Mean Value, Taylor, and all that

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Careful: Not proofread!

Derivative

Recall the definition of the derivative of a function f at a point p:

$$f'(p) = \lim_{w \to p} \frac{f(w) - f(p)}{w - p} \tag{1}$$

Derivative

Thus, to say that

$$f'(p) = 3$$

means that if we take any neighborhood U of 3, say the interval $(1, 5)$, then the ratio

$$\frac{f(w) - f(p)}{w - p}$$

falls inside U when w is close enough to p, i.e. in some neighborhood of p. (Of course, we can't let w be equal to p, because of the $w - p$ in the denominator.)

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So if

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then the ratio

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lies in $(1, 5)$ when w is close enough to p, i.e. in some neighborhood of p, but not equal to p.

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lies in $(1, 5)$ when w is close enough to p, i.e. in some neighborhood of p, but not equal to p.
In particular,

$$\frac{f(w) - f(p)}{w - p} > 0 \qquad \text{if } w \text{ is close enough to } p\text{, but } \neq p.$$

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- when $w > p$, but near p, the value $f(w)$ is $> f(p)$.
- when $w < p$, but near p, the value $f(w)$ is $< f(p)$.

Positive Derivative and Increasing behavior

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Local Maxima and Minima

A function f is said to have a *local maximum* at a point p if there is a neighborhood U of p such that that for all $x \in U$ in the domain of f, the value $f(x)$ is $\geq f(p)$.

A function f is said to have a *local minimum* at a point p if there is a neighborhood U of p such that that for all $x \in U$ in the domain of f, the value $f(x)$ is $\leq f(p)$.

Local Maxima and Minima

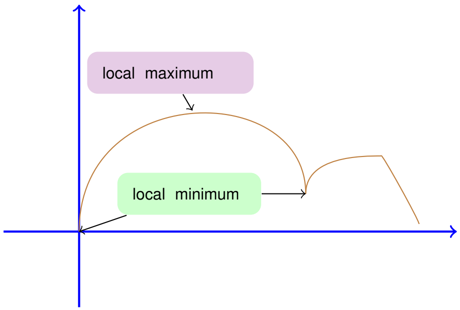

Figure: Local Maxima and Minima

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If $f(p) \leq f(x)$ for all x in a neighborhood of p, and $f'(p)$ exists, then $f'(p)$ is 0.

Note that we are requiring that f be defined in a neighborhood of p, and so on *both sides* of p.

Proof of the Local Max/Min Theorem

Proof Suppose $f'(p)$ exists but is not 0. Then $f'(p)$ is either > 0 or < 0.
If $f'(p) > 0$ then we know that to the right of p, but close to p, the values of f are $>$ than $f(p)$,

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Rolle's Theorem

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Consider a function

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where $a, b \in \mathbb{R}$ *with* $a < b$. *Suppose*

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- f *is continuous function*
- f *is differentiable on* (a, b)
- $f(a) = f(b)$.

Then there is a point c strictly between a and b where the derivative of f *is* 0*:*

$$f'(c) = 0 \text{ for some } c \in (a, b).$$

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The only other possibility is that both the max and the min value occur at the end points a and b. But h has the *same value* at a and at b. So then the max and the min value must be the same. Thus in this case h is constant and so its derivative is 0 everywhere.

Useful consequence Rolle's Theorem

Suppose now that f and g are functions on a compact interval $[a, b]$, and are differentiable in (a, b).

Next suppose also that f and g have the same value at a, and also the same value at b:

$$f(a) = g(a), \qquad \text{and} \qquad f(b) = g(b).$$

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Then f' and g' agree at some point c between a and b:

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To see this simply apply Rolle's theorem to the function $h = f - g$.

Mean Value Theorem

Theorem

Suppose f is continuous on a compact interval $[a, b]$ and differentiable in (a, b). Then there is a point c in (a, b) where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof of Mean Value Theorem

Proof Compare f with the straight line function L which agrees with f at the points a and b:

$$L(a) = f(a), \qquad L(b) = f(b),$$

and the slope of L is constant given by

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Hence:

$$f'(c) = L'(c) = \frac{f(b) - f(a)}{b - a}$$

Polynomials: coefficients and derivatives at 0

Consider a polynomial

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Observe that

$$P'(x) = a_1 + 2a_2 x + 3a_3 x^2$$
$$P^{(2)}(x) = 2a_1 + 3 * 2a_3 x$$
$$P^{(3)}(x) = 3 * 2 * 1a_3$$

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Observe now that if we put in $x = 0$ we can recover the values of a_0, a_1, a_2, a_3:

$$\begin{aligned} a_0 &= P(0) \\ a_1 &= P'(0) \\ a_2 &= \frac{1}{2!} P^{(2)}(0) \\ a_3 &= \frac{1}{3!} P^{(3)}(0) \qquad \text{of course, } P^{(3)}(x) \text{ is constant for all } x. \end{aligned}$$

Polynomials with specified derivatives derivatives at 0

In general, we have for any polynomial of degree n:

$$P(x) = P(0) + P'(0)x + \frac{P^{(2)}(0)}{2!}x^2 + \ldots + \frac{P^{(n)}(0)}{n!}x^n \qquad (2)$$

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Moreover, *the n-th derivative of this polynomial is a constant.*

Polynomials with specified derivatives derivatives at 0

Exercise. Find a polynomial function P for which

$$P(0) = 1, \quad P'(0) = 1, \quad P''(0) = -2, \qquad P^{(3)}(0) = 12$$

Solution: The simplest choice is

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We could also take, for instance,

$$1 + 1.x + \frac{-2}{2!}x^2 + \frac{12}{3!}x^3 + \frac{K}{4!}x^4,$$

where K is any constant.

Polynomials with specifications

Exercise. Find a polynomial function P for which

$$P(0) = -4, \quad P'(0) = 3, \quad P''(0) = -4, \qquad P^{(3)}(0) = 6$$

and also

$$P(1) = 5$$

Solution: To satisfy the conditions at 0 we can take the polynomial

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Now tune the constant K to the requirement that $P(1)$ be 5,i.e. choose K in such a way that

$$5 = -4 + 3 * 1 + \frac{-4}{2!}1^2 + \frac{6}{3!}1^3 + \frac{K}{4!}1^4,$$

which we can solve for K.

Taylor Polynomial of a Function

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We know that we can choose a polynomial function P whose value and derivatives at 0 up to order the 14th order match those for f:

$$P(0) = f(0), \qquad P'(0) = f'(0), \quad \ldots, \quad P^{(14)}(0) = f^{(14)}(0)$$

For instance, we can take

$$P(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(14)}(0)}{14!}x^{14} + \frac{K}{15!}x^{15} \quad (3)$$

where K is any constant (could be 0 too in the simplest case).

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But then...f' and P' agree at both 0 and c_1, hence their derivatives agree at a point c_2 in between:

$$f^{(2)}(c_2) = P^{(2)}(c_2)$$

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and on and on .... until ...

Taylor Polynomial of a Function

we have a point c, of course still between 0 and 4, where $f^{(15)}$ and $P^{(15)}$ agree:

$$f^{(15)}(c) = P^{(15)}(c)$$

Now if you look back at (3) to see what $P(x)$ was, you can see that the 15th-derivative of P is the constant K:

$$P^{(15)}(x) = \frac{K}{15!}15!x^0 = K$$

Hence,

$$K = f^{(15)}(c)$$

Thus, the constant K happens to be the 15-th derivative of f at some point c between 0 and 4.

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Theorem

Suppose f is a function defined in a neighborhood of 0 *and is n times differentiable on this neighborhood, where n is some positive integer (i.e.* $n \in \{1, 2, 3, ...\}$*).Then for any x in this neighborhood there is a point c lying between* 0 *and x such that*

$$f(x) = f(0) + f'(0)x) + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(c)}{n!}x^n \tag{4}$$

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The main point here is the *remainder* or *error* term

$$R_n = \frac{f^{(n)}(c)}{n!}x^n$$

when f is approximated by the *Taylor polynomial*

$$f(0) + f'(0)x) + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}$$

Analytic Functions

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Thus for such functions f we have, for x in some neighborhood of 0,

$$f(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k \tag{5}$$

The function f for which thise holds for all x in a neighborhood U of 0 is said to be *analytic* on U.

Analytic Functions

In class, we proved that the functions e^x and $\sin x$ are analytic, by showing that the Taylor remainder goes to 0 in each case. Polynomials are, of course, analytic, because the remainder term becomes 0 for them eventually.