TWISTED EXPONENTS AND TWISTED FROBENIUS–SCHUR INDICATORS FOR HOPF ALGEBRAS

DANIEL S. SAGE AND MARIA D. VEGA

Abstract. Classically, the exponent of a group is the least common multiple of the orders of its elements. This notion was generalized by Etingof and Gelaki to the context of Hopf algebras. Kashina, Sommerhäuser and Zhu later observed that there is a strong connection between exponents and Frobenius–Schur indicators. In this paper, we introduce the notion of twisted exponents and show that there is a similar relationship between the twisted exponent and the twisted Frobenius–Schur indicators defined in previous work of the authors. In particular, we exhibit a new formula for the twisted Frobenius–Schur indicators and use it to prove periodicity and rationality statements for the twisted indicators.

1. Introduction

Classically, the exponent of a group G is the least common multiple of the orders of its elements. More generally, the exponent of a representation (V, ρ) of G is the exponent of $\rho(G) \subset \mathrm{GL}(V)$. In [EG99], Etingof and Gelaki, building on work of Kashina [Kas99], extended this notion from groups to Hopf algebras. Kashina, Sommerhäuser and Zhu later observed that there is a strong connection between exponents and Frobenius–Schur indicators [KSZ06]. Suppose that H is a semisimple Hopf algebra over an algebraically closed field of characteristic zero, and let V be a representation of H. They found a formula for the FS indicators $\nu_m(V)$ which implies that the indicators are $\exp(V)$-periodic and that the indicators lie in the cyclotomic ring generated by the $\exp(V)$-th roots of unity.

Suppose that H is a Hopf algebra endowed with a fixed automorphism τ of finite order. In this paper, we introduce a new invariant called the twisted exponent of V with respect to τ. For groups, the twisted exponent has a simple explicit description. Recall that the norm of an element $g \in G$ (in the sense of [BG04]) is the product $g\tau(g)\dots\tau^{r-1}(g)$; here, r is the order of τ. The twisted exponent of G is then the least common multiple of the orders of the norms of the group elements. Our primary goal is to show that for H semisimple over an algebraically closed field of characteristic zero, the results of [KSZ06] generalize to the twisted context. More precisely, we exhibit a new formula for the twisted Frobenius–Schur indicators defined in [SV12] and use it to prove periodicity and rationality statements for the twisted indicators.

2010 *Mathematics Subject Classification.* Primary:16T05; Secondary: 20C15.

Key words and phrases. Hopf algebra, exponent, Frobenius–Schur indicator.

The first author was partially supported by NSF grant DMS-0946431, and the second author was partially supported by a grant from the Simons Foundation (#281502).

2. Notation

Let H be a Hopf algebra over a field $\Bbbk$ with comultiplication Δ, counit ε, and bijective antipode S. We will use the usual Sweedler notation for iterated comultiplication: if $h \in H$, then

$$\Delta^{m-1}(h) = \sum_{(h)} h_1 \otimes h_2 \otimes \cdots \otimes h_m.$$

We will denote iterated multiplication by $\mu^m : H^{\otimes m} \to H$, i.e., $\mu(l_1 \otimes l_2 \otimes \ldots \otimes l_m) = l_1 l_2 \ldots l_m$. Let $\mathrm{Rep}(H)$ be the category of finite-dimensional left H-modules; we only consider such H-modules in this paper.

Let τ be a Hopf algebra automorphism of H; in particular, τ commutes with S. We will always assume that τ has finite order r. For all $j \in \mathbb{N}$, we define the *j-th twisted Sweedler power* of $h \in H$ to be

$$\tilde{h}_\tau^{[j]} := \mu^j \circ \left(\mathrm{Id} \otimes \tau \otimes \cdots \otimes \tau^{j-1}\right) \circ \Delta^{j-1}(h).$$

More explicitly, we have

$$\tilde{h}_\tau^{[j]} = \sum_{(h)} \left(h_1 \tau\left(h_2\right) \cdots \tau^{j-1}\left(h_j\right)\right).$$

We will write $\tilde{h}^{[j]}$ for $\tilde{h}_\tau^{[j]}$ when the automorphism is clear from context.

Suppose that A is a linear endomorphism of H. To simply notation, we write $A^{\otimes[k,k+m)} := A^k \otimes A^{k+1} \otimes \cdots \otimes A^{k+m-1} \in \mathrm{End}(H^{\otimes m})$. In particular, $A^{\otimes[k,k+m)} = (A^k)^{\otimes m} \circ A^{\otimes[0,m)}$. As an example, note that $\tilde{h}^{[j]} = \mu^j \circ \tau^{\otimes[0,j)} \circ \Delta^{j-1}(h)$.

3. Twisted exponents

3.1. Definition and first examples.

Definition 3.1. The *twisted exponent* $\exp_\tau(H)$ of H is the smallest $k \in \mathbb{N}$ such that

$$\mu^{kr} \circ \left(\mathrm{Id} \otimes (S^{-2}\tau) \otimes \cdots \otimes (S^{-2}\tau)^{(kr-1)}\right) \circ \Delta^{kr-1} = \varepsilon \cdot \mathrm{Id} \tag{3.1}$$

or ∞ if no such k exists. When $\exp_\tau(H)$ is finite, we set $d_\tau = r \exp_\tau(H)$. We will denote the endomorphism on the left side of (3.1) by $\Gamma^\tau_{kr}(H)$ or simply Γ^τ_{kr} when H is clear from context. More generally, the twisted exponent $\exp_\tau(V)$ of an H-module (V, ρ) is the smallest k such that

$$\rho(\mu^{kr} \circ (\mathrm{Id} \otimes (S^{-2}\tau) \otimes \cdots \otimes (S^{-2}\tau)^{kr-1}) \circ \Delta^{kr-1}(h)) = \varepsilon(h) \cdot 1_V \tag{3.2}$$

for all $h \in H$.

It is obvious that the twisted exponent of the regular representation of H coincides with $\exp_\tau(H)$.

If $S^2 = \mathrm{Id}$, then the defining formula for the twisted exponent reduces to $\tilde{h}^{[kr]} \cdot v = \varepsilon(h)v$ for all $h \in H$ and $v \in V$. In particular, this is the case when H is finite-dimensional, semisimple, and cosemisimple.

As a first example, we consider twisted exponents for group algebras. If $G(H)$ is the group of group-like elements in H, then the automorphism τ induces a norm map $N : G(H) \to G(H)$ given by $N(g) = g\tau(g) \ldots \tau^{r-1}(g)$.

Proposition 3.2.

(1) If $g \in G(H)$, then $\mathrm{ord}(N(g))$ divides $\exp_\tau(H)$.

(2) If G is a group, then $\exp_\tau \Bbbk[G]$ *is the least common multiple of* $\mathrm{ord}(N(g))$ *for $g \in G$.*

(Throughout the paper, we make the convention that any positive integer divides ∞.)

Proof. Since $g \in G(H)$, $\Delta(g) = g \otimes g$ and $S^2(g) = g$. This implies that $S^{-2}\tau(g) = \tau(g)$, so that $(S^{-2}\tau)^k(g) = \tau^k(g)$ for all $k \in \mathbb{N}$. We now compute:

$$\begin{aligned} N(g)^{\exp_\tau(H)} &= \mu^{d_\tau}(g \otimes \tau(g) \otimes \cdots \otimes \tau^{d_\tau - 1}(g)) \\ &= \mu^{d_\tau} \circ \left(\mathrm{Id} \otimes (S^{-2}\tau) \otimes \cdots \otimes (S^{-2}\tau)^{(d_\tau - 1)}\right) \circ \Delta^{d_\tau - 1}(g) = \varepsilon(g) \cdot 1 = 1. \end{aligned}$$

This implies that $\mathrm{ord}(N(g))$ divides $\exp_\tau(H)$, as desired.

In the group algebra case, one need only observe that $\exp_\tau \Bbbk[G]$ is the smallest k such that $1 = \tilde{g}^{[kr]} = N(g)^k$. $\square$

The same proof shows that if (V, ρ) is a representation of G, then $\exp_\tau(V)$ is the least common multiple of $\mathrm{ord}(\rho(N(g)))$.

If τ is an involution of G and $\exp_\tau(\Bbbk[G]) = 1$, then it is immediate that $\tau(g) = g^{-1}$ and G is commutative. A similar result holds for Hopf algebras.

Proposition 3.3. *If $d_\tau \leq 2$, then H is commutative and cocommutative. Moreover, τ is either the identity or the antipode.*

Proof. If $d_\tau = 1$ or $d_\tau = 2$ and $r = 1$, then $\tau = \mathrm{Id}$, and the conclusion follows from [EG99, Proposition 2.2(6)]. Now suppose that $d_\tau = 2$ and $r = 2$, so $\exp_\tau(H) = 1$. It follows that $h_1(S^{-2}\tau)h_2 = \varepsilon(h)1$ for all $h \in H$, so by definition of the antipode, $S^{-2}\tau = S$, i.e., $\tau = S^3$. This implies that τ is an algebra and coalgebra antiautomorphism as well as a bialgebra automorphism. Hence, $\tau(x)\tau(y) = \tau(xy) = \tau(y)\tau(x)$ for all $x, y \in H$. Since τ is a bijection, H is commutative. Also, $\tau(h_1) \otimes \tau(h_2) = \Delta\tau(h) = \tau(h_2) \otimes \tau(h_1)$ for any $h \in H$, and one obtains the cocommutativity of H by applying $\tau^{-1} \otimes \tau^{-1}$ to this equation. Note that in this case, since $S^2 = \mathrm{Id}$, $\tau = S$. $\square$

3.2. **Properties of the twisted exponent.**

In this section, we derive twisted analogues of the basic properties of the exponent.

We first examine the relationship between $\exp_\tau(H)$ and $\exp_\tau(V)$ for $(V, \rho) \in \mathrm{Rep}(H)$.

Proposition 3.4. *Suppose that* $\exp_\tau(V)$ *is finite. Then $k \in \mathbb{N}$ satisfies* (3.2) *if and only if k divides* $\exp_\tau(V)$.

Proof. Set $e = \exp_\tau(V)$. Note that for any $\ell \geq 0$,

$$\begin{aligned} &\rho(\mu^{er} \circ (S^{-2}\tau)^{\otimes[\ell, er+\ell)} \circ \Delta^{er-1}(h)) \\ &\qquad = \rho(\mu^{er} \circ (S^{-2}\tau)^{\otimes[0,er)}((S^{-2}\tau)^\ell(h_1) \otimes \ldots \otimes (S^{-2}\tau)^\ell(h_{er})) \\ &\qquad = \rho(\mu^{er} \circ (S^{-2}\tau)^{\otimes[0,er)}((S^{-2}\tau)^\ell(h)_1 \otimes \ldots \otimes (S^{-2}\tau)^\ell(h)_{er}) \\ &\qquad = \varepsilon((S^{-2}\tau)^\ell(h))1_V = \varepsilon(h)1_V. \end{aligned}$$

Now, suppose that k satisfies (3.2), and write $k = et + \ell$ with $0 \leq \ell < e$. Applying the previous observation repeatedly, we obtain

$$\begin{aligned}
&\varepsilon((S^{-2}\tau)^{ter}(h))1_V = \varepsilon(h)1_V = \rho(\mu^{kr} \circ (S^{-2}\tau)^{\otimes[0,kr)} \circ \Delta^{kr-1}(h)) \\
&= \rho(\mu^{kr} \circ ((S^{-2}\tau)^{\otimes[0,er)} \otimes (S^{-2}\tau)^{\otimes[(t-1)er,ter)} \otimes (S^{-2}\tau)^{\otimes[ter,(te+\ell)r)}) \circ \Delta^{kr-1}(h)) \\
&= \left(\prod_{i=1}^{t} \rho(\mu^{er} \circ (S^{-2}\tau)^{\otimes[(i-1)er,ier)}(h_{(i-1)er+1} \otimes \ldots \otimes h_{ier})\right) \\
&\qquad\qquad \rho(\mu^{\ell r} \circ (S^{-2}\tau)^{\otimes[ter,(te+\ell)r)}(h_{ter+1} \otimes \ldots \otimes h_{kr})) \\
&= \left(\prod_{i=1}^{t} \varepsilon(h_i)\right) \rho(\mu^{\ell r} \circ (S^{-2}\tau)^{\otimes[ter,(te+\ell)r))}(h_{t+1} \otimes \ldots \otimes h_{t+\ell r})) \\
&= \rho(\mu^{\ell r} \circ (S^{-2}\tau)^{\otimes[0,\ell r)} \circ (S^{-2}\tau)^{\otimes ter}(\Delta^{\ell r-1}(h))) \\
&= \rho(\mu^{\ell r} \circ (S^{-2}\tau)^{\otimes[0,\ell r)}(\Delta^{\ell r-1}((S^{-2}\tau)^{ter}(h))).
\end{aligned}$$

It follows that $\ell < r$ satisfies (3.2), so $\ell = 0$.

The same computation with $k = et$ gives the reverse implication. □

Corollary 3.5. *The twisted exponent* $\exp_\tau(H)$ *is the least common multiple of* $\exp_\tau(V)$ *for* $V \in \mathrm{Rep}(H)$.

Proof. Assume that $\exp_\tau(H)$ is finite, as otherwise there is nothing to prove. It is obvious that $\exp_\tau(H)$ satisfies (3.2) for any V, so the Proposition implies that $\exp_\tau(H) = \mathrm{lcm}\{\exp_\tau(V) \mid V \in \mathrm{Rep}(H)\} \mid \exp_\tau(H)$. □

We next consider the behavior of the twisted exponent under various algebraic operations.

Proposition 3.6.

(1) *Let* $A \subseteq H$ *be a Hopf subalgebra such that* $\tau(A) = A$, *and suppose that* $r' = \mathrm{ord}(\tau|_A)$ *(so* $r' \mid r$*). Then,* $\exp_\tau(A)$ *divides* $\exp_\tau(H)\frac{r}{r'}$*. In particular, if* $r = r'$*, then* $\exp_\tau(A) \mid \exp_\tau(H)$.

(2) *Let* $\phi : H \to H'$ *be a surjective map of Hopf algebras with* $\tau(\ker(\phi)) = \ker(\phi)$*, and let* τ' *be the Hopf algebra automorphism of* H' *defined by* $\tau'(\phi(h)) = \phi(\tau(h))$*. Then,* $\exp_{\tau'}(H')$ *divides* $\exp_\tau(H)\frac{r}{r'}$.

(3) K *is a field extension of* $\Bbbk$*, then* $\exp_\tau(H \otimes_\Bbbk K) = \exp_\tau(H)$.

(4) *If* H *is finite-dimensional, then* $\exp_{\tau^*}(H^*) = \exp_\tau(H)$.

Proof. Set $e = \exp_\tau(H)$. To prove the first statement, it suffices by Proposition 3.4 to show that $\Gamma^\tau_{er}(A) = \varepsilon\,\mathrm{Id}_A$. This is clear, since $\Gamma^\tau_{er}(A) = \Gamma^\tau_{er}(H)|_A$. The proof of (2) is similar, using the fact that $\Gamma^{\tau'}_{er}(H') \circ \phi = \phi \circ \Gamma^\tau_{er}(H) = \phi \circ (\varepsilon\,\mathrm{Id}_H) = \varepsilon'\,\mathrm{Id}_{H'}$. The fact that the twisted exponent is preserved by extension of scalars in obvious. Finally, if H is finite-dimensional, $\Gamma^\tau_{kr}(H)^* = \Gamma^{\tau^*}_{kr}(H^*)$, which implies that $\exp_{\tau^*}(H^*) = \exp_\tau(H)$. □

Let H' be another Hopf algebra with bijective antipode endowed with an automorphism τ' of finite order r'. Then $\gamma = \tau \otimes \tau'$ is an automorphism of $H \otimes H'$ with order $\mathrm{lcm}(r, r')$, and $\exp_\gamma(H \otimes H')$ is defined.

Proposition 3.7. *With* γ *as above,*

$$\exp_\gamma(H \otimes H') = \frac{\mathrm{lcm}(d_\tau, d_{\tau'})}{\mathrm{lcm}(r, r')}.$$

In particular, if τ and τ' have the same order, then $\exp_\gamma(H \otimes H') = \operatorname{lcm}(d_\tau, d_{\tau'})$.

Proof. By definition of the tensor product that $\Gamma_t^\gamma = \Gamma_t^\tau \otimes \Gamma_t^{\tau'}$ for all $t \in \mathbb{N}$. Moreover, $\Gamma^\gamma_{k\,\mathrm{ord}(\gamma)} = \varepsilon_{H\otimes H'} \operatorname{Id}_{H\otimes H'}$ if and only if $\Gamma^\tau_{k\,\mathrm{ord}(\gamma)} = \varepsilon \operatorname{Id}_H$ and $\Gamma^{\tau'}_{k\,\mathrm{ord}(\gamma)} = \varepsilon' \operatorname{Id}_{H'}$, so this condition holds if and only if d_τ and $d_{\tau'}$ both divide $k\,\mathrm{ord}(\gamma)$. Hence, $d_\gamma = \operatorname{lcm}(d_\tau, d_{\tau'})$, and $\exp_\gamma(H \otimes H')$ is obtained by dividing this by $\mathrm{ord}(\gamma)$. □

3.3. **The case of H finite-dimensional and involutory.** Assume that H is finite-dimensional and involutory. It is proved in [KSZ06] that the exponent of any module V is given by the order of the action on $V \otimes H^*$ of a certain element of $H \otimes H^*$. We show that the twisted exponent can be computed in a similar way.

Let $\operatorname{coev} : \Bbbk \to H \otimes H^*$ be the coevaluation map. Recall that if $b_1, \dots, b_n$ is a basis of H with dual basis $b_1^*, \dots, b_n^*$, then $\operatorname{coev}(1) = \sum b_i \otimes b_i^*$. We now define $q_\tau \in H \otimes H^*$ via $q_\tau = \prod_{i=0}^{r-1} (\operatorname{Id}_H \otimes (\tau^*)^{-i}) \operatorname{coev}(1)$. More explicitly,

$$q_\tau = \sum b_{i_1} \cdots b_{i_r} \otimes b_{i_1}^* (\tau^*)^{-1}(b_{i_2}^*) \cdots (\tau^*)^{-(r-1)}(b_{i_r}^*). \tag{3.3}$$

We denote by $q_\tau|_{V\otimes H^*}$ the action of q_τ on $V \otimes H^*$ by left multiplication.

Proposition 3.8. *The order of $q_\tau|_{V\otimes H^*}$ equals $\exp_\tau(V)$.*

Proof. Given $k \in \mathbb{N}$, let $m = kr$, so that

$$q_\tau^k = \sum b_{i_1} \cdots b_{i_m} \otimes b_{i_1}^* (\tau^*)^{-1}(b_{i_2}^*) \cdots (\tau^*)^{-(m-1)}(b_{i_m}^*).$$

Note that $f \in \operatorname{End}(V \otimes H^*)$ is uniquely determined by the maps $(\operatorname{Id}_V \otimes \operatorname{ev}_h) \circ f$. Thus, the order of q_τ is the smallest k such that $(\operatorname{Id}_V \otimes \operatorname{ev}_h)(q_\tau^k \cdot (v \otimes \alpha)) = v \otimes \alpha$ for all $v \in V$, $h \in H$, and $\alpha \in H^*$. Since left multiplication by q_τ^k commutes with the right H^*-action, it suffices to consider $\alpha = \varepsilon$. Using basic properties of dual bases, we obtain

$$\begin{aligned}
&(\operatorname{Id}_V \otimes \operatorname{ev}_h)(q_\tau^k \cdot (v \otimes \varepsilon)) \\
&\quad = \sum b_{i_1}^*(h_1)((\tau^*)^{-1}(b_{i_2}^*))(h_2) \cdots ((\tau^*)^{-(m-1)}(b_{i_m}^*))(h_m)\varepsilon(h_{m+1})(b_{i_1} \cdots b_{i_m} \cdot v) \\
&\quad = \sum b_{i_1}^*(h_1) b_{i_2}^*(\tau(h_2)) \cdots b_{i_m}^*(\tau^{(m-1)}(h_m)) b_{i_1} \cdots b_{i_m} \cdot v \\
&\quad = \sum h_1 \tau(h_2) \cdots \tau^{(m-1)}(h_m) \cdot v = \tilde{h}^{[kr]} \cdot v.
\end{aligned}$$

Since $(\operatorname{Id}_V \otimes \operatorname{ev}_h)(\operatorname{Id}_{V\otimes H^*}(v \otimes \varepsilon) = \varepsilon(h)v$, we conclude that the order of $q_\tau|_{V\otimes H^*}$ is the minimum k such that $\tilde{h}^{[kr]} \cdot v = \varepsilon(h)v$ for all h and v. However, when H is involutory, this is the definition of $\exp_\tau(V)$.

□

4. Twisted FS indicators and twisted exponents

In this section, we assume that $\Bbbk$ is an algebraically closed field of characteristic zero and that H is a semisimple Hopf algebra. In particular, H is finite-dimensional, so S is bijective; moreover, a theorem of Radford states that any Hopf algebra automorphism has finite order [Rad90]. We denote by Λ the unique (two-sided) integral of H such that $\varepsilon(\Lambda) = 1$.

Let V be a representation of H with character χ. Then, for any $m \in \mathbb{N}$ divisible by r, the mth twisted Frobenius-Schur indicator was defined in [SV12] to be the character sum

$$\nu_m(\chi, \tau) = \chi\left(\tilde{\Lambda}^{[m]}\right).$$

We usually will write $\widetilde{\nu}_m(\chi)$ instead of $\nu_m(\chi, \tau)$ when the automorphism is clear from context. This generalizes both the FS indicators defined for semisimple Hopf algebras by Linchenko and Montgomery [LM00] and the twisted FS indicators for groups introduced by Bump and Ginzburg [BG04].

The mth twisted FS indicator can be computed as the trace of a certain order m operator [SV12]. This implies that $\widetilde{\nu}_m(\chi) \in \mathbb{Z}[\zeta_m]$, where ζ_m is a primitive mth root of unity. A priori, this allows the possibility that the field extension of $\mathbb{Q}$ generated by all the $\widetilde{\nu}_m(\chi)$'s is infinite. However, this is not the case when the twisted exponent is finite. Indeed, we now exhibit another trace formula for $\widetilde{\nu}_m(\chi)$, which can be used to show that the function $m \mapsto \widetilde{\nu}_m(\chi)$ is $r \exp_\tau(V)$-periodic with image lying in $\mathbb{Z}[\zeta_{\exp_\tau(V)}]$.

Let q_τ be the element of $H \otimes H^*$ defined in (3.3).

Theorem 4.1. *If $m = kr$, then the mth twisted Frobenius-Schur indicator is given by*

$$\widetilde{\nu}_m(\chi) = \frac{1}{\dim H} \operatorname{tr}\left(q_\tau^k|_{V\otimes H^*}\right).$$

Proof. Since the trace of left multiplication by $\alpha \in H^*$ on H^* equals $\dim(H)\alpha(\Lambda)$ [LR88, Proposition 2.4], the trace of left multiplication by $h \otimes \alpha$ on $V \otimes H^*$ is $\dim(H)\chi(h)\alpha(\Lambda)$. We now apply this fact to compute the trace of $q_\tau^k|_{V\otimes H^*}$:

$$\begin{aligned}
\operatorname{tr}\left(q_\tau^k|_{V\otimes H^*}\right) &= \dim(H) \sum \chi(b_{i_1} b_{i_2} \dots b_{i_m})(b^*_{i_1}((\tau^*)^{-1}(b^*_{i_2})) \cdots ((\tau^*)^{-(m-1)}(b^*_{i_m})))(\Lambda) \\
&= \dim(H) \sum \chi(b_{i_1} b_{i_2} \dots b_{i_m}) b^*_{i_1}(\Lambda_1) b^*_{i_2}(\tau(\Lambda_2)) \cdots b^*_{i_m}(\tau^{(m-1)}(\Lambda_m)) \\
&= \dim(H) \sum \chi(\Lambda_1 \tau(\Lambda_2) \dots \tau^{m-1}(\Lambda_m)) \\
&= \dim(H)\chi(\tilde{\Lambda}^{[m]}).
\end{aligned}$$

Therefore,

$$\widetilde{\nu}_m(\chi) = \chi(\tilde{\Lambda}^{[m]}) = \frac{1}{\dim H} \operatorname{tr}\left(q_\tau^k|_{V\otimes H^*}\right).$$

□

When $\tau = \mathrm{Id}$, this result is due to [KSZ06].

Corollary 4.2.

(1) *If $\exp_\tau(V)$ is finite, then the function $m \mapsto \widetilde{\nu}_m(\chi_V)$ is $r \exp_\tau(V)$-periodic with image lying in $\mathbb{Z}[\zeta_{\exp_\tau(V)}]$.*

(2) *If $\exp_\tau(H)$ is finite, then this function is $r \exp_\tau(H)$-periodic for any $V \in \operatorname{Rep}(H)$ and $\widetilde{\nu}_m(\chi_V) \in \mathbb{Z}[\zeta_{\exp_\tau(H)}]$ for all $m \in r\mathbb{N}$.*

Proof. Since H is semisimple, it is involutory. Accordingly, by Proposition 3.8, $e = \exp_\tau(V) = \operatorname{ord}(q_\tau|_{V\otimes H^*})$, whence the periodicity of the indicators. Moreover, this implies that the eigenvalues of $q_\tau^k|_{V\otimes H^*}$ are eth roots of unity for each k. Since $\widetilde{\nu}_m(\chi)$ is an algebraic integer [SV12, Corollary 3.6], the first statement now follows immediately from the theorem while the second is a consequence of Corollary 3.5.

□

We remark that it is not true that the field $\mathbb{Q}(\widetilde{\nu}_m(\chi_V) \mid m \in r\mathbb{N}, V \in \operatorname{Rep}(H))$ coincides with $\mathbb{Q}(\zeta_{\exp_\tau(H)})$. For example, if H is the group algebra of a finite group and $\tau = \mathrm{Id}$, then the FS indicators are all integers.

Let $\mathbb{Q}_d$ denote the cyclotomic field $\mathbb{Q}(\zeta_d)$. The previous corollary implies that $\widetilde{\nu}_m(\chi) \in \mathbb{Q}_m \cap \mathbb{Q}_{\exp_\tau(V)} = \mathbb{Q}_{\gcd(m,\exp_\tau(V))}$. Accordingly, $\widetilde{\nu}_m(\chi) \in \mathbb{Z}$ when m and $\exp_\tau(V)$ are relatively prime. However, more can be said.

Given $m, d \in \mathbb{N}$, we follow the terminology of [KSZ06] and say that m is *large compared to* d if $d/\gcd(d,m)$ and m are relatively prime. Equivalently, for any p dividing m, the p-adic valuation of m is at least as great as the p-adic valuation of d.

Theorem 4.3. *Let $d_\tau(V) = r\exp_\tau(V)$.*

(1) If $m \in r\mathbb{N}$ is large compared to $d_\tau(V)$, then $\widetilde{\nu}_m(\chi) \in \mathbb{Z}$.

(2) If $d_\tau(V)$ is square-free, then $\widetilde{\nu}_m(\chi) \in \mathbb{Z}$ for all $m \in r\mathbb{N}$. In particular, this is the case when $d_\tau = d_\tau(H)$ is square-free.

Proof. Suppose that $m = kr$ is large compared to $d_\tau(V)$. Let $e = \exp_\tau(V)$, and set $e' = \gcd(k, e)$. By Theorem 4.1,

$$\widetilde{\nu}_m(\chi) = \frac{1}{\dim H} \operatorname{tr}\left((q_\tau^{e'}|_{V\otimes H^*})^{k/e'} \right),$$

and since $q_\tau^{e'}|_{V\otimes H^*}$ has order e/e', we see that $\widetilde{\nu}_m$ is an algebraic integer in $\mathbb{Q}_{e/e'} \cap \mathbb{Q}_m = \mathbb{Q}_{\gcd(e/e',m)}$. To prove the first statement, it suffices to show that e/e' and m are relatively prime. However, $e/e' = d_\tau(V)/re'$ and $\gcd(d_\tau(V), m) = re'$, so $\gcd(d_\tau(V)/re', m) = 1$ by assumption. Any m is large compared to a square-free integer, so $\widetilde{\nu}_m \in \mathbb{Z}$ for all m when $d_\tau(V)$ is square-free. Finally, divisors of square-free integers are square-free, so if d_τ is square-free, so is $d_\tau(V)$. $\square$

The untwisted version of this result is due to Kashina, Sommerhäuser, and Zhu [KSZ06].

5. The twisted exponents for H_8

Let H_8 denote the Kac algebra of dimension 8. It is the smallest semisimple Hopf algebra which is neither commutative nor cocommutative has dimension 8. As an algebra, H_8 is generated by elements x, y and z, with relations:

$$x^2 = y^2 = 1,\ z^2 = \frac{1}{2}(1 + x + y - xy),\ xy = yx,\ xz = zy,\ \text{and } yz = zx.$$

The coalgebra structure of H_8 is given by the following:

$$\Delta(x) = x \otimes x,\ \varepsilon(x) = 1,\ \text{and } S(x) = x,$$
$$\Delta(y) = y \otimes y,\ \varepsilon(y) = 1,\ \text{and } S(y) = y,$$
$$\Delta(z) = \frac{1}{2}(1\otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),$$
$$\varepsilon(z) = 1,\ \text{and } S(z) = z.$$

This Hopf algebra was first introduced by Kac and Paljutkin [KP66] and revisited later by Masuoka [Mas95].

The Hopf algebra H_8 has 4 one-dimensional representations and a single two-dimensional simple module. The characters for the irreducible representations of H_8 are listed in Table 1, where χ_i corresponds to representation V_i. The automorphism group of H_8 is the Klein four-group. These automorphisms are given in Table 2.

	1	x	y	xy	z	xz	yz	xyz
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1	-1
χ_3	1	-1	-1	1	i	$-i$	$-i$	i
χ_4	1	-1	-1	1	$-i$	i	i	$-i$
χ_5	2	0	0	-2	0	0	0	0

Table 1. Characters for the Irreducible Representations of H_8

	1	x	y	z
$\tau_1 = \mathrm{Id}$	1	x	y	z
τ_2	1	x	y	xyz
τ_3	1	y	x	$\frac{1}{2}(z + xz + yz - xyz)$
τ_4	1	y	x	$\frac{1}{2}(-z + xz + yz + xyz)$

Table 2. Automorphisms of H_8

	V_1	V_2	V_3	V_4	V_5	H_8
$\exp_{\tau_1}(V_i) = \exp(V_i)$	1	2	2	2	8	8
$\exp_{\tau_2}(V_i)$	1	1	1	1	4	4
$\exp_{\tau_3}(V_i)$	1	1	2	2	2	2
$\exp_{\tau_4}(V_i)$	1	1	2	2	2	2

Table 3. Twisted Exponents for H_8

All four automorphisms satisfy $\tau^2 = \mathrm{Id}$. Furthermore, $\mathrm{ord}(\tau_1) = 1$ and $\mathrm{ord}(\tau_j) = 2$ for $j = 2, 3$ and 4. The twisted exponents are given in Table 3. The sixth column corresponds to the regular representation of H_8, which can easily be computed from the twisted exponents of the V_i's and Corollary 3.5.

References

[BG04] D. Bump and D. Ginzburg, *Generalized Frobenius–Schur numbers*, J. Algebra **278** (2004), 294–313.

[EG99] P. Etingof and S. Gelaki, *On the exponent of finite-dimensional Hopf algebras*, Mathematical Research Letters **6** (1999), 131–140.

[Kas99] Y. Kashina, *On the order of the antipode of Hopf algebras in* ${}^H_H\mathrm{YD}$, Comm. Algebra **27** (1999), no. 3, 1261–1273.

[KP66] G. I. Kac and V. G. Paljutkin, *Finite ring-groups*, Trudy Moscov. Mat. Ob. **15** (1966), 224–261.

[KSZ06] Y. Kashina, Y. Sommerhäuser, and Y. Zhu, *On higher Frobenius–Schur indicators*, Mem. Amer. Math. Soc. **181** (2006), no. 855, viii+65.

[LM00] V. Linchenko and S. Montgomery, *A Frobenius–Schur theorem for Hopf algebras*, Algebr. Represent. Theory **3** (2000), 347–355.

[LR88] R. G. Larson and D. E. Radford, *Finite dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple*, J. Algebra **117** (1988), 267–289.

[Mas95] A. Masuoka, *Semisimple Hopf algebras of dimension* 6*,* 8, Israel J. Math. **93** (1995), 361–373.

[Rad90] D. E. Radford, *The group of automorphisms of a semisimple Hopf algebra over a field of characteristic 0 is finite*, Amer. J. Math. **112** (1990), 331–357.

[SV12] D. S. Sage and M. D. Vega, *Twisted Frobenius–Schur indicators for Hopf algebras*, J. Algebra **354** (2012), 136–147.

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803
E-mail address: `sage@math.lsu.edu`

Department of Mathematics, North Carolina State University, Raleigh, NC 27695
E-mail address: `mdvega@ncsu.edu`