Math 1553 Practice Set 4

Fall 2009 **Name**:

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Present your work neatly, with <u>*complete explanations*</u> *as needed.*

1. Consider the vectors

$$\mathbf{u} = 4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}, \qquad \mathbf{v} = (12, 0, -5), \qquad \mathbf{w} = 3\mathbf{j} - 4\mathbf{k}$$

   (i) Work out the lengths of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

   (ii) Work out the unit vector in the direction of $\mathbf{u}$.

   (iii) Express the angle between $\mathbf{v}$ and $\mathbf{w}$ in the form $\cos^{-1}[\cdots]$.

   (iv) Work out the vector products:

$$\mathbf{v} \times \mathbf{w}$$

   and

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$$

   (v) Work out

$$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} \; - \; (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

   (vi) Work out the area enclosed by the parallelogram formed by the vectors $\mathbf{v}$ and $\mathbf{w}$.

   (vii) What is the angle between $\mathbf{v} \times \mathbf{w}$ and the plane containing $\mathbf{v}$ and $\mathbf{w}$?

   (viii) Work out

$$\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$$

   (ix) Work out the volume of the parallelepiped formed by the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

2. Consider the points

$$A = (1, 2, 2), \qquad B = (-1, 0, 2), \qquad C = (2, 3, 0)$$

   Find a point P such that AP is perpendicular to the plane containing the points A, B, C.

3. Consider the points

$$P = (1, 2, 3), \qquad Q = (-1, 0, 3), \qquad R = (4, -1, 2), \qquad S = (2, 1, 1)$$

Find the angle between the vectors $\vec{PQ}$ and $\vec{RS}$.

4. If the scalar product of two unit vectors is 0 then the angle between these vectors is:

5. If the cross product of two unit vectors is 0 then these vectors are:

6. Use brute force calculation to show that

$$(\mathbf{a} \cdot \mathbf{b})^2 \, + \, |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 \, |\mathbf{b}|^2$$

holds for all vectors $\mathbf{a}$ and $\mathbf{b}$. Using this result show that the magnitude of

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta,$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (we choose the angle θ to be in $[0, \pi]$).

1. Consider the vectors

$$\mathbf{u} = 4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}, \qquad \mathbf{v} = (12, 0, -5), \qquad \mathbf{w} = 3\mathbf{j} - 4\mathbf{k}$$

(i) Work out the lengths of the vectors **u**, **v**, and **w**.

Sol:

$$|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{4^2 + (-4)^2 + 2^2} = \sqrt{36} = 6$$
$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{(12)^2 + (0)^2 + (-5)^2} = \sqrt{169} = 13$$

$$|\mathbf{w}| = \sqrt{\mathbf{w} \cdot \mathbf{w}} = \sqrt{(0)^2 + (3)^2 + (-4)^2} = \sqrt{25} = 5$$

(ii) Work out the unit vector in the direction of **u**.

Sol: The unit vector along **u** is obtained by dividing **u** by its length, i.e. it is

$$\frac{1}{|\mathbf{u}|}\mathbf{u}$$

Using the length value from (i), we then obtain the vector

$$\frac{1}{6}(4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$$

(iii) Express the angle between **v** and **w** in the form $\cos^{-1}[\cdots]$.

Sol: If the angle between the vectors is θ, then

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos\theta.$$

Hence

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}||\mathbf{w}|}$$

The scalar product if **v** and **w** is:

$$\mathbf{v} \cdot \mathbf{w} = (12, 0, -5) \cdot (0, 3, -4) = 12 * 0 \; + \; 0 * 3 \; + \; (-5) * (-4) = 20$$

We already know the lengths of **v** and **w** from (i); using all this we have

$$\cos\theta = \frac{20}{13 * 5} = \frac{20}{65}$$

So the angle between the vectors is

$$\theta = \cos^{-1}\frac{20}{65}$$

(iv) Work out the vector products:

$$\mathbf{v} \times \mathbf{w}$$

and

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$$

Sol:

$$\begin{aligned}\mathbf{v} \times \mathbf{w} &= (12, 0, -5) \times (0, 3, -4) \\ &= (0 * (-4) - (-5) * 3\,,\ (-5) * 0 - 12 * (-4)\,,\ 12 * 3 - 0 * 0) \\ &= (15,\ 48,\ 36)\end{aligned}$$

Thus,

$$\mathbf{v} \times \mathbf{w} = (15,\ 48,\ 36) \tag{1}$$

Then:

$$\begin{aligned}\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= (4, -4, 2) \times (15,\ 48,\ 36) \\ &= ((-4) * 36 - 2 * (48)\,,\ 2 * (15) - 4 * (36)\,,\ 4 * (48) - (-4) * (15)) \\ &= (-240, -114, 252)\end{aligned}$$

(v) Work out

$$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} \ - \ (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

Sol: First, we have the scalar products

$$\mathbf{u} \cdot \mathbf{v} = (4, -4, 2) \cdot (12, 0, -5) \ = \ 4 * (12) \ + \ (-4) * 0 \ + \ 2 * (-5) \ = \ 38$$

and

$$\mathbf{u} \cdot \mathbf{w} = (4, -4, 2) \cdot (0, 3, -4) \ = \ 4 * 0 \ + \ (-4) * 3 \ + \ 2 * (-4) \ = \ -20$$

Then

$$\begin{aligned}(\mathbf{u} \cdot \mathbf{w})\mathbf{v} \ - \ (\mathbf{u} \cdot \mathbf{v})\mathbf{w} &= (-20) * (12, 0, -5) \ - \ (38) * (0, 3, -4) \\ &= (-240,\ -114,\ 252)\end{aligned}$$

Compare with the answer in (iv)!

(vi) Work out the area enclosed by the parallelogram formed by the vectors $\mathbf{v}$ and $\mathbf{w}$.

Sol: The area of the parallelogram is the length of the cross product $\mathbf{v} \times \mathbf{w}$. Let us first work out

$$\mathbf{v} \times \mathbf{w} = (15,\ 48,\ 36)$$

(Okay, this is copied and pasted from (iv).) The norm of this vector is

$$|\mathbf{v} \times \mathbf{w}| = \sqrt{(15)^2 + (48)^2 + (36)^2} = \sqrt{3825}$$

(vii) What is the angle between $\mathbf{v} \times \mathbf{w}$ and the plane containing $\mathbf{v}$ and $\mathbf{w}$?

Sol: The cross product of two vectors is perpendicular to both the vectors. Thus, the answer is 90^0.

(viii) Work out

$$\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$$

Sol: We have seen in class that this works out to

$$\det[\mathbf{u}, \mathbf{v}, \mathbf{w}]\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}],$$

where the $\det[\cdots]$ term is the determinant:

$$\begin{aligned} \det \begin{bmatrix} 4 & 12 & 0 \\ -4 & 0 & 3 \\ 2 & -5 & -4 \end{bmatrix} &= 4 * \big(0 * (-4) - 3 * (-5)\big) - (-4) * \big((12) * (-4) - 0 * (-5)\big) \\ &\qquad + 2 * \big((12) * 3 - 0 * 0\big) \\ &= 4 * (15) + (-4) * (48) + 2 * (36) \\ &= 324 \end{aligned}$$

Note that the determinant here is exactly the same as the scalar triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$:

$$\det[\mathbf{u}, \mathbf{v}, \mathbf{w}] = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

We can see this happening here:

$$\mathbf{u} \cdot \underbrace{(\mathbf{v} \times \mathbf{w})}_{\text{see eqn. (1)}} = (4,\ -4,\ 2) \cdot (15, 48, 36) = 4 * (15) + (-4) * (48) + 2 * (36)$$

(ix) Work out the volume of the parallelepiped formed by the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

Sol: The volume is the absolute value of

$$\det[\mathbf{u}, \mathbf{v}, \mathbf{w}]$$

and so it is 324.

2. Consider the points

$$A = (1, 2, 2), \qquad B = (-1, 0, 2), \qquad C = (2, 3, 0)$$

Find a point P such that AP is perpendicular to the plane containing the points A, B, C.

Sol: The vector $\vec{AP}$ would be perpendicular to the plane containing the vectors $\vec{AB}$ and $\vec{AC}$.

Thus, a strategy would be to work out the cross product

$$\mathbf{N} = \vec{AB} \times \vec{AC}$$

and then obtain P by adding $\mathbf{N}$ to P.

First we have the vectors

$$\vec{AB} = (-1, 0, 2) \; - \; (1, 2, 2) \; = \; (-2, -2, 0)$$

$$\vec{AC} = (2, 3, 0) \; - \; (1, 2, 2) \; = \; (1, 1, -2)$$

So

$$\begin{aligned} \vec{AB} \times \vec{AC} &= ((-2) * (-2) - 0 * 1 \, , \, (0) * (1) - (-2) * (-2) \, , \, (-2) * (1) - (-2) * 1) \\ &= (4, -4, 0) \end{aligned}$$

This vector, call it $\mathbf{N}$, is perpendicular to the plane containing the vectors $\vec{AB}$ and $\vec{AC}$.

Now for the point

$$P = A + \mathbf{N} = (1, 2, 2) + (4, -4, 0) = (5, -2, 2)$$

the vector $\vec{AP}$ is simply $\mathbf{N}$, and so is perpendicular to the plane containing the points A, B, C.

3. Consider the points

$$P = (1, 2, 3), \qquad Q = (-1, 0, 3), \qquad R = (4, -1, 2), \qquad S = (2, 1, 1)$$

Find the angle between the vectors $\vec{PQ}$ and $\vec{RS}$.

Sol: First we work out the vectors

$$\vec{QP} = (-1, 0, 3) \; - \; (1, 2, 3) = (-2, -2, 0),$$

and

$$\vec{RS} = (2, 1, 1) - (4, -1, 2) = (-2, 2, -1)$$

To find the angle θ between them we use the formula

$$\vec{QP} \cdot \vec{RS} = |\vec{QP}||\vec{RS}| \cos\theta$$

The scalar product is

$$\vec{QP} \cdot \vec{RS} = (-2, -2, 0) \cdot (-2, 2, -1) = (-2)(-2) + (-2)2 + 0 * (-1) = 0$$

We can stop at this stage, because we have two non-zero vectors whose scalar product is 0: the angle between them must by 90^0.

4. If the scalar product of two unit vectors is 0 then the angle between these vectors is:

Sol: The scalar product of two vectors is 0 if and only if either at least one of them is zero or the angle between them is 90^0. In the present case we have unit vectors, and so the conclusion is that they must be at right angles to each other.

5. If the cross product of two unit vectors is 0 then these vectors are:

Sol: We know (and prove in item 5 below) that the length of the cross product of two vectors is the product of the lengths of the vectors with the sin of the angle between them. The lengths for unit vectors is 1, and so the length of the cross product is simple sin of the angle between them. So if the cross product is 0 then this angle must be 0 or π. Thus, the vectors must be either parallel or anti-parallel (i.e. pointing in the same direction or in opposite directions).

6. Show that

$$(\mathbf{a} \cdot \mathbf{b})^2 + |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 \, |\mathbf{b}|^2$$

holds for all vectors $\mathbf{a}$ and $\mathbf{b}$. Using this result show that the magnitude of

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta,$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (we choose the angle θ to be in $[0, \pi]$).

Sol: Let

$$\mathbf{a} = (a_1, a_2, a_3) \qquad \mathbf{b} = (b_1, b_2, b_3)$$

Then

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$$

and

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

So

$$
\begin{aligned}
(\mathbf{a}\cdot\mathbf{b})^2 \;+\; |\mathbf{a}\times\mathbf{b}|^2 &= (a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2 \\
&\quad + (a_1b_1+a_2b_2+a_3b_3)^2 \\
\\
&= a_2^2b_3^2-2a_2b_3a_3b_2+a_3^2b_2^2 \;+\; a_3^2b_1^2-2a_3b_1a_1b_3+a_1^2b_3^2 \\
&\qquad\qquad\qquad\qquad + \; a_1^2b_2^2-2a_1b_2a_2b_1+a_2^2b_1^2 \\
&\quad + a_1^2b_1^2+a_2^2b_2^2+a_3^2b_3^2+2a_2b_2a_3b_3+2a_3b_3a_1b_1+2a_1b_1a_2b_2 \\
&= a_2^2b_3^2+a_3^2b_2^2 \;+\; a_3^2b_1^2+a_1^2b_3^2 \;+\; a_1^2b_2^2+a_2^2b_1^2 \\
&\qquad + a_1^2b_1^2+a_2^2b_2^2+a_3^2b_3^2 \\
\\
&= a_1^2(b_1^2+b_2^2+b_3^2) \;+\; a_2^2(b_1^2+b_2^2+b_3^2) \\
&\qquad + a_3^2(b_1^2+b_2^2+b_3^2) \\
&= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) \\
\\
&= |\mathbf{a}|^2\,|\mathbf{b}|^2 \qquad \text{:amazing!}
\end{aligned}
$$

Next, we know that

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta,$$

where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. We say 'the' angle, but we could just either the angle value which lies between 0^0 and 180^0 or the other one, between 180^0 and 360^0. To be reasonable, just choose θ in $[0,\pi]$. As far as $\cos\theta$ goes the choice actually makes no difference since $\cos\theta$ is equal to $\cos(2\pi-\theta)$. But choosing θ in $[0,\pi]$ makes $\sin\theta$ be non-negative

$$\sin\theta \geq 0.$$

Now returning to the identity

$$(\mathbf{a}\cdot\mathbf{b})^2 \;+\; |\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2\,|\mathbf{b}|^2$$

we have

$$|\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta \;+\; |\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2\,|\mathbf{b}|^2$$

and so

$$|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2\,|\mathbf{b}|^2(1-\cos^2\theta) = |\mathbf{a}|^2\,|\mathbf{b}|^2\sin^2\theta.$$

Taking the square-root, and keeping in mind that $\sin\theta \geq 0$, we conclude that

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta.$$