_1 = \tau^r_{\geq 0}\mathcal{F}$. Then we have a distinguished triangle

$$(\tau^q_{\geq 1}\mathcal{F}_1)[-1] \to J_{!*}(\mathcal{F}) \to \mathcal{F}_1 \to \tau^q_{\geq 1}\mathcal{F}_1.$$

Note that $(\tau^q_{\geq 1}\mathcal{F}_1)[-1] \in {}^{q}\mathcal{D}(\overline{U})^{\geq 2}$. Now, the condition $r(x) - q(x) \leq 2$ implies that ${}^{q}\mathcal{D}(\overline{U})^{\geq 2} \subset {}^{r}\mathcal{D}(\overline{U})^{\geq 0}$. Clearly, $\mathcal{F}_1 \in {}^{r}\mathcal{D}(\overline{U})^{\geq 0}$, so it follows that $J_{!*}\mathcal{F} \in {}^{r}\mathcal{D}(\overline{U})^{\geq 0}$. Since it obviously takes values in ${}^{q}\mathcal{D}(\overline{U})^{\leq 0}$, $J_{!*}\mathcal{F} \in \mathcal{M}^{q,r}(\overline{U})$.

Next, note that if $\mathcal{F} \in \mathcal{D}(\overline{U})$ is such that $\mathcal{F}|_U \in \mathcal{M}^{q,r}(U)$, then both $(\tau^r_{\geq 0}\mathcal{F})|_U$ and $(\tau^q_{\leq 0}\mathcal{F})|_U$, and hence $(J_{!*}\mathcal{F})|_U$, are isomorphic to $\mathcal{F}|_U$. In particular, we can see now that j^* is essentially surjective. Given $\mathcal{F} \in \mathcal{M}^{q,r}(U)$, let $\tilde{\mathcal{F}}$ be any object on $\mathcal{D}(\overline{U})$ such that $j^*\tilde{\mathcal{F}} \simeq \mathcal{F}$. (Such an object exists by [6, Corollary 2].) Then $\mathcal{F}' = J_{!*}\tilde{\mathcal{F}}$ is an object of $\mathcal{M}^{q,r}(\overline{U})$ such that $j^*\mathcal{F}' \simeq \mathcal{F}$.

Now, if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism in $\mathcal{M}^{q,r}(U)$, then by [6, Corollary 2], we can find objects $\mathcal{F}'$ and $\mathcal{G}'$ in $\mathcal{D}(\overline{U})$ and a morphism $\phi' : \mathcal{F}' \to \mathcal{G}'$ such that $j^*\mathcal{F}' \simeq \mathcal{F}$, $j^*\mathcal{G}' \simeq \mathcal{G}$, and $j^*\phi' \simeq \phi$. By applying $J_{!*}$, we may assume that $\mathcal{F}'$, $\mathcal{G}'$, and ϕ' actually belong to $\mathcal{M}^{q,r}(\overline{U})$. This shows that j^* is full.

To show that j^* is faithful, it suffices to show that if ϕ is an isomorphism, then ϕ' must be as well. Since $\phi'|_U$ is an isomorphism, the kernel and cokernel of ϕ' must be supported on Z. But by [6, Lemma 6], the fact that $q(x) < p(x) < r(x)$

for $x \in Z^{G\text{-gen}}$ implies that $\mathcal{F}'$ and $\mathcal{G}'$ have no subobjects or quotients supported on Z. Thus, ϕ' is an isomorphism. Since j^* is fully faithful and essentially surjective, it is an equivalence of categories. □

Remark 2.5. It follows from the above proof that for any $\mathcal{F} \in \mathcal{M}^{q,r}(U)$, $\mathcal{IC}(\overline{U}, \mathcal{F})$ is isomorphic to $\tau^q_{\leq 0}\tau^r_{\geq 0}\tilde{\mathcal{F}}$, where $\tilde{\mathcal{F}}$ is any object of $\mathcal{D}(\overline{U})$ whose restriction to U is isomorphic to $\mathcal{F}$.

The above proof could also have been carried out using the functor $J'_{!*} = \tau^r_{\geq 0} \circ \tau^q_{\leq 0}$ instead of $J_{!*}$. From that version of the proof, one sees that $\mathcal{IC}(\overline{U}, \mathcal{F})$ is also isomorphic to $\tau^r_{\geq 0}\tau^q_{\leq 0}\tilde{\mathcal{F}}$.

Proposition 2.6. *Let p be a perversity, and let z_0 be a generic point of an irreducible component of Z of minimal codimension. Among all perversities q which, together with some r, satisfy the assumptions of Proposition 2.3, there is a unique maximal one, denoted p^-. It is given by*

$$p^-(x) = \begin{cases} p(x) - 1 & \text{if } p(x) \geq p(z_0), \\ p(x) & \text{if } p(x) < p(z_0). \end{cases} \tag{3}$$

Similarly, there is a unique minimal perversity among all r of that proposition, denoted p^+, and given by

$$p^+(x) = \begin{cases} p(x) + 1 & \text{if } \operatorname{codim} \bar{x} - p(x) \geq \operatorname{codim} \bar{z}_0 - p(z_0), \\ p(x) & \text{if } \operatorname{codim} \bar{x} - p(x) < \operatorname{codim} \bar{z}_0 - p(z_0). \end{cases} \tag{4}$$

Remark 2.7. Although our formulas for p^- and p^+ appear to be different from those of [6, Theorem 2], they do in fact coincide under the assumptions of loc. cit. Those assumptions are that U is open and dense in X and that for any $x \in U^{G\text{-gen}}$ and any $z \in \bar{x} \cap Z^{G\text{-gen}}$, we have

$$p(z) > p(x) \qquad \text{and} \qquad \operatorname{codim} \bar{z} - p(z) > \operatorname{codim} \bar{x} - p(x).$$

These inequalities cannot hold simultaneously unless $\operatorname{codim} \bar{x} \leq \operatorname{codim} \bar{z} - 2$. In particular, this means $U^{G\text{-gen}}$ cannot contain any closed points of U, so one must necessarily be in an equivariant setting.

Proposition 2.6, on the other hand, applies with no a priori restrictions on X or U. This really does allow us to use the intermediate extension functor in nonequivariant settings, but in practice, it is still necessary to require that $\operatorname{codim} U \leq \operatorname{codim} Z - 2$; indeed, if this condition fails, then $\mathcal{M}^{p^-,p^+}(U)$ will be reduced to the zero object. To see this, note that $p^-(x) = p^+(x)$ implies that $\operatorname{codim} \bar{x} \leq \operatorname{codim} \bar{z}_0 - 2$, so if $\operatorname{codim} U > \operatorname{codim} Z - 2$, then we have $p^-(x) < p^+(x)$ for all points $x \in U$. It follows that ${}^{p^-}\mathcal{D}(U)^{\leq 0} \subset {}^{p^+}\mathcal{D}(U)^{\leq -1}$, so any object in $\mathcal{M}^{p^-,p^+}(U)$ will belong to ${}^{p^+}\mathcal{D}(U)^{\leq -1} \cap {}^{p^+}\mathcal{D}(U)^{\geq 0}$. The latter category contains only the zero object.

Proof of Propositionn 2.6. Let us first show that p^- is a perversity. Suppose $\operatorname{codim} \bar{x} \geq \operatorname{codim} \bar{y}$, so $p(x) \geq p(y)$. If $p(x) \geq p(y) \geq p(z_0)$ or $p(z_0) > p(x) \geq p(y)$, then the conditions (2) obviously hold because they hold for p. Now suppose $p(x) \geq p(z_0) > p(y)$. The strictness of the second inequality implies that $\operatorname{codim} \bar{x} > \operatorname{codim} \bar{y}$. In this situation, we clearly have $p^-(x) = p(x) - 1 \geq p(y) = p^-(y)$ and

$$\operatorname{codim} \bar{x} - p^-(x) = \operatorname{codim} \bar{x} - p(x) + 1 > \operatorname{codim} \bar{y} - p(y) = -\operatorname{codim} \bar{y} - p^-(y).$$

Thus, p^- is a perversity.

Let q and r be perversities satisfying the assumptions of Proposition 2.3. The requirement that $q(x) = p(x) - 1$ for all $x \in Z^{G\text{-gen}}$ implies that $q(x) = p(x) - 1 = p^-(x)$ for all x with $\operatorname{codim} \bar{x} \geq \operatorname{codim} \bar{z}_0$. For all such points, of course, we have $p(x) \geq p(z_0)$. Now suppose x is such that $\operatorname{codim} \bar{x} < \operatorname{codim} \bar{z}_0$, so that $p(z_0) \geq p(x)$. If $p(z_0) > p(x)$, it is trivial that $q(x) \leq p^-(x)$, while if $p(z_0) = p(x)$, then $q(x) \leq q(z_0) = p(z_0) - 1 = p(x) - 1 = p^-(x)$. Thus, $q(x) \leq p^-(x)$ for all $x \in X^{G\text{-gen}}$, so $q \leq p^-$, and p^- has the desired maximality property.

The proofs of the corresponding statements for p^+ are similar. □

Remark 2.8. If we were to change the definition of "perversity" by imposing the inequalities (2) only when $y \in \bar{x}$, then this result could be improved, i.e., p^- could be replaced be a larger perversity and p^+ by a smaller one, resulting in a larger domain category for $\mathcal{IC}(\overline{U}, \cdot)$. Let us call a sequence of points $x_1, y_1, x_2, \ldots, y_k, x_{k+1}$ in $X^{G\text{-gen}}$ a *lower chain* (resp. *upper chain*) if the following conditions hold:

(1) $x_i, x_{i+1} \in \bar{y}_i$ for all i, and $x_{k+1} \in \bar{y}_k \cap Z^{G\text{-gen}}$, and
(2) $p(x_{i+1}) = p(y_i)$ and $p(x_i) = p(y_i) - \operatorname{codim} \bar{y}_i + \operatorname{codim} \bar{x}_i$ (resp. $p(x_{i+1}) = p(y_i) - \operatorname{codim} \bar{y}_i + \operatorname{codim} \bar{x}_{i+1}$ and $p(x_i) = p(y_i)$) for all i.

Let S (resp. T) be the set of all points of $X^{G\text{-gen}}$ occurring in some lower (resp. upper) chain, and define

$$p^{\ominus}(x) = \begin{cases} p(x) - 1 & \text{if } x \in S, \\ p(x) & \text{otherwise,} \end{cases} \qquad \text{and} \qquad p^{\oplus}(x) = \begin{cases} p(x) + 1 & \text{if } x \in T, \\ p(x) & \text{otherwise.} \end{cases}$$

It is not difficult to prove an analogue of Proposition 2.6 using these formulas instead of p^- and p^+.

3. Notation and Preliminaries

In this section, we introduce some useful notation and terminology, and we prove a number of lemmas on perverse coherent sheaves. To simplify the discussion, we henceforth assume that U is actually an open dense subscheme of X and that Z has codimension at least 2. Let $j : U \hookrightarrow X$ be the inclusion map. For the most part, we will consider only "standard" perversities, defined as follows.

Definition 3.1. A perversity p is said to be *standard* if

$$p(x) = p^-(x) = p^+(x) = 0 \qquad \text{if } \operatorname{codim} \bar{x} = 0. \tag{5}$$

Note that if p is standard, so is its dual $\bar{p}$.

Remark 3.2. The assumption that $\operatorname{codim} Z \geq 2$ is essential: if this condition fails, there are no standard perversities.

Given a perversity p, when there is no risk of ambiguity, we write

$$\mathcal{D}(X)^{-,\leq 0} = {}^{p^-}\mathcal{D}(X)^{\leq 0} \qquad \text{and} \qquad \mathcal{D}(X)^{+,\geq 0} = {}^{p^+}\mathcal{D}(X)^{\geq 0},$$

or even simply $\mathcal{D}^{-,\leq 0}$ and $\mathcal{D}^{+,\geq 0}$. Next, let

$$\mathcal{M}^{p,\pm}(U) = \mathcal{D}(U)^{-,\leq 0} \cap \mathcal{D}(U)^{+,\geq 0} \qquad \text{and} \qquad \mathcal{M}^{p,\pm}(X) = \mathcal{D}(X)^{-,\leq 0} \cap \mathcal{D}(X)^{+,\geq 0}.$$

Then we have an intermediate extension functor $\mathcal{IC}(X, \cdot) : \mathcal{M}^{p,\pm}(U) \to \mathcal{M}^{p,\pm}(X)$. These categories will usually be denoted simply $\mathcal{M}^{\pm}(U)$ and $\mathcal{M}^{\pm}(X)$, respectively.

Let $\mathbb{D}$ denote the coherent (Serre-Grothendieck) duality functor $R\mathcal{H}om(\cdot, \omega_X)$. By [6, Lemma 5], $\mathbb{D}$ takes $\mathcal{M}^p(X)$ to $\mathcal{M}^{\bar{p}}(X)$ and $\mathcal{M}^{p,\pm}(\overline{U})$ to $\mathcal{M}^{\bar{p},\pm}(\overline{U})$.

Two specific standard perversities will be particularly useful in the sequel:

$$s(x) = \begin{cases} 0 \\ 1 \end{cases} \qquad c(x) = \begin{cases} \operatorname{codim} \bar{x} & \text{if } \operatorname{codim} \bar{x} < \operatorname{codim} Z, \\ \operatorname{codim} \bar{x} - 1 & \text{if } \operatorname{codim} \bar{x} \geq \operatorname{codim} Z. \end{cases}$$

We call s the "S_2 perversity" and c the "Cohen–Macaulay perversity" for reasons that are made clear in Lemma 3.9. These two perversities are dual to one another, and they are extremal among all standard perversities (see Lemma 3.3). For convenience, we also record the corresponding "$-$" and "$+$" perversities:

$$s^-(x) = 0 \qquad c^+(x) = \operatorname{codim} \bar{x}$$

$$s^+(x) = \begin{cases} 0 \\ 1 \\ 2 \end{cases} \qquad c^-(x) = \begin{cases} \operatorname{codim} \bar{x} & \text{if } \operatorname{codim} \bar{x} < \operatorname{codim} Z + 1, \\ \operatorname{codim} \bar{x} - 1 & \text{if } \operatorname{codim} \bar{x} = \operatorname{codim} Z + 1, \\ \operatorname{codim} \bar{x} - 2 & \text{if } \operatorname{codim} \bar{x} \geq \operatorname{codim} Z. \end{cases}$$

It is clear that s^- and c^+ are the smallest and largest possible perversities, respectively, that take the value 0 on generic points of X. Since the "$-$" and "$+$" operations respect the partial order on perversities, we have the following result.

Lemma 3.3. *Every standard perversity p satisfies $s \leq p \leq c$.* □

We will use the following observation repeatedly.

Lemma 3.4. *Let $\mathcal{F}$ be a coherent sheaf on U. The complex $\mathcal{IC}(X, \mathcal{F})$ is defined if and only if $\mathcal{F} \in \mathcal{D}(U)^{+,\geq 0}$, or equivalently, if $\operatorname{depth}_{\mathcal{O}_x} \mathcal{F}_x \geq p^+(x)$ for all $x \in U^{G\text{-gen}}$.*

If $\mathcal{IC}(X, \mathcal{F})$ is defined, then given a coherent extension $\mathcal{G}$ of $\mathcal{F}$ to X, we have $\mathcal{G} \simeq \mathcal{IC}(X, \mathcal{F})$ if and only if $\mathcal{G} \in \mathcal{D}(X)^{+,\geq 0}$, or equivalently, if $\operatorname{depth}_{\mathcal{O}_x} \mathcal{G}_x > p(x)$ for all $x \in Z^{G\text{-gen}}$.

Proof. The exactness of i_x^* implies that $H^k(i_x^*\mathcal{F}) = 0$ unless $k = 0$, and since we are assuming that p is a standard perversity, this cohomology vanishes for $k > p^-(x) \geq 0$. Thus, $\mathcal{F} \in \mathcal{D}(U)^{-,\leq 0}$ automatically. The depth-condition characterization of $\mathcal{D}(U)^{+,\geq 0}$ comes from the well-known fact that the lowest degree in which $H^k(i_x^!\mathcal{F})$ is nonzero is $\operatorname{depth}_{\mathcal{O}_x} \mathcal{F}_x$. The same arguments apply to $\mathcal{G}$ as well. □

Next, recall (see [13, §I.3.3]) that to any quasicoherent sheaf of algebras $\mathcal{F}$ on X, one can canonically associate a new scheme Y and an affine morphism $f : Y \to X$ such that $f_*\mathcal{O}_Y \simeq \mathcal{F}$. Moreover, f is finite if and only if $\mathcal{F}$ is coherent. This procedure is often called "global Spec"; we will use the notation $Y = \mathbf{Spec}\, \mathcal{F}$.

Coherent sheaves of algebras and the global Spec operation play a major role in the sequel. The following proposition relates these to the $\mathcal{IC}$ functor.

Proposition 3.5. *Let $\mathcal{F}$ be a coherent sheaf of algebras on X. Form $Y = \mathbf{Spec}\, \mathcal{F}$, and let $f : Y \to X$ be the canonical map. Then $\mathcal{IC}(Y, \mathcal{O}_{f^{-1}(U)})$ is defined if and only if $\mathcal{IC}(X, \mathcal{F}|_U)$ is. If both are defined, then $\mathcal{IC}(Y, \mathcal{O}_{f^{-1}(U)}) \simeq \mathcal{O}_Y$ if and only if $\mathcal{IC}(X, \mathcal{F}|_U) \simeq \mathcal{F}$.*

Remark 3.6. Here, the notation $\mathcal{IC}(Y, \mathcal{O}_{f^{-1}(U)})$ is to be understood in terms of the intermediate extension functor associated to the open inclusion $f^{-1}(U) \hookrightarrow Y$ and the perversity $p' = p \circ f : Y^{G\text{-gen}} \to Z$. The fact that f is a finite morphism implies that the complement of $f^{-1}(U)$ has the same codimension as Z. In addition, $\operatorname{codim} f^{-1}(\bar{x}) = \operatorname{codim} \bar{x}$ for any $x \in X^{G\text{-gen}}$, so p' does indeed satisfy the inequalities (2).

Moreover, since Y is finite over X, it satisfies our basic hypotheses for defining perverse coherent sheaves—it is of finite type over S, and $\mathfrak{Coh}(Y)$ has enough locally free objects. (To see the latter, note that if $\mathcal{G}$ is a coherent $\mathcal{O}_Y$-module, then there is a locally free $\mathcal{O}_X$-module $\mathcal{F}$ which surjects to $f_*\mathcal{G}$. This gives a surjective morphism of $\mathcal{O}_Y$-modules $f^*\mathcal{F} \to f^*f_*\mathcal{G}$. Composing with the surjection $f^*f_*\mathcal{G} \to \mathcal{G}$ exhibits $\mathcal{G}$ as a quotient of the locally free $\mathcal{O}_Y$-module $f^*\mathcal{F}$.)

Proof of Proposition 3.5. By Lemma 3.4, it suffices to show $\mathcal{O}_{f^{-1}(U)} \in \mathcal{D}(f^{-1}(U))^{+,\geq 0}$ if and only if $\mathcal{F}|_U \in \mathcal{D}(U)^{+,\geq 0}$, and then that $\mathcal{O}_Y \in \mathcal{D}(Y)^{+,\geq 0}$ if and only if $\mathcal{F} \in \mathcal{D}(X)^{+,\geq 0}$. We prove both assertions simultaneously.

Let $x \in X^{G\text{-gen}}$, and let $Y_x = f^{-1}(x)$. The latter is a finite set of points, and $(f|_{Y_x})_*$ is clearly an exact functor that kills no nonzero sheaf. Now, we have

$$R(f|_{Y_x})_* i^!_{Y_x}\mathcal{O}_Y \simeq i^!_x f_*\mathcal{O}_Y \simeq i^!_x\mathcal{F},$$

so the lowest degree in which $H^k(i^!_x\mathcal{F})$ is nonzero is the same as the lowest degree in which $H^k(i^!_{Y_x}\mathcal{O}_Y)$ is nonzero. Let i_{y,Y_x} be the inclusion of a point y into Y_x. Then $i^!_{y,Y_x} = i^*_{y,Y_x}$ is also an exact functor; it kills no nonzero sheaf whose support contains y. Since $i^!_y = i^!_{y,Y_x} \circ i^!_{Y_x}$, we conclude that the lowest degree in which some $H^k(i^!_y\mathcal{O}_Y) = H^k(i^!_{y,Y_x} i^!_{Y_x}\mathcal{O}_Y)$ is nonzero is the same as the lowest degree in which $H^k(i^!_x\mathcal{F})$ is nonzero. In particular, considering the degree $k = p^+(x)$, we see that $\mathcal{O}_Y \in \mathcal{D}(Y)^{+,\geq 0}$ if and only if $\mathcal{F} \in \mathcal{D}(X)^{+,\geq 0}$, and likewise for $\mathcal{O}_{f^{-1}(U)}$ and $\mathcal{F}|_U$. □

Note that the proof in fact shows that if f is a finite morphism, then f_* is t-exact.

In the remainder of the section, we prove a handful of results specific to the S_2 and Cohen–Macaulay perversities.

Proposition 3.7. *For any coherent sheaf $\mathcal{E}$ on U such that $\mathcal{IC}^s(X,\mathcal{E})$ is defined, there is a canonical isomorphism $\mathcal{IC}^s(X,\mathcal{E}) \simeq j_*\mathcal{E}$. In particular, $j_*\mathcal{E}$ is coherent.*

Proof. We begin by observing that $\mathcal{IC}^s(X,\mathcal{E})$ is actually a sheaf (i.e., that it is concentrated in degree 0). Indeed, $\mathcal{IC}^s(X,\mathcal{E})$ is perverse with respect to s^-. This perversity is constant with value 0, so the resulting t-structure is just the standard t-structure.

Next, we show that $j_*\mathcal{E}$ is coherent. Note that the smallest value taken by s^+ on $Z^{G\text{-gen}}$ is 2. Now, $\mathcal{E}$ is, by assumption, a perverse coherent sheaf on U with respect to s^+. According to [6, Corollary 3], the complex $\tau^{\mathrm{std}}_{\leq 0}(Rj_*\mathcal{E})$ has coherent cohomology. But that object is simply $j_*\mathcal{E}$, the nonderived push-forward of $\mathcal{E}$.

Since $j_*\mathcal{E}$ is concentrated in degree 0, it obviously lies in $\mathcal{D}^{-,\leq 0}(X)$, so by Remark 2.5, $\mathcal{IC}^s(X,\mathcal{E})$ can be calculated as $\tau^+_{\geq 0}(j_*\mathcal{E})$. Thus, the truncation functor gives us a canonical morphism $j_*\mathcal{E} \to \mathcal{IC}^s(X,\mathcal{E})$. On the other hand, we have the usual adjunction morphism $\mathcal{IC}^s(X,\mathcal{E}) \to j_*j^*\mathcal{IC}^s(X,\mathcal{E}) \simeq j_*\mathcal{E}$. Both these morphisms have the property that their restrictions to U are simply the identity morphism of $\mathcal{E}$. The compositions

$$\mathcal{IC}^s(X,\mathcal{E}) \to j_*\mathcal{E} \to \mathcal{IC}^s(X,\mathcal{E}) \qquad \text{and} \qquad j_*\mathcal{E} \to \mathcal{IC}^s(X,\mathcal{E}) \to j_*\mathcal{E}$$

are then both identity morphisms of the appropriate objects, because their restrictions to U are the identity morphism of $\mathcal{E}$, and the functors $\mathcal{IC}^s(X,\cdot)$ and j_* are both fully faithful. Thus, $\mathcal{IC}^s(X,\mathcal{E}) \simeq j_*\mathcal{E}$. □

We remark that the notation "$\mathcal{IC}^s(X, \mathcal{E})$" is still useful, in spite of the above proposition, because $\mathcal{IC}^s(X, \mathcal{E})$ is not always defined, whereas $j_*\mathcal{E}$ is. Most of the statements in Section 4 become false if we drop the assumption that $\mathcal{IC}(X, \mathcal{E})$ be defined and replace that object by $j_*\mathcal{E}$, which is not coherent in general. Moreover, most of the proofs rely, at least implicitly, on the fact that the S_2-perversity gives rise to a nontrivial t-structure on $\mathcal{D}(X)$.

Definition 3.8. A scheme X is said to be *locally* S_2 at $x \in X$ if $\operatorname{depth} \mathcal{O}_x \geq \min\{2, \dim \mathcal{O}_x\}$. X is S_2 if it is locally S_2 at every point.

Lemma 3.9. $\mathcal{IC}^s(X, \mathcal{O}_U)$ *is defined if and only if U is locally S_2 at all points $x \in U^{G\text{-gen}}$ such that* $\operatorname{codim} \bar{x} \geq \operatorname{codim} Z$. *In that case, the following conditions are equivalent:*

(1) $\mathcal{IC}^s(X, \mathcal{O}_U) \simeq \mathcal{O}_X$.

(2) *X is locally S_2 at all points of $Z^{G\text{-gen}}$.*

Similarly, $\mathcal{IC}^c(X, \mathcal{O}_U)$ *is defined if and only if U is locally Cohen–Macaulay at all points of $U^{G\text{-gen}}$. In that case, the following conditions are equivalent:*

(1) $\mathcal{IC}^c(X, \mathcal{O}_U) \simeq \mathcal{O}_X$.

(2) *X is locally Cohen–Macaulay at all points of $Z^{G\text{-gen}}$.*

Proof. The proofs of the two parts of this lemma are essentially identical; we will treat only the S_2 case. By Lemma 3.4, $\mathcal{IC}^s(X, \mathcal{O}_U)$ is defined if and only if $\operatorname{depth} \mathcal{O}_x \geq s^+(x)$ for all $x \in U^{G\text{-gen}}$, i.e., if

$$\operatorname{depth} \mathcal{O}_x \geq \begin{cases} 0 & \text{if } \operatorname{codim} \bar{x} < \operatorname{codim} Z - 1, \\ 1 & \text{if } \operatorname{codim} \bar{x} = \operatorname{codim} Z - 1, \\ 2 & \text{if } \operatorname{codim} \bar{x} \geq \operatorname{codim} Z. \end{cases}$$

The first two cases above hold trivially. (In the case $\operatorname{codim} \bar{x} = \operatorname{codim} Z - 1$, we have $\dim \mathcal{O}_x \geq 1$, and any local ring of positive dimension has positive depth.) Since Z has codimension at least 2, the last case holds only if $\mathcal{O}_x$ is S_2. Thus, $\mathcal{IC}^s(X, \mathcal{O}_U)$ is defined if and only if U is locally S_2 at points $x \in X^{G\text{-gen}}$ with $\operatorname{codim} \bar{x} \geq \operatorname{codim} Z$. The same argument applied to $\mathcal{O}_X$ shows the equivalence of conditions (1) and (2) above. ☐

A similar proof gives the following result:

Lemma 3.10. $\mathcal{IC}^s(X, \mathcal{F})$ *is defined if and only if* $\operatorname{depth} \mathcal{F}_x \geq \min\{2, \dim \mathcal{F}_x\}$ *at all points $x \in U^{G\text{-gen}}$ such that* $\operatorname{codim} \bar{x} \geq \operatorname{codim} Z$.

Finally, for the last two lemmas of this section, we assume that G is a linear algebraic group over $S = \operatorname{Spec} k$ for some algebraically closed field k, and that X is a variety over k. In this case, we can extract a bit more geometric information from the preceding results. Recall that in this setting, the notion of "orbit" is well-behaved: X is a union of orbits, each of which is a smooth locally closed subvariety, isomorphic to a homogeneous space for G. The following lemma deals with local cohomology on an orbit.

Lemma 3.11. *Let $\mathcal{F} \in \mathcal{D}(X)$, and let C be a G-orbit in X. Suppose that there is some $p \in \mathbb{Z}$ such that for any generic point x of C, we have $H^k(i_x^! \mathcal{F}) = 0$ for all $k < p$. Then, for any $y \in C$, we have $H^k(i_y^! \mathcal{F}) = 0$ for all $k < p +$* $\operatorname{depth} \mathcal{O}_{y,C}$, *where $\mathcal{O}_{y,C}$ is the local ring at y of the reduced induced scheme structure on C.*

Proof. We begin by noting that any equivariant coherent sheaf on C is locally free. Indeed, given a coherent sheaf $\mathcal{E}$ on C, consider the function $\phi : C \to \mathbb{Z}$ defined by $\phi(y) = \dim_{k(y)} k(y) \otimes_{\mathcal{O}_{y,C}} \mathcal{E}_y$, where $k(y)$ is the residue field of the local ring $\mathcal{O}_{y,C}$. This function is constant on closed points of C (because they form a single G-orbit), and hence, by the semicontinuity theorem, on all of C. By, for instance, [17, Ex. II.5.8], since ϕ is constant and C is reduced, $\mathcal{E}$ is locally free.

Now, let $i_{y,C} : \{y\} \hookrightarrow C$ and $j_C : C \hookrightarrow X$ be the inclusion maps. (The former is merely a topological map; the latter is a morphism of schemes.) Recall that $i^!_{y,C}(j^!_C\mathcal{F}) \simeq R\mathcal{H}om(\mathcal{O}_{y,C}, i^!_y\mathcal{F})$, where $j^!_C$ is the right adjoint to Rj_{C*} in the setting of Grothendieck duality for coherent sheaves (as constructed in, say, [16]), but $i^!_{y,C}$ and $i^!_y$ are the Verdier-duality right adjoints to $(i_{y,C})_!$ and $(i_y)_!$, respectively.

By the argument given in [6, Lemma 2(b)], the vanishing assumptions on $H^k(i^!_x\mathcal{F})$ for x a generic point of C imply that $H^k(j^!_C\mathcal{F})$ vanishes for all $k < p$; furthermore, the lowest nonzero cohomologies of $i^!_{y,C}j^!_C\mathcal{F}$ and of $i^!_y\mathcal{F}$ occur in the same degree. Now, $j^!_C\mathcal{F}$ is a bounded complex of locally free sheaves on C, so there is some open subscheme $C_0 \subset C$ containing y such that $j^!_C\mathcal{F}|_{C_0}$ is in fact a complex of free sheaves. Recall, as in the proof of Lemma 3.4, that $H^k(i^!_{y,C_0}\mathcal{O}_{y,C_0})$ vanishes in degrees $k < \operatorname{depth} \mathcal{O}_{y,C_0} = \operatorname{depth} \mathcal{O}_{y,C}$. It follows that the cohomology of $i^!_{y,C}j^!_C\mathcal{F} = i^!_{y,C_0}(j^!_C\mathcal{F}|_{C_0})$ vanishes in degrees $k < p + \operatorname{depth} \mathcal{O}_{y,C}$. □

We conclude with the following refinement of Lemma 3.9.

Lemma 3.12. *Assume that G acts on X with finitely many orbits. If $\mathcal{IC}^s(X, \mathcal{O}_U)$ is defined, then the following conditions are equivalent:*

1. $\mathcal{IC}^s(X, \mathcal{O}_U) \simeq \mathcal{O}_X$.
2. *X is locally S_2 at all points of Z.*

Similarly, if $\mathcal{IC}^c(X, \mathcal{O}_U)$ is defined, the following conditions are equivalent:

1. $\mathcal{IC}^c(X, \mathcal{O}_U) \simeq \mathcal{O}_X$.
2. *X is locally Cohen–Macaulay at all points of Z.*

Proof. As in the proof of Lemma 3.9, we treat only the S_2 case. Since G acts with finitely many orbits, every closed G-invariant subvariety contains an open orbit, so every point of $X^{G\text{-gen}}$ is a generic point of some G-orbit. It suffices to show that part (2) of Lemma 3.9 is equivalent to part (2) of the present lemma. That assertion follows from Lemma 3.11: we see that for any $x \in Z^{G\text{-gen}}$ and any y in the G-orbit C containing x, we have $\operatorname{depth} \mathcal{O}_y \geq \operatorname{depth} \mathcal{O}_x$, since $\operatorname{depth} \mathcal{O}_{x,C} \geq 0$. □

4. S_2-Extension

Our goal in this section is to use coherent intermediate extension with respect to the S_2-perversity to construct new schemes and then to use powerful general properties of the intermediate extension functor to deduce various properties of those schemes. Throughout this section, all $\mathcal{IC}$'s will be with respect to the S_2-perversity unless otherwise specified.

The construction involves the global Spec operation (see Proposition 3.5 and the comments preceding it) on coherent sheaves of commutative algebras. Henceforth, all sheaves of algebras that we consider will be assumed to be coherent and commutative. We reemphasize the fact that we are working in the equivariant setting, so that schemes are G-schemes, morphisms are G-morphisms, and sheaves are G-equivariant.

Proposition 4.1. *Let $\mathcal{E}$ be a sheaf of $\mathcal{O}_U$-algebras on U. Then $\mathcal{IC}(X, \mathcal{E})$ can be made into a sheaf of $\mathcal{O}_X$-algebras in a unique way that is compatible with the algebra structure on $\mathcal{E}$.*

Proof. This is immediate from Proposition 3.7 and the fact that the algebra structure on $\mathcal{E}$ determines a unique algebra structure on $j_*\mathcal{E}$. □

Definition 4.2. Let $U \subset X$ be an open subscheme whose complement has codimension at least 2. A morphism of schemes $f : Y \to X$ is said to be S_2 *relative to* U if for all $x \in X^{G\text{-gen}}$ such that $\operatorname{codim} \bar{x} \geq \operatorname{codim} Z$, we have $H^k(i_x^! f_* \mathcal{O}_Y) = 0$ if $k < 2$.

Remark 4.3. Note that if f is finite and S_2 relative to U, then the image under f of any generic point of an irreducible component of Y must lie in U. Indeed, if y is such a generic point and $x = f(y)$, then $H^0(i_y^! \mathcal{O}_Y) \neq 0$, which implies that $H^0(i_x^! f_* \mathcal{O}_Y) \neq 0$ by the argument given in the proof of Proposition 3.5. In particular, $f^{-1}(U)$ cannot be empty; in fact, it is open dense.

Remark 4.4. If f is finite, the definition of "S_2 relative to U" is equivalent to requiring that $f_*\mathcal{O}_Y \simeq \mathcal{IC}(X, f_*\mathcal{O}_{f^{-1}(U)})$, and hence, according to Proposition 3.5, to requiring that $\mathcal{IC}(Y, \mathcal{O}_{f^{-1}(U)}) \simeq \mathcal{O}_Y$. (Note that the proposition applies since $f^{-1}(U)$ is open dense by the previous remark.)

In particular, by Lemma 3.9, $\mathrm{id} : X \to X$ is S_2 relative to U if and only if $\mathcal{IC}(X, \mathcal{O}_U)$ is defined and X is locally S_2 outside U. Moreover, if $f : Y \to X$ is a finite morphism with Y S_2 and $f^{-1}(U)$ dense, then f is S_2 relative to U.

Theorem 4.5. *Let $\rho_1 : \tilde{U} \to U$ be a finite morphism such that $\mathcal{IC}(X, \rho_{1*}\mathcal{O}_{\tilde{U}})$ is defined, and let $\tilde{X}$ denote the scheme* $\mathbf{Spec}\, \mathcal{IC}(X, \rho_{1*}\mathcal{O}_{\tilde{U}})$*. The natural morphism $\rho : \tilde{X} \to X$ is universal with respect to finite morphisms $f : Y \to X$ which are S_2 relative to U and whose restriction $f|_{f^{-1}(U)}$ factors through ρ_1. In other words, if $f : Y \to X$ is any finite morphism that is S_2 relative to U and such that $f|_{f^{-1}(U)}$ factors through ρ_1, then f factors through ρ in a unique way.*

In addition, ρ is a finite morphism, $\rho^{-1}(U) \simeq \tilde{U}$, and $\rho|_{\tilde{U}} = \rho_1$. Moreover, $\tilde{U}$ is a dense open subscheme of $\tilde{X}$, and $\mathrm{id} : \tilde{X} \to \tilde{X}$ *is S_2 relative to $\tilde{U}$.*

Here is a diagram:

$$\begin{tikzcd}
f^{-1}(U) \arrow[r, hook] \arrow[dr, "g_1"] \arrow[ddr, "f|_{f^{-1}(U)}"'] & Y \arrow[dr, dotted, "g"] \arrow[ddr, bend left, "f"'] & \\
& \tilde{U} \arrow[r, hook, "\tilde{j}"'] \arrow[d, "\rho_1"] & \tilde{X} \arrow[d, "\rho"] \\
& U \arrow[r, hook, "j"'] & X
\end{tikzcd}$$

As usual, the universal property enjoyed by $\tilde{X}$ and ρ characterizes them uniquely up to unique isomorphism.

Note that by Lemma 3.10, the condition that $\mathcal{IC}(X, \rho_{1*}\mathcal{O}_{\tilde{U}})$ be defined is equivalent to requiring that $\operatorname{depth}(\rho_{1*}\mathcal{O}_{\tilde{U}})_x \geq \min\{2, \dim(\rho_{1*}\mathcal{O}_{\tilde{U}})_x\}$ at all points $x \in U^{G\text{-gen}}$ such that $\operatorname{codim} \bar{x} \geq \operatorname{codim} Z$. Moreover, since $\mathcal{IC}(X, \rho_{1*}\mathcal{O}_{\tilde{U}}) \in \mathcal{M}^{\pm}(X)$, this sheaf satisfies the analogous depth conditions for all $x \in X^{G\text{-gen}}$ with $\operatorname{codim} \bar{x} \geq \operatorname{codim} Z$. By Proposition 3.7, this is equivalent to saying that $j \circ \rho_1 : \tilde{U} \to X$ is S_2 relative to U.

Definition 4.6. The scheme $\tilde{X}$ constructed in Theorem 4.5 is called the S_2-*extension* of $\rho_1 : \tilde{U} \to U$.

It should be noted that in general, an S_2-extension may be locally S_2 only at the points in $X^{G\text{-gen}}$. It is in fact an S_2 scheme when G is trivial (so that $X^{G\text{-gen}} = X$) or when Lemma 3.12 can be invoked. Moreover, if $\mathcal{IC}(X, \rho_{1*}\mathcal{O}_{\tilde{U}})$ is defined in the nonequivariant case, i.e., the depth condition on $(\rho_{1*}\mathcal{O}_{\tilde{U}})_x$ described above holds for $x \in U$ and not just $x \in U^{G\text{-gen}}$, then both equivariant and nonequivariant S_2-extensions of ρ_1 are defined. Since the nonequivariant universal mapping property is stronger than the equivariant universal property, we obtain the following corollary.

Corollary 4.7. *If $\mathcal{IC}(X, \rho_{1*}\mathcal{O}_{\tilde{U}})$ is defined in the nonequivariant case, then the nonequivariant and equivariant S_2-extensions of ρ_1 are canonically isomorphic.*

Before proving the theorem, we consider a few examples in which S_2-extension has an elementary description.

Example 4.8. If the map $j \circ \rho_1 : \tilde{U} \to X$ is already finite (for example, if $\tilde{U}$ is a single point), then $\tilde{X} = \tilde{U}$.

Example 4.9. Note that the complement of $\tilde{U}$ in $\tilde{X}$ must have codimension at least 2, since Z has codimension at least 2 in X and ρ is finite. Recall that according to Serre's criterion, a scheme is normal if and only if it is S_2 and regular in codimension 1. Thus, if $\tilde{U}$ is normal, the fact that $\tilde{X}$ is locally S_2 outside $\tilde{U}$ implies that $\tilde{X}$ is also normal.

In particular, suppose that U is a normal subscheme of the integral scheme X and that $\rho_1 : \tilde{U} \to U$ is an isomorphism. Then $\rho : \tilde{X} \to X$ is simply the usual normalization of X. In view of Proposition 3.7, we see that the normalization of X has a remarkably simple description as $\mathbf{Spec}(j_*\mathcal{O}_U)$.

Example 4.10. As a slight generalization of the previous example, let us now suppose only that $\tilde{U}$ is normal and that X is integral. An elementary construction of $\tilde{X}$ is given as follows. Given an affine open subscheme $V = \operatorname{Spec} A$ of X, let K be the fraction field of $\rho_1^{-1}(V)$, and let B be the integral closure of the image of the natural map $A \to K$ induced by ρ_1. Let $\tilde{V} = \operatorname{Spec} B$. The various $\tilde{V}$'s obtained in this way as V ranges over affine open subschemes of X can be glued together to form a scheme $\tilde{X}'$. This scheme enjoys a universal property similar to that of the normalization of a scheme. By comparing with the universal property of $\tilde{X}$, it is easy to verify that $\tilde{X}$ and $\tilde{X}'$ are in fact canonically isomorphic. We thus obtain an alternative elementary description of $\mathcal{IC}(X, \rho_{1*}\mathcal{O}_{\tilde{U}})$; it is the sheaf $V \mapsto B$.

Proof of Theorem 4.5. We begin by establishing various properties of ρ and $\tilde{X}$. Since $\mathcal{IC}(X, \rho_{1*}\mathcal{O}_{\tilde{U}})$ is coherent, ρ is finite. From the definition of **Spec**, we know that $\rho^{-1}(U) \simeq \mathbf{Spec}\, \mathcal{IC}(X, \rho_{1*}\mathcal{O}_{\tilde{U}})|_U \simeq \mathbf{Spec}\, \rho_{1*}\mathcal{O}_{\tilde{U}}$. Now, ρ_1 is finite, and therefore affine, so $\mathbf{Spec}\, \rho_{1*}\mathcal{O}_{\tilde{U}}$ is canonically isomorphic to $\tilde{U}$. Identifying these two schemes, we also see that $\rho|_{\tilde{U}} = \rho_1$. Moreover, Proposition 3.5 tells us that $\mathcal{IC}(\tilde{X}, \mathcal{O}_{\tilde{U}}) \simeq \mathcal{O}_{\tilde{X}}$, and then by Lemma 3.9 and Remark 4.4, we see that $\mathrm{id} : \tilde{X} \to \tilde{X}$ is S_2 relative to $\tilde{U}$.

As we have previously observed, ρ finite implies that ρ_* is exact and t-exact. Thus, $\rho_*\mathcal{O}_{\tilde{X}} \simeq \mathcal{IC}(X, \rho_{1*}\mathcal{O}_{\tilde{U}})$, and ρ is S_2 relative to U. We have already seen that $\rho|_{\tilde{U}}$ factors through ρ_1; indeed, with the obvious identifications, it equals ρ_1.

Finally, Remark 4.3 tells us that all generic points of irreducible components of $\tilde{X}$ lie in $\tilde{U}$, and hence that $\tilde{U}$ is dense in $\tilde{X}$.

It remains to show that $\tilde{X}$ and ρ are universal with respect to these properties. Let $f : Y \to X$ be a finite morphism that is S_2 relative to U, and assume that $f|_{f^{-1}(U)}$ factors through ρ_1. Let $V = f^{-1}(U)$, and let $g_0 : V \to \tilde{U}$ be the morphism such that $f|_V = \rho_1 \circ g_0$. Then g_0 gives rise to a morphism of sheaves $\rho_{1*}\mathcal{O}_{\tilde{U}} \to f_*\mathcal{O}_V$ on U and therefore to a morphism of perverse coherent sheaves

$$\rho_*\mathcal{O}_{\tilde{X}} \simeq \mathcal{IC}(X, \rho_{1*}\mathcal{O}_{\tilde{U}}) \to \mathcal{IC}(X, f_*\mathcal{O}_V) \simeq f_*\mathcal{O}_Y.$$

Applying the global Spec functor to this morphism of sheaves $\rho_*\mathcal{O}_{\tilde{X}} \to f_*\mathcal{O}_Y$, we obtain a morphism of schemes $g : Y \to \tilde{X}$; this is the desired morphism such that $f = \rho \circ g$. The uniqueness of g follows from the fact that there is a unique morphism $\mathcal{IC}(X, \rho_{1*}\mathcal{O}_{\tilde{U}}) \to \mathcal{IC}(X, f_*\mathcal{O}_V)$ whose restriction to U is the morphism $\rho_{1*}\mathcal{O}_{\tilde{U}} \to f_*\mathcal{O}_V$ induced by g_0. □

As usual, any object characterized by a universal property comes with a uniqueness theorem, but for S_2-extension, there is an even stronger uniqueness property.

Proposition 4.11. *Let $\hat{X}$ be a scheme containing $\tilde{U}$ as a dense open set, and let $\hat{\rho} : \hat{X} \to X$ be a finite morphism extending $\rho_1 : \tilde{U} \to U$. If $\hat{X}$ is locally S_2 outside of $\tilde{U}$, then $\hat{X}$ is isomorphic to $\tilde{X}$.*

Proof. Since $\hat{X}$ is locally S_2 outside of $\tilde{U}$, its structure sheaf is an $\mathcal{IC}$ sheaf: $\mathcal{O}_{\hat{X}} \simeq \mathcal{IC}(\hat{X}, \mathcal{O}_{\tilde{U}})$. The functor $\hat{\rho}_*$ is exact and t-exact, so $\hat{\rho}_*\mathcal{O}_{\hat{X}} \simeq \mathcal{IC}(X, \hat{\rho}_*\mathcal{O}_{\tilde{U}})$. But now $\hat{X} \simeq \mathbf{Spec}\, \rho_*\mathcal{O}_{\hat{X}} \simeq \tilde{X}$. □

Remark 4.12. The developments of this section are closely related to the ideas in Section 5.10 of EGA4, Part II [15], which (translated into our notation) deals with the S_2 condition for sheaves of the form $j_*\mathcal{F}$ and schemes of the form $\mathbf{Spec}\, j_*\mathcal{O}_U$. The assumptions in loc. cit. are a bit different (e.g., the last part of our Proposition 3.7 must be imposed as a hypothesis), and the specific setting of Theorem 4.5 is not treated there. Nevertheless, it is easy to imagine adapting the methods used there to prove that $\rho : \tilde{X} \to X$ is S_2 relative to U. However, the universal property of ρ and the uniqueness statement in Proposition 4.11 are consequences of the fact that the $\mathcal{IC}$ functor is an equivalence of categories with its essential image. Proving those statements in the language of [15, Section 5.10] would likely amount to unwinding the proof of Proposition 2.3 and the construction of the perverse coherent t-structure in [6]. The conciseness and clarity of the uniqueness arguments are perhaps the main benefit of using perverse coherent sheaves here.

Remark 4.13. Our main goal in this paper is to apply the S_2-extension construction in the setting of special pieces, where the hypotheses of Example 4.10 hold. Indeed, U will be normal, and $\rho_1 : \tilde{U} \to U$ will be a surjective étale morphism; see Section 7. Since an elementary construction of the S_2-extension is available in that setting (as was known to Procesi [27]), one could in principle forego developing the machinery of the functor $\mathbf{Spec}\, \mathcal{IC}(X, \cdot)$. However, we will also require the results of Section 5, and those seem to be much easier to state and prove in the context of perverse coherent sheaves than in a purely ring-theoretic setting.

We conclude this section with a remark on the "Macaulayfication" problem: given a scheme, find a Cohen–Macaulay scheme that is birationally equivalent to it. (For varieties over a field of characteristic 0, this problem is solved by Hironaka's theorem.) Kawasaki, extending early work of Faltings [14], has shown how to construct a Macaulayfication of any Noetherian scheme over a ring (of arbitrary characteristic) with a dualizing complex [24], but the resulting scheme is not canonical. (It does not have the obvious universal property.) Indeed, just by considering varieties that fail to be Cohen–Macaulay at a single closed point, Brodmann has exhibited a family of examples which do not have a universal Macaulayfication [7]. There may, however, be a finite Macaulayfication that is universal among appropriate finite morphisms from Cohen-Macaulay schemes. The following theorem addresses the problem in a way that is reminiscent of Example 4.9.

Theorem 4.14 (Macaulayfication)**.** *Let X be a scheme of finite type over a Noetherian base scheme S admitting a dualizing complex, and suppose $\mathfrak{Coh}(X)$ has enough locally free sheaves. Let U be an open Cohen–Macaulay subscheme whose complement has codimension at least 2. Then X has a finite Macaulayfication if and only if $\mathcal{IC}^c(X, \mathcal{O}_U)$ is a sheaf (where c is the Cohen–Macaulay perversity). In this case, the unique finite Macaulayfication is $\tilde{X}^c \stackrel{\text{def}}{=} \mathbf{Spec}\, \mathcal{IC}^c(X, \mathcal{O}_U)$ and coincides with the S_2-extension of $\mathrm{id} : U \to U$. The scheme $\tilde{X}^c$ is universal with respect to finite morphisms to X which are S_2 relative to U. In particular, any finite morphism $f : Y \to X$ with Y Cohen-Macaulay and $f^{-1}(U)$ dense factors uniquely though $\tilde{X}^c$.*

This theorem is stated without a group action for convenience. An equivariant version akin to Theorem 4.5 can be proved by a similar argument.

Proof. Let $f : Y \to X$ be a finite morphism such that $f^{-1}(U)$ is Cohen–Macaulay and dense in Y. The last condition allows us to apply Lemma 3.9: Y is Cohen–Macaulay if and only if $\mathcal{IC}^c(Y, \mathcal{O}_{f^{-1}(U)}) \simeq \mathcal{O}_Y$. Next, by Proposition 3.5, we see that Y is Cohen–Macaulay if and only if we have $\mathcal{IC}^c(X, f_*\mathcal{O}_{f^{-1}(U)}) \simeq f_*\mathcal{O}_Y$. In particular, if f is an isomorphism over U, then Y is a Macaulayfication of X if and only if $\mathcal{IC}^c(X, \mathcal{O}_U)$ is a sheaf. However, since $c \geq s$, we know that whenever $\mathcal{IC}^c(X, \mathcal{O}_U)$ is a sheaf, it must coincide with $\mathcal{IC}^s(X, \mathcal{O}_U)$. We now see that $\tilde{X}^c$ is just the "S_2-ification" of X, and the universal property follows from Theorem 4.5. Finally, a finite morphism $f : Y \to X$ with Y Cohen–Macaulay and $f^{-1}(U)$ dense is S_2 relative to U, so the universal property applies in this situation. □

Note that this construction does not coincide with Kawasaki's Macaulayfication; the latter involves blow-ups and accordingly is never finite. Instead, this theorem generalizes a result of Schenzel [29] relating finite Macaulayfications and "S_2-ifications" for a certain class of local rings. (Schenzel's result is essentially Theorem 4.14 in the special case $X = \operatorname{Spec} A$, where A is a local domain that is a quotient of a Gorenstein ring.) However, the above scheme-theoretic statement appears not to have been previously known.

Finally, we remark that Theorem 4.5 can be generalized to define a Cohen–Macaulay extension (or a p-extension for any perversity p with $s \leq p \leq c$) of appropriate $\rho_1 : \tilde{U} \to U$. Let ρ_1 be a finite morphism such that $\mathcal{IC}^p(X, \rho_{1*}\mathcal{O}_{\tilde{U}})$ is defined and a sheaf, and let $\tilde{X}^p$ denote the scheme $\mathbf{Spec}\, \mathcal{IC}^p(X, \rho_{1*}\mathcal{O}_{\tilde{U}})$. A similar argument to that given in the proof of Theorem 4.5 shows that the natural morphism $\rho : \tilde{X}^p \to X$ is universal with respect to finite morphisms $f : Y \to X$ which are "p relative to U" (defined in the obvious way) and whose restriction

$f|_{f^{-1}(U)}$ factors through ρ_1. However, since $\mathcal{IC}^p(X, \rho_{1*}\mathcal{O}_{\tilde{U}})$ and $\mathcal{IC}^s(X, \rho_{1*}\mathcal{O}_{\tilde{U}})$ coincide when the former is a sheaf, we see that $\tilde{X}^p$ is just the S_2-extension, and so ρ is in fact universal with respect to finite morphisms S_2 relative to U for which U has dense preimage.

5. Middle Extension in the Derived Category

Recall that the intermediate extension functor $\mathcal{IC}(X, \cdot) : \mathcal{M}^{\pm}(U) \to \mathcal{M}^{\pm}(X)$ is an equivalence of categories. (We make no assumptions about the perversity in this section.) In particular, we have

$$\mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{IC}(X, \mathcal{E}), \mathcal{IC}(X, \mathcal{F})) \simeq \mathrm{Hom}_{\mathcal{M}(U)}(\mathcal{E}, \mathcal{F}),$$

for $\mathcal{E}, \mathcal{F}$ in $\mathcal{M}^{\pm}(U)$. The goal of this section is introduce a derived version of this functor. We will construct a functor of triangulated categories from a suitable subcategory of $\mathcal{D}(U)$ to $\mathcal{D}(X)$ that "extends" $\mathcal{IC}(X, \cdot)$, and then prove a generalization of Proposition 2.3 in this setting.

For most of this section, we make the following assumption:

(Q) There is a class of projective objects $\mathcal{Q}$ in $\mathcal{M}^{\pm}(U)$ such that
  (i) every object of $\mathcal{M}^{\pm}(U)$ is a quotient of some object in $\mathcal{Q}$, and
  (ii) for every object $\mathcal{A}$ in $\mathcal{Q}$, $\mathcal{IC}(X, \mathcal{A})$ is a projective sheaf on X.

For example, condition (Q) holds if X is a quasiaffine scheme that is locally S_2 outside U, and p is the S_2 perversity. In that case, $\mathcal{IC}(X, \mathcal{O}_U) = \mathcal{O}_X$. Every object in $\mathcal{M}^{\pm}(U)$ is in fact a sheaf. Moreover, since U is quasiaffine, every coherent sheaf on U is a quotient of a free sheaf, so we can take $\mathcal{Q}$ to be the class of free sheaves on U.

Let $\mathcal{M}^{\pm}_0(U)$ be the abelian category $\mathcal{M}^{\pm}(U) \cap \mathfrak{Coh}(U)$. (In some cases, such as with the S_2 perversity, it happens that $\mathcal{M}^{\pm}_0(U) = \mathcal{M}^{\pm}(U)$.) Let $\mathcal{M}^{\pm}_0(X)$ be the subcategory of $\mathcal{M}^{\pm}(X)$ consisting of objects $\mathcal{F}$ such that $j^*\mathcal{F} \in \mathcal{M}^{\pm}_0(U)$. Clearly, $\mathcal{IC}(X, \cdot)$ and j^* restrict to give equivalences of categories between $\mathcal{M}^{\pm}_0(U)$ and $\mathcal{M}^{\pm}_0(X)$.

Now, $\mathcal{M}^{\pm}_0(U)$ is a full subcategory of $\mathfrak{Coh}(U)$ with enough projective objects that are also projective in $\mathfrak{Coh}(U)$ (namely, the objects of $\mathcal{Q}$). It follows that the bounded derived category $\mathcal{DM}^{\pm}_0(U)$ can be identified with a full triangulated subcategory of $\mathcal{D}(U)$. For brevity, we henceforth write $\mathcal{D}^{\pm}_0(U)$ for $\mathcal{DM}^{\pm}_0(U)$.

Let $\mathcal{D}^{\pm}_0(X)$ be the full subcategory of $\mathcal{D}(X)$ consisting of those objects $\mathcal{A}$ for which ${}^pH^n(\mathcal{A}) \in \mathcal{M}^{\pm}_0(X)$ for all n. This is a triangulated subcategory of $\mathcal{D}(X)$ (because the subcategory $\mathcal{M}^{\pm}_0(X)$ of $\mathcal{M}(X)$ is stable under extensions). The perverse t-structure on $\mathcal{D}^{\pm}_0(X)$ (that is, the t-structure induced by the perverse t-structure on $\mathcal{D}(X)$) has heart $\mathcal{M}^{\pm}_0(X)$.

Following [3, §3.1], given any t-structure on a full triangulated subcategory of the derived category of an abelian category, there is a *realization functor* from the derived category of the heart of the t-structure to the original derived category. In our situation, we obtain a functor $\mathrm{real} : \mathcal{DM}^{\pm}_0(X) \to \mathcal{D}(X)$, where $\mathcal{DM}^{\pm}_0(X)$ is the bounded derived category of the abelian category $\mathcal{M}^{\pm}_0(X)$. We now briefly review its construction. This requires the machinery of filtered derived categories; we refer the reader to [3, §3.1] for complete definitions and details. Let $\mathcal{DF}(X)$ be the bounded filtered derived category of coherent sheaves on X, and let $\mathcal{DF}_{\text{bête}}(X)$ be the full subcategory of $\mathcal{DF}(X)$ consisting of objects whose filtration is stupid ("bête") with respect to the perverse t-structure on $\mathcal{D}^{\pm}_0(X)$. Forgetting the filtration gives us a functor $\omega : \mathcal{DF}_{\text{bête}}(X) \to \mathcal{D}(X)$; on the other hand, by [3, Proposition 3.1.8], there is an equivalence of categories $G : \mathcal{DF}_{\text{bête}}(X) \to \mathcal{CM}_0^{\pm}(X)$, where $\mathcal{CM}_0^{\pm}(X)$ is the category of complexes of objects of $\mathcal{M}_0^{\pm}(X)$. We first define $\overline{\text{real}} : \mathcal{CM}_0^{\pm}(X) \to \mathcal{D}(X)$ by $\overline{\text{real}} = \omega \circ G^{-1}$. By [3, Proposition 3.1.10], $\overline{\text{real}}$ factors through $\mathcal{DM}_0^{\pm}(X)$ and thus gives rise to a functor $\text{real} : \mathcal{DM}_0^{\pm}(X) \to \mathcal{D}(X)$. This functor is compatible with cohomology in the following sense: for all n, there is an isomorphism of functors $H^n \simeq {}^pH^n \circ \text{real} : \mathcal{DM}_0^{\pm}(X) \to \mathcal{M}_0^{\pm}(X)$.

We define ${}^D\mathcal{IC}(X, \cdot) : \mathcal{D}_0^{\pm}(U) \to \mathcal{D}(X)$ by ${}^D\mathcal{IC}(X, \cdot) = \text{real} \circ \mathcal{IC}(X, \cdot)$.

Lemma 5.1. *The functor* ${}^D\mathcal{IC}(X, \cdot) : \mathcal{D}_0^{\pm}(U) \to \mathcal{D}(X)$ *takes values in* $\mathcal{D}_0^{\pm}(X)$*, and there are isomorphisms of functors* ${}^pH^n({}^D\mathcal{IC}(X, \cdot)) \simeq \mathcal{IC}(X, {}^pH^n(\cdot))$ *for all* n*.*

Proof. Since $\mathcal{IC}(X, \cdot) : \mathcal{M}_0^{\pm}(U) \to \mathcal{M}(X)$ is exact, the functor on derived categories $\mathcal{IC}(X, \cdot) : \mathcal{D}_0^{\pm}(U) \to \mathcal{DM}(X)$ respects cohomology: $\mathcal{IC}(X, {}^pH^n(\cdot))$ (or, equivalently, $\mathcal{IC}(X, {}^{\text{std}}H^n(\cdot)))$ is isomorphic to $H^n(\mathcal{IC}(X, \cdot))$. Next, $H^n \simeq {}^pH^n \circ \text{real}$, so

$$\mathcal{IC}(X, {}^pH^n(\cdot)) \simeq H^n(\mathcal{IC}(X, \cdot)) \simeq {}^pH^n(\text{real}(\mathcal{IC}(X, \cdot))) \simeq {}^pH^n({}^D\mathcal{IC}(X, \cdot)).$$

Since $\mathcal{IC}(X, \cdot)$ takes values in $\mathcal{M}_0^{\pm}(X)$, it is obvious that ${}^D\mathcal{IC}(X, \cdot)$ takes values in $\mathcal{D}_0^{\pm}(X)$. □

Lemma 5.2. *There is an isomorphism of functors* $j^{*\,D}\mathcal{IC}(X, \cdot) \simeq \text{id} : \mathcal{D}_0^{\pm}(U) \to \mathcal{D}_0^{\pm}(U)$*.*

Proof. Since $\mathcal{IC}(X, \cdot) : \mathcal{M}_0^{\pm}(U) \to \mathcal{M}_0^{\pm}(X)$ and $j^* : \mathcal{M}_! * (U) \to \mathcal{M}_0^{\pm}(U)$ are inverse equivalences of categories, they give rise to equivalences of the corresponding categories of complexes, as well as of the the corresponding derived categories. These equivalences are such that the square in the center of the diagram below commutes.

$$\begin{tikzcd}
\mathcal{DF}_{\text{bête}}(X) \arrow[r, "\sim", "G"'] \arrow[d, "j^*"', "\wr"] \arrow[rrr, bend left, "\omega"] & \mathcal{CM}_0^{\pm}(X) \arrow[r] \arrow[d, shift right, "j^*"', "\wr"] & \mathcal{DM}_0^{\pm}(X) \arrow[r, "\text{real}"] \arrow[d, shift right, "j^*"', "\wr"] & \mathcal{D}(X) \arrow[dl, bend left, "j^*"] \\
\mathcal{DF}_{\text{bête}}(U) \arrow[r, "G", "\sim"'] \arrow[rr, bend right, "\omega"'] & \mathcal{CM}_0^{\pm}(U) \arrow[r] \arrow[u, shift right, "\mathcal{IC}(X,\cdot)"'] & \mathcal{D}_0^{\pm}(U) \arrow[u, shift right, "j^*"'] \arrow[ur, bend left, "{}^D\mathcal{IC}(X,\cdot)"] &
\end{tikzcd}$$

In the setting of filtered derived categories, the restriction functor $j^* : \mathcal{DF}(X) \to \mathcal{DF}(U)$ respects stupidity of the filtration (because j^* takes $\mathcal{M}_0^{\pm}(X)$ to $\mathcal{M}_0^{\pm}(U)$) and so gives rise to a functor $j^* : \mathcal{DF}_{\text{bête}}(X) \to \mathcal{DF}_{\text{bête}}(U)$ that makes the leftmost square in the diagram above commute. Here $\mathcal{DF}_{\text{bête}}(U)$ is defined with respect to the perverse t-structure on $D_0^{\pm}(U)$ (which is simply a shift of the standard t-structure). It is clear that restriction commutes with forgetting the filtration, so $j^* \circ \omega \simeq \omega \circ j^*$. Together, these statements imply that $j^* \circ \text{real} : \mathcal{DM}_0^{\pm}(X) \to \mathcal{D}_0^{\pm}(U)$ is isomorphic to $j^* : \mathcal{DM}_0^{\pm}(X) \to \mathcal{D}_0^{\pm}(U)$. Composing with $\mathcal{IC}(X, \cdot) : \mathcal{D}_0^{\pm}(U) \to \mathcal{DM}_0^{\pm}(X)$, we find that $j^* \circ \text{real} \circ \mathcal{IC}(X, \cdot) \simeq j^* \circ \mathcal{IC}(X, \cdot)$, or in other words, $j^* \circ {}^D\mathcal{IC}(X, \cdot) \simeq \text{id}$. □

Definition 5.3. An object $\mathcal{F}$ of $\mathcal{M}_0^{\pm}(U)$ is said to be *short* if $\mathcal{IC}(X, \mathcal{F}) \in {}^{\text{std}}\mathcal{D}(X)^{\leq 1}$.

For example, if p is the S_2 perversity, all objects in $\mathcal{M}(U)$ are short. Indeed, they, as well as all other short objects we will actually encounter, satisfy the stronger condition that their images under $\mathcal{IC}(X, \cdot)$ belong to ${}^{\text{std}}\mathcal{D}(X)^{\leq 0}$, but the weaker condition above suffices for the statements we wish to prove.

Proposition 5.4. *If $\mathcal{F} \in \mathcal{M}_0^{\pm}(U)$ is short, there are natural isomorphisms*

$$\mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{IC}(X,\mathcal{E}),\mathcal{IC}(X,\mathcal{F})[n]) \simeq \mathrm{Hom}_{\mathcal{D}(U)}(\mathcal{E},\mathcal{F}[n]) \tag{6}$$

$$R\mathcal{H}om_{\mathcal{D}(X)}(\mathcal{IC}(X,\mathcal{E}),\mathcal{IC}(X,\mathcal{F})) \simeq Rj_* \, R\mathcal{H}om_{\mathcal{D}(U)}(\mathcal{E},\mathcal{F}) \tag{7}$$

for all $n \in \mathbb{Z}$ and all $\mathcal{E} \in \mathcal{M}_0^{\pm}(U)$.

Remark 5.5. According to [3, Remarque 3.1.17(ii)], the isomorphism (6) always exists for $n \leq 1$, without assuming condition (Q) or the shortness of $\mathcal{F}$.

Proof of Proposition 5.4. As we have just remarked, the natural morphism

$$\mathrm{Hom}_{\mathcal{D}(U)}(\mathcal{E},\mathcal{F}[n]) \to \mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{IC}(X,\mathcal{E}),\mathcal{IC}(X,\mathcal{F})[n]). \tag{8}$$

induced by $\mathcal{IC}(X,\cdot)$ is an isomorphism for $n \leq 1$. Lemma 5.2 implies that this morphism is always injective (in other words, that $\mathcal{IC}(X,\cdot)$ is faithful), so it simply remains to show that it is surjective for $n > 1$. We proceed by induction.

Given $f : {}^D\mathcal{IC}(X,\mathcal{E}) \to {}^D\mathcal{IC}(X,\mathcal{F})[n]$, choose a surjective map $g : \mathcal{A} \to \mathcal{E}$ with $\mathcal{A} \in \mathcal{Q}$. By assumption, $\mathcal{IC}(X,\mathcal{A})$ is a projective sheaf, so $\mathrm{Hom}(\mathcal{IC}(X,\mathcal{A}),\mathcal{G}) = 0$ for all $\mathcal{G} \in {}^{\mathrm{std}}\mathcal{D}(X)^{\leq -1}$. Since $\mathcal{F}$ is short, we have $\mathcal{IC}(X,\mathcal{F})[n] \in {}^{\mathrm{std}}\mathcal{D}(X)^{\leq -n+1}$, and since $n > 1$, we have

$$\mathrm{Hom}(\mathcal{IC}(X,\mathcal{A}),\mathcal{IC}(X,\mathcal{F})[n]) = 0.$$

Now, let $\mathcal{H} = \ker g$, and consider the exact sequence

$$\cdots \to \mathrm{Hom}(\mathcal{IC}(X,\mathcal{H})[1],\mathcal{IC}(X,\mathcal{F})[n]) \to \mathrm{Hom}(\mathcal{IC}(X,\mathcal{E}),\mathcal{IC}(X,\mathcal{F})[n]) \\ \to \mathrm{Hom}(\mathcal{IC}(X,\mathcal{A}),\mathcal{IC}(X,\mathcal{F})[n]) \to \cdots.$$

We see that f must be the image of some morphism $f' : \mathcal{IC}(X,\mathcal{H})[1] \to \mathcal{IC}(X,\mathcal{F})[n]$, that is, $f = f' \circ d$, where $d : \mathcal{IC}(X,\mathcal{E}) \to \mathcal{IC}(X,\mathcal{H})[1]$ comes from the distinguished triangle associated to the short exact sequence $0 \to \mathcal{H} \to \mathcal{A} \to \mathcal{E} \to 0$. Now, $f'[-1] : \mathcal{IC}(X,\mathcal{H}) \to \mathcal{IC}(X,\mathcal{F})[n-1]$ is in the image of $\mathcal{IC}(X,\cdot)$ by the inductive hypothesis, and d is in its image by Remark 5.5, so f is in its image as well.

It remains to prove the corresponding fact for $R\mathcal{H}om$. For the remainder of the proof, we assume that we are working in the nonequivariant setting (or that G is trivial). As remarked in [6] immediately preceding Lemma 2, $R\mathcal{H}om$ commutes with the forgetful functor from an equivariant category to the nonequivariant one, so we lose nothing by making this assumption.

In the nonequivariant setting, all of the preceding arguments also apply to any open set $V \subset X$. In particular, since ${}^D\mathcal{IC}(X,\mathcal{E})|_V \simeq {}^D\mathcal{IC}(V,\mathcal{E}|_{V\cap U})$ for any object $\mathcal{E} \in \mathcal{M}_0^{\pm}(U)$, we have, for any $n \in \mathbb{Z}$, an isomorphism

$$\mathrm{Hom}_{\mathcal{D}(V)}(\mathcal{IC}(X,\mathcal{E})|_V,\mathcal{IC}(X,\mathcal{F})[n]|_V) \simeq \mathrm{Hom}_{\mathcal{D}(V\cap U)}(\mathcal{E}|_{V\cap U},\mathcal{F}[n]|_{V\cap U}).$$

Now, for any two objects $\mathcal{A},\mathcal{B} \in \mathcal{D}(X)$, there is a canonical morphism

$$j^* R\mathcal{H}om_{\mathcal{D}(X)}(\mathcal{A},\mathcal{B}) \to R\mathcal{H}om_{\mathcal{D}(U)}(j^*\mathcal{A}, j^*\mathcal{B}).$$

Let us take $\mathcal{A} = \mathcal{IC}(X,\mathcal{E})$ and $\mathcal{B} = \mathcal{IC}(X,\mathcal{F})$. Of course, we then have $j^*\mathcal{A} \simeq \mathcal{E}$ and $j^*\mathcal{B} \simeq \mathcal{F}$. Now, by adjointness, the above morphism gives rise to a canonical morphism

$$\phi : R\mathcal{H}om_{\mathcal{D}(X)}(\mathcal{IC}(X,\mathcal{E}),\mathcal{IC}(X,\mathcal{F})) \to Rj_* \, R\mathcal{H}om_{\mathcal{D}(U)}(\mathcal{E},\mathcal{F}).$$

To show that ϕ is in fact an isomorphism, it suffices to show that it induces isomorphisms on all hypercohomology groups over all open sets. For any open set $V \subset X$, we have

$$\begin{aligned} &H^n(R\Gamma(V, R\mathcal{H}om_{\mathcal{D}(X)}(\mathcal{IC}(X, \mathcal{E}), \mathcal{IC}(X, \mathcal{F})))) \\ &\quad \simeq \mathrm{Hom}_{\mathcal{D}(V)}(\mathcal{IC}(X, \mathcal{E})|_V, \mathcal{IC}(X, \mathcal{F})[n]|_V) \simeq \mathrm{Hom}_{\mathcal{D}(V\cap U)}(\mathcal{E}|_{V\cap U}, \mathcal{F}[n]|_{V\cap U}) \\ &\qquad \simeq H^n(R\Gamma(V, Rj_* \, R\mathcal{H}om(\mathcal{E}, \mathcal{F}))). \end{aligned}$$

Thus, ϕ is an isomorphism. □

Theorem 5.6. *If all objects in $\mathcal{M}_0^{\pm}(U)$ are short, then ${}^D\mathcal{IC}(X, \cdot) : \mathcal{D}_0^{\pm}(U) \to \mathcal{D}_0^{\pm}(X)$ is an equivalence of categories, with inverse given by j^*. Moreover, for any two objects $\mathcal{E}, \mathcal{F} \in \mathcal{D}_0^{\pm}(U)$, there are natural isomorphisms*

$$\mathrm{Hom}_{\mathcal{D}(X)}({}^D\mathcal{IC}(X, \mathcal{E}), {}^D\mathcal{IC}(X, \mathcal{F})) \simeq \mathrm{Hom}_{\mathcal{D}(U)}(\mathcal{E}, \mathcal{F}) \tag{9}$$

$$R\mathcal{H}om_{\mathcal{D}(X)}({}^D\mathcal{IC}(X, \mathcal{E}), {}^D\mathcal{IC}(X, \mathcal{F})) \simeq Rj_* \, R\mathcal{H}om_{\mathcal{D}(U)}(\mathcal{E}, \mathcal{F}) \tag{10}$$

Proof. If all objects in $\mathcal{M}_0^{\pm}(U)$ are short, the isomorphism (6) holds for all objects $\mathcal{E}, \mathcal{F} \in \mathcal{M}_0^{\pm}(U)$. As observed in the proof of [3, Proposition 3.1.16], the realization functor is an equivalence of categories if and only if (6) holds for all objects in $\mathcal{M}_0^{\pm}(U)$. Since $\mathcal{IC}(X, \cdot) : \mathcal{D}_0^{\pm}(U) \to \mathcal{DM}_0^{\pm}(X)$ is an equivalence of categories, we see that ${}^D\mathcal{IC}(X, \cdot) = \mathrm{real} \circ \mathcal{IC}(X, \cdot)$ is as well. By Lemma 5.2, its inverse must be j^*.

Once we know that ${}^D\mathcal{IC}(X, \cdot)$ is an equivalence of categories, (9) is immediate. We deduce (10) from it by an argument identical to that given for (7) above. □

For applications of this result, we must consider certain categories whose objects are dual to coherent sheaves. Given a perversity p, let

$$\mathcal{M}_*^{p,\pm}(U) = \mathbb{D}(\mathcal{M}_0^{\bar{p},\pm}(U)) \subset \mathcal{M}^{p,\pm}(U).$$

Corollary 5.7. *Suppose the dualizing complexes on U and X have the following properties: with respect to some perversity p, ω_U is a short object in $\mathcal{M}_0^{p,\pm}(U)$, and $\omega_X \simeq \mathcal{IC}^p(X, \omega_U)$. Then, with respect to the dual perversity $\bar{p}$, we have $\mathcal{IC}^{\bar{p}}(X, \mathcal{E}) \simeq Rj_*\mathcal{E}$ for all $\mathcal{E} \in \mathcal{M}_*^{\bar{p},\pm}(U)$.*

Proof. Let $\mathcal{E}$ be an object of $\mathcal{M}_*^{\bar{p},\pm}(U)$. Then $\mathbb{D}\mathcal{E} = R\mathcal{H}om_{\mathcal{D}(U)}(\mathcal{E}, \omega_U)$ is an object in $\mathcal{M}_0^{p,\pm}(U)$. It follows from [6, Lemma 5(a)] that

$$\mathbb{D}\mathcal{IC}^p(X, \mathbb{D}\mathcal{E}) = R\mathcal{H}om(\mathcal{IC}^p(X, \mathbb{D}\mathcal{E}), \omega_X) \simeq \mathcal{IC}^{\bar{p}}(X, \mathcal{E}).$$

But we also have

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}(X)}(\mathcal{IC}^p(X, \mathbb{D}\mathcal{E}), \omega_X) &\simeq R\mathcal{H}om_{\mathcal{D}(X)}(\mathcal{IC}^p(X, \mathbb{D}\mathcal{E}), \mathcal{IC}^p(X, \omega_U)) \\ &\simeq Rj_* \, R\mathcal{H}om_{\mathcal{D}(U)}(\mathbb{D}\mathcal{E}, \omega_U). \end{aligned}$$

Since $R\mathcal{H}om_{\mathcal{D}(U)}(\mathbb{D}\mathcal{E}, \omega_U) \simeq \mathcal{E}$, we see that $\mathcal{IC}^{\bar{p}}(X, \mathcal{E}) \simeq Rj_*\mathcal{E}$. □

Corollary 5.8. *Suppose X is a Gorenstein scheme.*

1. *If $p^+(x) = \operatorname{codim} \bar{x}$ for all $x \in U^{G\text{-gen}}$, then there is an isomorphism of functors $\mathcal{IC}(X, \cdot) \simeq Rj_*$.*
2. *If $\mathcal{F}$ is a Cohen–Macaulay sheaf on U, then $\mathcal{IC}(X, \mathcal{F}) \simeq Rj_*\mathcal{F}$ with respect to any perversity.*

Proof. On a Gorenstein scheme, we may take $\omega_X \simeq \mathcal{O}_X$. A Gorenstein scheme is, in particular, Cohen–Macaulay, so by Lemma 3.4, $\omega_X \simeq \mathcal{IC}(X, \omega_U)$ with respect to every perversity. Corollary 5.7 now tells us that on $\mathcal{M}^{\pm}_*(U)$, $\mathcal{IC}(X, \cdot) \simeq Rj_*$ for every perversity.

For part (1) of the corollary, we must simply show that $\mathcal{M}^{p,\pm}_*(U) = \mathcal{M}^{p,\pm}(U)$, or, equivalently, that $\mathcal{M}^{\bar{p},\pm}_0(U) = \mathcal{M}^{\bar{p},\pm}(U)$. The assumption that $p^+(x) = \operatorname{codim} \bar{x}$ implies that $\bar{p}^-(x) = 0$ for all $x \in U^{G\text{-gen}}$. It follows that $\mathcal{M}^{\bar{p},\pm}(U) \subset \mathcal{M}^{p^-}(U) = \mathfrak{Coh}(U)$, as desired.

For part (2), we first note that $\mathcal{IC}(X, \mathcal{F})$ is defined with respect to any perversity by Lemma 3.4. In particular, we see that $\mathcal{F} \in \mathcal{M}^{c,\pm}(U)$, so $\mathcal{DF} \in \mathcal{M}^{s,\pm}(U) = \mathcal{M}^{s,\pm}_0(U)$. Since $\mathcal{DF} \in \mathfrak{Coh}(U)$, $\mathcal{F}$ is in $\mathcal{M}^{\pm}_*(U)$ with respect to any perversity, and the result follows by Corollary 5.7. □

Corollary 5.9. *Let X be a Gorenstein scheme, $U \subset X$ an open subscheme, and $\rho_1 : \tilde{U} \to U$ a finite morphism. If ρ_1 admits an S_2-extension, let $\tilde{X}$ be the scheme thus obtained.*

1. *If $\tilde{U}$ is Cohen–Macaulay, then $\tilde{X}$ is as well.*
2. *If $\rho_{1*}\mathcal{O}_{\tilde{U}}$ is isomorphic to its own Serre–Grothendieck dual, then $\tilde{X}$ is Gorenstein.*

In particular, part of the content of this corollary is the assertion that if either $\tilde{U}$ is Cohen–Macaulay or $\rho_{1*}\mathcal{O}_{\tilde{U}}$ is self-dual, then ρ_1 necessarily admits an S_2-extension.

Proof. For part (1), it follows from Proposition 3.5, Lemma 3.9, and Corollary 5.8 that the S_2-extension $\tilde{X}$ exists and is locally Cohen–Macaulay at least at all points of $\tilde{X}^{G\text{-gen}}$. This reasoning can be repeated in the nonequivariant category to obtain a nonequivariant S_2-extension that is in fact Cohen–Macaulay. The latter variety must coincide with $\tilde{X}$ by Corollary 4.7.

Henceforth, assume that $\rho_{1*}\mathcal{O}_{\tilde{U}} \simeq \mathbb{D}(\rho_{1*}\mathcal{O}_{\tilde{U}})$. Evidently, $\rho_{1*}\mathcal{O}_{\tilde{U}} \in {}^{s^-}\mathcal{D}(U)^{\leq 0}$, and since the dual perversity to s^- is c^+, we have $\rho_{1*}\mathcal{O}_{\tilde{U}} \in {}^{c^+}\mathcal{D}(U)^{\geq 0}$ as well. It follows that the intermediate extension of $\rho_{1*}\mathcal{O}_{\tilde{U}}$ is defined with respect to any perversity; furthermore, by Corollary 5.8, it is independent of perversity. Let $\mathcal{F} = \mathcal{IC}(X, \rho_{1*}\mathcal{O}_{\tilde{U}})$. By [6, Lemma 5], $\mathbb{D}\mathcal{F} \simeq \mathcal{IC}(X, \mathbb{D}(\rho_{1*}\mathcal{O}_{\tilde{U}}))$, and hence $\mathcal{F} \simeq \mathbb{D}\mathcal{F}$.

Now, by Proposition 3.5 and Lemma 3.9, we know that $\tilde{X} = \mathbf{Spec}\,\mathcal{F}$ is Cohen–Macaulay. In particular, given a point $x \in X$ and a point $y \in \rho^{-1}(x)$, we know that the local ring $\mathcal{O}_{y,\tilde{X}}$ is a finite Cohen–Macaulay extension of the Gorenstein local ring $\mathcal{O}_{x,X}$. According to [11, Theorem 3.3.7], $\mathcal{O}_{y,\tilde{X}}$ is Gorenstein if and only if

$$\mathcal{O}_{y,\tilde{X}} \simeq \mathrm{Hom}_{\mathcal{O}_{x,X}}(\mathcal{O}_{y,\tilde{X}}, \mathcal{O}_{x,X}). \tag{11}$$

Consider the fact that

$$i_x^*\mathcal{F} = i_x^*\rho_*\mathcal{O}_{\tilde{X}} \simeq \bigoplus_{y \in \rho^{-1}(x)} \mathcal{O}_{y,\tilde{X}}.$$

Obviously, (11) implies that

$$i_x^*\mathcal{F} \simeq \mathrm{Hom}_{\mathcal{O}_{x,X}}(i_x^*\mathcal{F}, \mathcal{O}_{x,X}). \tag{12}$$

Conversely, if (12) holds, then by considering the action of each $\mathcal{O}_{y,\tilde{X}}$ on each side of this isomorphism, we see that (11) must hold as well. Thus, (11) and (12) are equivalent. On the other hand, by [26, Proposition 7.24(iii)] (for instance),

$$\mathrm{Hom}_{\mathcal{O}_{x,X}}(i_x^*\mathcal{F}, \mathcal{O}_{x,X}) \simeq i_x^* \mathcal{H}om(\mathcal{F}, \mathcal{O}_X) \simeq i_x^*(\mathbb{D}\mathcal{F}).$$

Now, (12) is true for all x because it is equivalent to the statement that $i_x^*\mathcal{F} \simeq i_x^*\mathbb{D}\mathcal{F}$. Therefore, (11) is true for all y, so $\tilde{X}$ is Gorenstein. □

We conclude this section with the statement of a purely ring-theoretic version of the preceding result. The authors are not aware of a direct proof of this statement in the setting of commutative algebra. Note that the implicit hypothesis that X satisfies condition (Q) is not needed here because only affine schemes are involved.

Corollary 5.10. *Let A be a Gorenstein domain. Let K be a finite extension of the fraction field of A, and let B be the integral closure of A in K. Let $I \subset A$ be a radical ideal of codimension at least 2, and let T be a set of generators for I.*

1. *If B_f is Cohen–Macaulay for all $f \in T$, then B is Cohen–Macaulay. (Equivalently, B is Cohen–Macaulay if $B_{\mathcal{P}}$ is for all prime (resp. maximal) ideals $\mathcal{P}$ of B lying over prime (resp. maximal) ideals of A not containing I.)*
2. *If $B_f \simeq \mathrm{Hom}_{A_f}(B_f, A_f)$ for all $f \in T$, then B is Gorenstein.*

6. Perverse Constructible Sheaves

We assume henceforth that X and G are separated schemes over $S = \mathrm{Spec}\, k$ for some field k and that U is smooth. In this section, we establish some results on ordinary (constructible) perverse sheaves on X which we will need in studying special pieces.

Fix a prime number ℓ different from the characteristic of k, and let $D(X)$ be the bounded G-equivariant derived category of constructible $\bar{\mathbb{Q}}_l$-sheaves on X (in the sense of Bernstein–Lunts [5]). By an abuse of notation, we use $\mathbb{D}$ to denote the Verdier duality functor in this category: here $\mathbb{D} = R\mathcal{H}om(\cdot, a^!\bar{\mathbb{Q}}_l)$, where $a : X \to \mathrm{Spec}\, k$ is the structure morphism.

Let $(D(X)^{\leq 0}, D(X)^{\geq 0})$ be the perverse t-structure on $D(X)$ with respect to the middle perversity:

$$D(X)^{\leq 0} = \{F \in D(X) \mid \dim \mathrm{supp}\, H^{-i}(F) \leq i\},$$
$$D(X)^{\geq 0} = \{F \in D(X) \mid \dim \mathrm{supp}\, H^{-i}(\mathbb{D}F) \leq i\}.$$

Let $M(X)$ be the heart of this t-structure. There is an intermediate extension functor $M(U) \to M(X)$. Given an equivariant local system E on U, we denote by $\mathrm{IC}(X, E)$ the object of $D(X)$ such that $\mathrm{IC}(X, E)[\dim X] \in M(X)$ is the intermediate extension of $E[\dim X] \in M(U)$.

In addition, let $({}^{\mathrm{std}}D(X)^{\leq 0}, {}^{\mathrm{std}}D(X)^{\geq 0})$ denote the standard t-structure on $D(X)$. Note that ${}^{\mathrm{std}}D(X)^{\leq -\dim X} \subset D(X)^{\leq 0}$ and ${}^{\mathrm{std}}D(X)^{\geq -\dim X} \supset D(X)^{\geq 0}$.

Proposition 6.1. *If X is irreducible and $\mathrm{IC}(X, \bar{\mathbb{Q}}_l)$ is a sheaf, then in fact we have $\mathrm{IC}(X, \bar{\mathbb{Q}}_l) \simeq \bar{\mathbb{Q}}_l$ (i.e., X is rationally smooth).*

Proof. Recall that there is a distinguished triangle

$$\mathrm{IC}(X, \bar{\mathbb{Q}}_l)[\dim X] \to Rj_*\bar{\mathbb{Q}}_l[\dim X] \to F \to \mathrm{IC}(X, \bar{\mathbb{Q}}_l)[\dim X + 1],$$

where F is supported on Z, and $F|_Z$ lies in $D(Z)^{\geq 0}$. In particular, this implies that $F \in {}^{\mathrm{std}}D(X)^{\geq -\dim Z}$. Taking the long exact sequence cohomology sequence associated to the above distinguished triangle, we see that $H^k(\mathrm{IC}(X,\bar{\mathbb{Q}}_l)[\dim X]) \simeq H^k(Rj_*\bar{\mathbb{Q}}_l[\dim X])$ for all $k < -\dim Z$. If we take $k = -\dim X$, we find that $H^0(\mathrm{IC}(X,\bar{\mathbb{Q}}_l)) \simeq H^0(Rj_*\bar{\mathbb{Q}}_l)$.

Since $\mathrm{IC}(X,\bar{\mathbb{Q}}_l)$ is assumed to be a sheaf, we have $H^0(\mathrm{IC}(X,\bar{\mathbb{Q}}_l)) \simeq \mathrm{IC}(X,\bar{\mathbb{Q}}_l)$. On the other hand, we have $H^0(Rj_*\bar{\mathbb{Q}}_l) \simeq j_*\bar{\mathbb{Q}}_l \simeq \bar{\mathbb{Q}}_l$, where the last isomorphism holds because X is assumed to be irreducible. □

Proposition 6.2. *Let $f : Y \to X$ be a finite morphism of irreducible varieties. Let $V = f^{-1}(U)$, and assume that $f_*(\bar{\mathbb{Q}}_l|_V)$ is a local system on U. If $\mathrm{IC}(X, f_*(\bar{\mathbb{Q}}_l|_V))$ is a sheaf, then Y is rationally smooth.*

Proof. Since f is finite (and hence affine and proper), f_* is exact and t-exact by [3, Corollaire 2.2.6], and in particular, $f_*\mathrm{IC}(Y,\bar{\mathbb{Q}}_l)$ is an intersection cohomology complex on X, namely, it is $\mathrm{IC}(X, f_*(\bar{\mathbb{Q}}_l|_V))$. This complex is, by assumption, actually a sheaf. Now, f_* kills no nonzero sheaf, so the fact that $f_*\mathrm{IC}(Y,\bar{\mathbb{Q}}_l)$ is a sheaf implies that $\mathrm{IC}(Y,\bar{\mathbb{Q}}_l)$ itself is a sheaf. By Proposition 6.1, Y is rationally smooth. □

Since the morphism obtained by S_2-extension of a finite morphism is also finite, the same argument as above gives us the following result relating intersection cohomology complexes on a scheme obtained by S_2-extension with those on the original scheme. This fact will be a vital step in the calculations of Section 7, as anticipated by Lusztig in his original formulation of Conjecture 1.1 [23, §0.4].

Proposition 6.3. *Let $\rho : \tilde{X} \to X$ be the S_2-extension of a finite morphism $\rho_1 : \tilde{U} \to U \subset X$. Let E be a local system on $\tilde{U}$, and assume that ρ_*E is a local system on U. Then we have $\rho_*\mathrm{IC}(\tilde{X}, E) \simeq \mathrm{IC}(X, \rho_*E)$.* □

We close this section with the following result expressing the size of fibers of the normalization map in terms of intersection cohomology.

Proposition 6.4. *Let X be an irreducible variety with rationally smooth normalization $\bar{X}$, and let $\nu : \bar{X} \to X$ be the normalization morphism. Then for any $x \in X$, $|\nu^{-1}(x)| = \dim H^0_x(\mathrm{IC}(X,\bar{\mathbb{Q}}_l))$. If X is also rationally smooth, then X is unibranch.*

Proof. Since ν is a finite morphism, it is exact and t-exact. This and the fact that ν is birational imply that $\nu_*\bar{\mathbb{Q}}_l \simeq \nu_*\mathrm{IC}(\bar{X},\bar{\mathbb{Q}}_l) \simeq \mathrm{IC}(X,\nu_*\bar{\mathbb{Q}}_l) \simeq \mathrm{IC}(X,\bar{\mathbb{Q}}_l)$. Taking stalks at x gives $\nu_*(\bar{\mathbb{Q}}_l|_{\nu^{-1}(x)}) \simeq \mathrm{IC}(X,\bar{\mathbb{Q}}_l)_x$, and hence, $H^0_x(\mathrm{IC}(X,\bar{\mathbb{Q}}_l)) \simeq \bar{\mathbb{Q}}_l^{|\nu^{-1}(x)|}$. The formula for the fiber size follows by taking dimensions. Finally, if X is rationally smooth, then $\mathrm{IC}(X,\bar{\mathbb{Q}}_l) \simeq \bar{\mathbb{Q}}_l$, so $|\nu^{-1}(x)| = 1$. □

Remark 6.5. This proposition is known, but we have provided a proof for lack of a suitable reference. The statement (without the assumption that $\bar{X}$ is rationally smooth) is given without proof in [4, 5E].

7. The Geometry of $\tilde{P}$

In this section, we prove Theorem 1.3. The field k is now assumed to be algebraically closed of good characteristic for the group G. We begin by observing that $\mathcal{IC}(P, \rho_{1*}\mathcal{O}_{\tilde{C}_1})$ is defined. Indeed, $\tilde{C}_1$ is open dense in P, and its complement has codimension at least 2. Also, the stalk of $\rho_{1*}\mathcal{O}_{\tilde{C}_1}$ at $x \in C_1$ is just the direct sum of $|F|$ copies of $\mathcal{O}_{C_1,x}$. Since C_1 is smooth, $\rho_{1*}\mathcal{O}_{\tilde{C}_1}$ certainly satisfies the condition of Lemma 3.10. We may thus define $\tilde{P}$ by S_2-extension:

$$\tilde{P} = \mathbf{Spec}\,\mathcal{IC}(P, \rho_{1*}\mathcal{O}_{\tilde{C}_1}).$$

Now, $\tilde{C}_1$ is regular and S_2 (because it is smooth), and its complement in $\tilde{P}$ (which has codimension at least 2) is S_2. By Serre's criterion, $\tilde{P}$ is normal. The first part of Theorem 1.3 is then immediate from Theorem 4.5.

The remainder of Theorem 1.3 is given by Propositions 7.1–7.4 below.

Proposition 7.1. (1) *The variety $\tilde{P}$ is endowed with natural actions of F and G, and these actions commute. If we regard F as acting trivially on P, then ρ is both G and F-equivariant.*
(2) *The variety $\tilde{P}$ is rationally smooth. Moreover, if $\operatorname{char} k = 0$, then $\tilde{P}$ is Gorenstein.*

Proof. First, note that $\rho_{1*}\mathcal{O}_{\tilde{C}_1}$ naturally has the structure of a $(G \times F)$-equivariant sheaf (where F acts trivially on P), so $\mathcal{IC}(P, \rho_{1*}\mathcal{O}_{\tilde{C}_1})$ acquires one as well. Applying the **Spec** construction to an equivariant sheaf produces a scheme carrying an group action and an equivariant morphism.

Next, the rational smoothness of $\tilde{P}$ follows from Proposition 6.2, together with the fact that $\mathrm{IC}(P, E)$ is a sheaf for any equivariant local system E on C_1. (See [23, Proposition 0.7(c)].)

Finally, observe that because $\tilde{P}$ is normal, the canonical morphism $\rho : \tilde{P} \to P$ factors through the normalization $\bar{P}$ of P:

$$\begin{tikzcd} \tilde{P} \arrow[r, "\bar{\rho}"] \arrow[rr, bend right, "\rho"'] & \bar{P} \arrow[r, "\nu"] & P \end{tikzcd}$$

By invoking Proposition 3.5, we see that $\tilde{P}$ can also be constructed as

$$\mathbf{Spec}\, \mathcal{IC}(\bar{P}, (\bar{\rho}|_{\tilde{C}_1})_* \mathcal{O}_{\tilde{C}_1}).$$

Now, $\bar{P}$ is Gorenstein in characteristic 0 by the theorem of Hinich–Panyushev [18, 25], so $\tilde{P}$ is Gorenstein as well by Corollary 5.9. (Note that $\bar{P}$ satisfies condition (Q), since it is normal and quasiaffine.) □

Proposition 7.2. *Each special piece P is unibranch.*

Proof. This follows from Proposition 6.4, the preceding result, and the fact that P is rationally smooth. □

Proposition 7.3. *The morphism $\bar{\rho} : \tilde{P} \to \bar{P}$ is the algebraic quotient of the F-action while $\rho : \tilde{P} \to P$ is the topological quotient. In particular, F acts transitively on the fibers of ρ.*

Proof. Since P is unibranch, P is homeomorphic to $\bar{P}$, and it suffices to show that $\bar{P} \simeq \tilde{P}/F$.

The functor IC is an equivalence of categories between appropriate categories of sheaves on C_1 and P, and it accordingly preserves finite limits. In particular, it preserves F-fixed objects, so $\mathcal{IC}(P, \rho_{1*}\mathcal{O}_{\tilde{C}_1})^F \simeq \mathcal{IC}(P, (\rho_{1*}\mathcal{O}_{\tilde{C}_1})^F) \simeq \mathcal{IC}(P, \mathcal{O}_{C_1})$. The result now follows, since we have $\tilde{P}/F \simeq \mathbf{Spec}\, \mathcal{IC}(P, \rho_{1*}\mathcal{O}_{\tilde{C}_1})^F$ and $\bar{P} \simeq \mathbf{Spec}\, \mathcal{IC}(P, \mathcal{O}_{C_1})$. □

Proposition 7.4. *For each parabolic subgroup $H \subset F$, we have $\rho^{-1}(C_H) \simeq \tilde{C}_H$.*

We prove this proposition in two steps. First, in Lemma 7.5, we obtain a general description of the varieties $\rho^{-1}(C_H)$ in terms of unknown F-stabilizers. This description will suffice to prove the proposition when F is abelian, and in particular, for the classical groups. Then, in Lemma 7.6, we show by case-by-case considerations that the F-stabilizers of points in $\rho^{-1}(C_H)$ are in fact conjugates of H for the exceptional groups.

Lemma 7.5. *Let H be a parabolic subgroup of F. Each connected component of $\rho^{-1}(C_H)$ is isomorphic to $(\tilde{C}_H)^\circ$. Let K_H be the stabilizer in F of some closed point of $\rho^{-1}(C_H)$. Then $|K_H| = |H|$, and there is a subgroup $L_H \subset N_F(K_H)$ such that $|L_H| = |N_F(H)|$ and $\rho^{-1}(C_H) \simeq (\tilde{C}_H)^\circ \times_{L_H} F$. Moreover, if H is conjugate to a subgroup of another parabolic H', then K_H is conjugate to a subgroup of $K_{H'}$.*

Proof. Let E denote either the regular representation of F or, by abuse of notation, the corresponding local system on C_1. We will calculate $\mathrm{IC}(P, E)|_{C_H}$ in a way that reflects the structure of $\rho^{-1}(C_H)$ and then compare with the known calculations following [23] to prove the result.

Consider the commutative diagram

$$\begin{tikzcd} \rho^{-1}(C_H) \arrow[r, hook, "\tilde{i}_H"] \arrow[d, "\rho"'] & \tilde{P} \arrow[d, "\rho"] \\ C_H \arrow[r, hook, "i_H"'] & P \end{tikzcd}$$

From Proposition 6.3, we have $\rho_* \mathrm{IC}(\tilde{P}, \bar{\mathbb{Q}}_l) \simeq \mathrm{IC}(P, \rho_* \bar{\mathbb{Q}}_l|_{C_1})$. Moreover, because $\rho|_{\rho^{-1}(C_1)}$ is a principal F-bundle, $\rho_* \bar{\mathbb{Q}}_l|_{C_1} = E$. On the other hand, $\mathrm{IC}(\tilde{P}, \bar{\mathbb{Q}}_l) \simeq \bar{\mathbb{Q}}_l$ as $\tilde{P}$ is rationally smooth, so we have

$$\rho_* \bar{\mathbb{Q}}_l \simeq \mathrm{IC}(P, E),$$

and hence $\mathrm{IC}(P, E)|_{C_H} \simeq (\rho_* \bar{\mathbb{Q}}_l)|_{C_H}$. Now, since ρ is proper, we know that $\rho_* \bar{\mathbb{Q}}_l|_{C_H} \simeq \rho_*(\bar{\mathbb{Q}}_l|_{\rho^{-1}(C_H)})$. We seek to understand $\rho_*(\bar{\mathbb{Q}}_l|_{\rho^{-1}(C_H)})$.

Choose a point $x \in C_H$ and a point $\tilde{x} \in \rho^{-1}(x)$. Since the map $\rho : \rho^{-1}(C_H) \to C_H$ is finite and G-equivariant, the stabilizer in G of $\tilde{x}$, which we denote $G^{\tilde{x}}$, must be a finite-index subgroup of the stabilizer G^x of x. The connected component of $\rho^{-1}(C_H)$ containing $\tilde{x}$, which will be denoted B, must be isomorphic to the homogeneous space $G/G^{\tilde{x}}$. Then, since F acts transitively on the fiber $\rho^{-1}(x)$, and the actions of F and G commute, it follows that every connected component of $\rho^{-1}(C_H)$ is isomorphic to $G/G^{\tilde{x}}$. Let L_H be the subgroup of F that preserves B (without necessarily fixing $\tilde{x}$). The preceding discussion shows that $\rho^{-1}(C_H)$ is isomorphic to $B \times_{L_H} F$ (where $a \in F$ acts on a pair $(b, f) \in B \times_{L_H} F$ by $a \cdot (b, f) = (b, fa^{-1})$). In particular, the number of connected components of $\rho^{-1}(C_H)$ is $[F : L_H]$.

Let K_H be the stabilizer in F of $\tilde{x}$. Since the actions of F and G commute, it follows that K_H is also the F-stabilizer of every other point in B. This implies that K_H is a normal subgroup of L_H. Now, the group L_H/K_H acts simply transitively on $\rho^{-1}(x) \cap B$, so this is the group of deck transformations of B over C_H. Let $A'(C_H) = L_H/K_H$. We also have $A'(C_H) \simeq G^x/G^{\tilde{x}}$, which is the quotient of $A(C_H) \simeq G^x/(G^x)^\circ$ by $G^{\tilde{x}}/(G^x)^\circ$.

The local system $(\rho|_B)_* \bar{\mathbb{Q}}_l$ on C_H corresponds to the regular representation of $A'(C_H)$, and the full local system $\rho_*(\bar{\mathbb{Q}}_l|_{\rho^{-1}(C_H)})$ is then clearly just the direct sum of $[F : L_H]$ copies of the regular representation of $A'(C_H)$. It is easily checked that the action of L_H/K_H on the space E^{K_H} of K_H-invariant vectors in

E is also the direct sum of $[F : L_H]$ copies of its regular representation. Thus, $\mathrm{IC}(P,E)|_{C_H} \simeq E^{K_H}$ as an $A(C_H)$-representation. Set $Q \subset A(C_H)$ equal to the kernel of this representation. Since E^{K_H} is a faithful representation of $A'(C_H)$, we obtain $A'(C_H) \simeq A(C_H)/Q$ as groups and left $A(C_H)$-spaces. (In other words, if $A'(C_H)$ is viewed as a quotient of $A(C_H)$, then $A'(C_H) = A(C_H)/Q$.)

On the other hand, according to [23, Proposition 0.7], $\mathrm{IC}(P,E)|_{C_H}$ is the representation of $A''(C_H) \stackrel{\text{def}}{=} N_F(H)/H$ on E^H. Following [23, 2], this group is a direct factor of $\bar{A}(C_H)$ and hence naturally a quotient of $A(C_H)$. The same argument now shows that $A''(C_H) \simeq A(C_H)/Q$ and hence $A'(C_H) \simeq A''(C_H)$ as groups and $A(C_H)$-spaces. Moreover, E^H and E^{K_H} are isomorphic representations via this isomorphism. In particular, we have $|K_H| = |H|$, since $\dim E^{K_H} = [F : K_H]$ and $\dim E^H = [F : H]$. It follows immediately that $|L_H| = |N_F(H)|$. (However, we cannot conclude that K_H is conjugate to H; see Remark 7.7.)

The fact that $A'(C_H) \simeq A''(C_H)$ as homogeneous spaces implies that $G^{\tilde{x}}$ is precisely the kernel G^x_F of the canonical map $G^x \to N_F(H)/H$. Thus, $B \simeq G^x/G^x_F = (\tilde{C}_H)^\circ$, and $\rho^{-1}(C_H) \simeq (\tilde{C}_H)^\circ \times_{L_H} F$.

Finally, the points of $\tilde{P}$ fixed by K_H form a closed subvariety. If we repeat the above argument with another parabolic H' with $H \subset H'$, so that $C_{H'} \subset \overline{C_H}$, we see that K_H must be contained in the F-stabilizers of points of $\overline{B} \cap \rho^{-1}(C_{H'})$. Every such stabilizer is conjugate to $K_{H'}$, so K_H is conjugate to a subgroup of $K_{H'}$. □

If F is abelian, then $|L_H| = |N_F(H)|$ implies that both groups are in fact equal to F. Thus, $\rho^{-1}(C_H) \simeq (\tilde{C}_H)^\circ \times_{N_F(H)} F = \tilde{C}_H$.

It remains to identify the K_H's and L_H's for the exceptional groups. There, the only nontrivial groups F that occur are symmetric groups $\mathfrak{S}_n$ with $2 \le n \le 5$. The following lemma gives us the required information about the K_H's.

Lemma 7.6. *Let $F = \mathfrak{S}_n$ with $2 \le n \le 5$. Let $\{K_H\}$ be a collection of subgroups of F, where H ranges over the parabolic subgroups of F. Assume that $|K_H| = |H|$ and that K_{H_1} is conjugate to a subgroup of K_{H_2} whenever H_1 is conjugate to a subgroup of H_2. Then each K_H is conjugate to H.*

Proof. If $F = \mathfrak{S}_2$ there are no nontrivial cases of H to consider.

If $F = \mathfrak{S}_3$, we must consider $H = \mathfrak{S}_2$. It is clear that every subgroup of F of order 2 is conjugate to H.

If $F = \mathfrak{S}_4$, then the nontrivial possibilities for H are $\mathfrak{S}_2$, $\mathfrak{S}_3$, and $\mathfrak{S}_2 \times \mathfrak{S}_2$. The last one is a Sylow 2-subgroup of F, so every subgroup of order 4 is conjugate to it. Next, it is easy to verify by hand calculation that every subgroup of $\mathfrak{S}_4$ generated by an element of order 3 and another of order 2 either has more than 6 elements or is conjugate to $\mathfrak{S}_3$. Finally, if $H = \mathfrak{S}_2$, we now know that K_H must be conjugate to a subgroup of $\mathfrak{S}_3$, so K_H is conjugate to $\mathfrak{S}_2$ by the preceding paragraph.

If $f = \mathfrak{S}_5$, there are five nontrivial parabolic subgroups up to conjugacy. Another hand calculation shows that any subgroup generated by an element of order 4 and another of order 2 either has size different from 24 or is conjugate to $\mathfrak{S}_4$. Next, if H is any of $\mathfrak{S}_3$, $\mathfrak{S}_2 \times \mathfrak{S}_2$, or $\mathfrak{S}_2$, then K_H must be conjugate to a subgroup of $\mathfrak{S}_4$, so by the previous paragraph, K_H is conjugate to H. Finally, suppose $H = \mathfrak{S}_3 \times \mathfrak{S}_2$. Then K_H must contain a subgroup conjugate to $\mathfrak{S}_3$. Again, an easy calculation shows that every subgroup generated by $\mathfrak{S}_3$ and an element of order 2 either has size different from 12 or is conjugate to $\mathfrak{S}_3 \times \mathfrak{S}_2$. □

Remark 7.7. The above lemma does not hold in general for F a finite Coxeter group. For example, suppose F is the Weyl group of type B_2, generated by simple reflections s and t with $(st)^4 = 1$. The groups $\langle s\rangle$ and $\langle t\rangle$ are representatives of the two conjugacy classes of nontrivial parabolic subgroups. If we set $K_{\langle s\rangle} = \langle s\rangle$ and $K_{\langle t\rangle} = \langle (st)^2\rangle$, the hypotheses of the lemma are satisfied, but evidently $K_{\langle t\rangle}$ is not conjugate to $\langle t\rangle$.

We now know that K_H is conjugate to H in all cases. Returning to the setting of Lemma 7.5, we see that since $L_H \subset N_F(K_H) \simeq N_F(H)$ and $|L_H| = |N_F(H)|$, we must in fact have $L_H = N_F(K_H)$, so $\rho^{-1}(C_H) \simeq (\tilde{C}_H)^\circ \times_{N_F(H)} F = \tilde{C}_H$. The proof of Proposition 7.4 is now complete, and hence, so is the proof of Theorem 1.3.

Finally, we observe that any smooth variety containing $\tilde{C}_1$ as a dense open set together with a finite morphism to P extending ρ_1 must coincide with $\tilde{P}$. In particular:

Corollary 7.8. *If* $\operatorname{char} k = 0$ *and G is classical, $\tilde{P}$ is isomorphic to the smooth variety over P constructed by Kraft and Procesi.*

Proof. This follows immediately from the theorem and Proposition 4.11. □

8. Normality of Special Pieces

Recall that Conjecture 1.1 contains implicitly the additional Conjecture 1.2 that all special pieces are normal. In the classical types in characteristic 0, this statement follows from the work of Kraft and Procesi [19]; they show that each special piece P is the algebraic quotient of $\tilde{P}$ by F, so by Proposition 7.3, P is normal.

In positive characteristic or for the exceptional types in characteristic 0, there is no uniform answer. Of course, some special pieces consist only of a single unipotent class, so those ones are obviously normal (and even smooth). In other cases, it is known that the full closure of a special unipotent class in the unipotent variety is normal. Since a special piece is an open subvariety of its closure, the normality of the closure implies the normality of the special piece. Normality of closures of unipotent classes (or, more typically, nilpotent orbits) has been studied extensively by a number of authors, so this technique gives information about a large number of special pieces. In this section, we list the normality results that can be obtained in this way.

The following proposition summarizes the situation for classical groups.

Proposition 8.1. *Let G be a simple algebraic group of classical type over an algebraically closed field k of good characteristic. Let C_1 be a special unipotent class, and let P be the corresponding special piece.*

1. *If* $\operatorname{char} k = 0$ *or G is of type A_n, then P is normal.*
2. *$P = C_1$ if and only if $\bar{A}(C_1) = 1$. In that case, of course, P is normal.*
3. *If G is of type B_n and C_1 is the subregular class, then P is normal.*

We remark that it is easy to determine whether $\bar{A}(C_1) = 1$ for a given special class, using the straightforward combinatorial descriptions of that group given in, say, [2] or [23].

Proof. As we remarked above, in characteristic 0, the result follows from the work of Kraft and Procesi [19]. In type A_n, every unipotent class is special, so every special piece consists of a single class.

Group	Smooth	Normal	Normal in char. 0	Normal if BPS conj. is true	Unknown
G_2	G_2, 1	$G_2(a_1)$			
F_4	F_4, $F_4(a_1)$, $F_4(a_2)$, C_3, B_3, $\tilde{A}_2$, A_2, $A_1+\tilde{A}_1$, 1		$F_4(a_3)$, $\tilde{A}_1$		
E_6	E_6, $E_6(a_1)$, D_5, $D_5(a_1)$, A_4+A_1, D_4, A_4, A_3, A_2+2A_1, $2A_2$, A_2+A_1, $2A_1$, A_1, 1	$E_6(a_3)$	$D_4(a_1)$, A_2		
E_7	E_7, $E_7(a_1)$, $E_7(a_2)$, E_6, $E_6(a_1)$, $E_7(a_4)$, $D_6(a_1)$, D_5+A_1, A_6, D_5, $D_5(a_1)+A_1$, A_4+A_2, A_4+A_1, $(A_5)''$, $A_3+A_2+A_1$, A_4, A_3+A_2, D_4, $(A_3+A_1)''$, A_2+3A_1, $2A_2$, A_3, A_2+2A_1, $(3A_1)''$, $2A_1$, A_1, 1	$E_7(a_3)$		$E_7(a_5)$, $E_6(a_3)$, $D_5(a_1)$, $D_4(a_1)$, A_2+A_1, A_2	$D_4(a_1)+A_1$
E_8	E_8, $E_8(a_1)$, $E_8(a_2)$, $E_8(a_4)$, $E_8(b_4)$, $E_7(a_1)$, $E_8(a_6)$, $D_7(a_1)$, $E_6(a_1)+A_1$, $D_7(a_2)$, E_6, D_5+A_2, $E_6(a_1)$, $E_7(a_4)$, A_6+A_1, A_6, D_5, D_4+A_2, $A_4+A_2+A_1$, $D_5(a_1)+A_1$, A_4+A_2, A_4+A_1, A_4, A_3+A_2, D_4, A_3, A_2+2A_1, $2A_1$, A_1, 1	$E_8(a_3)$		$E_8(a_5)$, $E_8(b_5)$, $E_8(b_6)$, $E_8(a_7)$, A_4+2A_1, $D_4(a_1)+A_2$, $D_4(a_1)+A_1$, $D_4(a_1)$, $2A_2$, A_2+A_1, A_2	$E_7(a_3)$, $D_6(a_1)$, $E_6(a_3)$, $D_5(a_1)$

Table 1. Normality of special pieces in the exceptional types

Next, it is obvious that $P = C_1$ if $\bar{A}(C_1) = 1$; the other implication follows from [2, Theorem 2.1].

Finally, the subregular classes in type B_n occur in Thomsen's list [34, §9] of classes known to have normal closure in any good characteristic. $\square$

Remark 8.2. Thomsen lists many more classes with normal closures in the classical types, including the subregular class in all types, but it happens that all other classes listed by him fall into case (2) of the proposition above.

In Table 1, we indicate what is known for special pieces in the exceptional groups. We name a special piece by giving the Bala–Carter label of the special class it contains. The column labelled "Smooth" lists all special pieces that contain only a single class (this is easily deduced from, say, the partial order diagram of unipotent classes in [12, Chapter 13]). Among the remaining special pieces, those with normal closure in any good characteristic (following Thomsen [34]) are listed in the next column, and those only known to have normal closure in characteristic 0 (following Broer [8, 9] and Sommers [31]) appear in the column after that.

Before explaining the last two columns, we remark that in types E_7 and E_8, the normality question has not been answered for all nilpotent orbit closures, even in characteristic 0. However, a number of specific orbits are known to have nonnormal closures (see, for example, [28]), and Broer, together with Panyushev and Sommers, has conjectured that all remaining orbits have normal closures (see the Remarks at the end of [10, §7.8]). Sommers has verified this conjecture in a large number of cases [32]. If the Broer–Panyushev–Sommers conjecture is true, it will imply the normality of a number of special pieces, listed in the penultimate column.

Finally, the last column lists special pieces whose closures are known to be nonnormal. To establish normality for these special pieces, some additional technique will be required.

The following proposition summarizes the information that can be found in Table 1.

Proposition 8.3. *Let G be a simple algebraic group of exceptional type over an algebraically closed field k of good characteristic.*

(1) *If G is of type G_2, all special pieces are normal.*
(2) *If G is of type F_4 or E_6 and $\operatorname{char} k = 0$, all special pieces are normal. If $\operatorname{char} k > 0$, all but two special pieces are known to be normal.*
(3) *If G is of type E_7 (resp. E_8), then all but seven (resp. fifteen) special pieces are known to be normal. If $\operatorname{char} k = 0$ and the Broer–Panyushev–Sommers conjecture holds, then all but one (resp. four) special pieces will be known to be normal.* □

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