Vector Spaces
Math 1553 Fall 2009

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A *vector space* V is a set of objects, called *vectors* on which there are two operations defined:

- *addition*

$$(v, w) \mapsto v + w$$

- *multiplication by scalar*

$$(k, v) \mapsto kv$$

satisfying the following long but natural list of conditions:

Axioms I: Addition

$$
\begin{aligned}
v + w &= w + v && \text{: addition is \textit{commutative}} \\
u + (v + w) &= (u + v) + w && \text{: addition is \textit{associatative}}
\end{aligned} \tag{1}
$$

Axioms I: Addition

$$\begin{aligned} v + w &= w + v && \text{: addition is \textit{commutative}} \\ u + (v + w) &= (u + v) + w && \text{: addition is \textit{associatative}} \end{aligned} \quad (1)$$

The associativity condition allows us to write either of $u + (v + w)$ and $(u + v) + w$ simply as

$$u + v + w,$$

without any ambiguity.

Axioms II: Zero and Negatives

There is a special vector **0**, the *zero vector*, for which

$$u + \mathbf{0} = u \quad \text{for all vectors } u \text{ in } V \tag{2}$$

Note that, because of commutativity of addition, we also then have

$$\mathbf{0} + u = u \quad \text{for all vectors } u \text{ in } V \tag{3}$$

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For every vector u there is a 'negative' $-u$ for which

$$u + (-u) = \mathbf{0} \tag{4}$$

Axioms III: Multiplication by Scalars

For the multiplication by scalars the conditions are

$$\begin{aligned} 1v &= v \\ (a+b)v & av + bv \\ a(v+w) &= av + aw \\ a(bv) &= (ab)v \end{aligned} \tag{5}$$

Vectors in plane geometry

Fix a point A in the plane.

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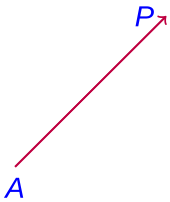
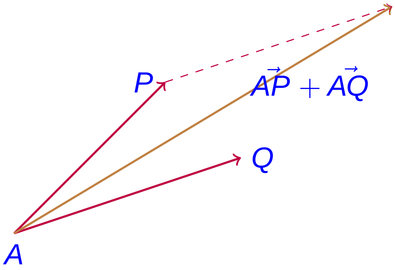
To each point P in the plane we then have the ordered pair (A, P), which we think of geometrically as a vector

$$\vec{AP}$$



Addition of geometric vectors

Geometric vectors are added by the *parallelogram law*:



Multiplication by scalars of geometric vectors

$$2\vec{AP}$$

is the vector $\vec{AQ}$, where Q is along the ray from A to P, but of twice the length of AP.

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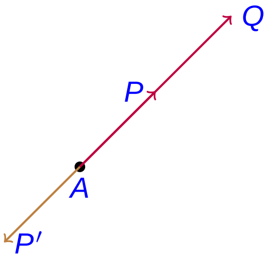
is the vector $\vec{AQ}$, where Q is along the ray from A to P, but of twice the length of AP.

$$(-1)\vec{AP} = -\vec{AP}$$

is the vector from A to the point P' on the ray away from $\vec{AP}$ but at equal distance from A as P:



Tangent space

The set of all geometric vectors in the plane starting at some point P is a vector space.

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A tangent vector $\vec{AP}$ is often identified with a tangent vector $\vec{BQ}$ if they are parallel, have the same direction, and magnitude.

The two-dimensional space $\mathbb{R}^2$

The vector space $\mathbb{R}^2$:

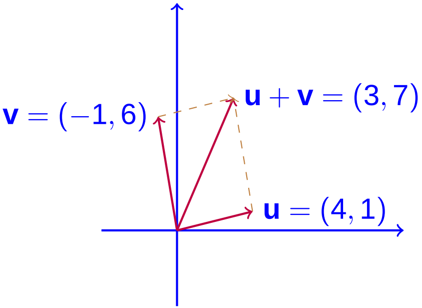
$$\mathbb{R}^2 = \{(x, y) \, : \, x, y \in \mathbb{R}\}$$

Addition:

$$(x, y) + (w, z) = (x + w, y + z)$$

Multiplication by scalar

$$k(x, y) = (kx, ky)$$



The three-dimensional space $\mathbb{R}^3$

The vector space $\mathbb{R}^2$:

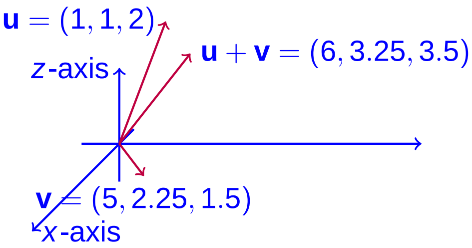
$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$$

Addition:

$$(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

Multiplication by scalar

$$k(x, y, z) = (kx, ky, kz)$$



Some Simple Theorems

Theorem

The zero vector is unique, i.e. if $\mathbf{0}'$ *is also vector for which*

$$\mathbf{v} + \mathbf{0}' = \mathbf{v} \qquad \textit{for all } \mathbf{v} \in V$$

then

$$\mathbf{0}' = \mathbf{0}$$

Proof of Uniqueness of the Zero Vector

Proof. The idea is to look at the sum of $\mathbf{0}$ and the potential other candidate $\mathbf{0}'$.

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On the other hand,

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because $\mathbf{0}'$ added to any vector is that vector.
Hence

$$\mathbf{0} = \mathbf{0}'$$

Done. □

Some Simple Theorems

Theorem

For any vector $\mathbf{u}$,

$$0\mathbf{u} = \mathbf{0}$$

Proof for Zero times any Vector is the Zero Vector

Proof. Let

$$\mathbf{x} = 0\mathbf{u}$$

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Then

$$\begin{aligned} \mathbf{x} + \mathbf{x} &= 0\mathbf{u} + 0\mathbf{u} \\ &= (0 + 0)\mathbf{u} \\ &= 0\mathbf{u} \\ &= \mathbf{x} \end{aligned} \tag{6}$$

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Thus

$$\mathbf{x} + \mathbf{x} = \mathbf{x}$$

Now add $-\mathbf{x}$ to both sides to get (using associativity)

$$\mathbf{x} + \mathbf{x} + (-\mathbf{x} = \mathbf{x} + (-\mathbf{x})$$

and so

$$\mathbf{x} = \mathbf{0}.$$

Done. $\square$

Uniqueness of the Negative of a Vector

Theorem

For any vector $\mathbf{u}$*, there is exactly one vector with the property that when added to* $\mathbf{u}$ *the result is* $\mathbf{0}$*.*

Proof for Uniquenss of Negative

Proof. Suppose $\mathbf{u}'$ and $\mathbf{u}''$ both have the property that when added to $\mathbf{u}$ the result is $\mathbf{0}$.

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$$\mathbf{u}' = \mathbf{u}' + \mathbf{0}$$

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Then

$$\begin{aligned} \mathbf{u}' &= \mathbf{u}' + \mathbf{0} \\ &= \mathbf{u}' + \quad (\mathbf{u} + \mathbf{u}'') \\ &= (\mathbf{u}' + \mathbf{u}) \quad + \mathbf{u}'' \\ &= \mathbf{0} + \mathbf{u}'' \\ &= \mathbf{u}'' \end{aligned} \tag{7}$$

Thus, $\mathbf{u}'$ is equal to $\mathbf{u}''$. $\square$

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Thus, $\mathbf{u}'$ is equal to $\mathbf{u}''$. $\square$

The unique vector which when added to $\mathbf{u}$ produces $-\mathbf{u}$ may thus be called *the* negative of $\mathbf{u}$, and it is denoted

$$-\mathbf{u}$$

Negative of a Vector and Multiplication by -1

Theorem

For any vector $\mathbf{u}$*,*

$$(-1)\mathbf{u} = -\mathbf{u}$$

Proof for $(-1)\mathbf{u} = -\mathbf{u}$

Proof. We have

$$\begin{aligned} \mathbf{u} + (-1)\mathbf{u} &= \big(1 + (-1)\big)\mathbf{u} \\ &= 0\mathbf{u} \\ &= \mathbf{0} \end{aligned} \tag{8}$$

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$$\begin{aligned} \mathbf{u} + (-1)\mathbf{u} &= \big(1 + (-1)\big)\mathbf{u} \\ &= 0\mathbf{u} \\ &= \mathbf{0} \end{aligned} \tag{8}$$

Thus, $(-1)\mathbf{u}$, when added to $\mathbf{u}$, gives the zero vector. Hence, $(-1)\mathbf{u}$ is the negative of $\mathbf{u}$. □

Linear combinations

A *linear combination* of vectors is a sum of multiples of the vectors.

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Thus,

$$2\mathbf{v} + (-3)\mathbf{w} + 14\mathbf{y}$$

is a linear combination of the vectors **v**, **w**, and **y**.

Basis

A *basis* for a vector space V is a set of vectors such that every vector can be expressed in a unique way as a linear combination of the basis vectors.

Thus, two vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ would form a basis of a vector space if every vector $\mathbf{v}$ in the space can be expressed as

$$\mathbf{v} = a\mathbf{u}_1 + b\mathbf{u}_2,$$

where a and b are scalars, and there is no other way to express $\mathbf{v}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

Standard Basis of $\mathbb{R}^2$

Any two non-zero vectors which are not along the same line form a basis of $\mathbb{R}^2$.

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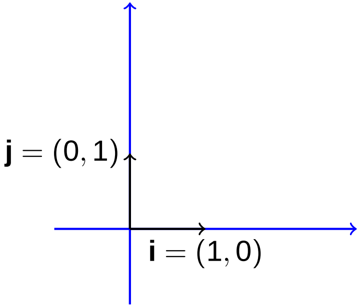
The *standard basis* of $\mathbb{R}^2$ is given by the vectors

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{i} = (1, 0) \\ \mathbf{e}_2 &= \mathbf{j} = (0, 1) \end{aligned} \tag{9}$$



Standard Basis of $\mathbb{R}^3$

Any three non-zero vectors which do not lie on the same plane form a basis of $\mathbb{R}^3$.

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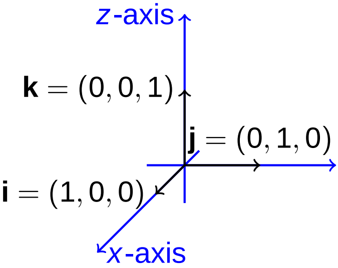
The *standard basis* of $\mathbb{R}^3$ is given by the vectors

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{i} = (1,0,0) \\ \mathbf{e}_2 &= \mathbf{j} = (0,1,0) \\ \mathbf{e}_3 &= \mathbf{k} = (0,0,1) \end{aligned} \tag{10}$$



Scalar Product

A *scalar product* on a vector space V associates to any pair of vectors $v, w \in V$ a scalar $v \cdot w$, satisfying:

$$\begin{aligned} v \cdot w &= w \cdot v \\ v \cdot (w + z) &= v \cdot w + v \cdot z \\ (kv) \cdot w &= k(v \cdot w) \end{aligned} \tag{11}$$

and we also require that

$$\mathbf{v} \cdot \mathbf{v} \geq 0 \qquad \text{for all } \mathbf{v} \in V \text{, and}$$

$$\mathbf{v} \cdot \mathbf{v} = 0 \text{ holds } \textit{only for the zero vector } \mathbf{v} = \mathbf{0}.$$

Scalar product with the zero vector is zero

We can check that

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To see this, let

$$x = \mathbf{v} \cdot \mathbf{0}$$

Then

$$\begin{aligned} x + x &= \mathbf{v} \cdot (\mathbf{0} + \mathbf{0}) \\ &= \mathbf{v} \cdot \mathbf{0} \\ &= x \end{aligned} \qquad (12)$$

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Thus,

$$x + x = x$$

and hence

$$x = 0$$

Scalar product of geometric vectors

$$\vec{AP} \cdot \vec{AQ} = |\vec{AP}|\,|\vec{AQ}|\,\cos(\text{angle between } \vec{AP} \text{ and } \vec{AQ})$$

Length

For a geometric vector $\vec{AP}$ then

$$\vec{AP} \cdot \vec{AP} = |\vec{AP}||\vec{AP}| \cos 0 = |\vec{AP}|^2$$

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Thus, *the scalar product of a vector with itself is the square of the length of the vector.*

Orthogonality

Notice that the *scalar product is* 0 *if and only if*:

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Notice that the *scalar product is* 0 *if and only if*:

- one of the vectors $\vec{AP}$ and $\vec{AQ}$ is $\mathbf{0}$; OR
- the vectors are *perpendicular*

Two vectors are said to be *orthogonal* if their scalar product is 0.

Scalar product in $\mathbb{R}^2$

$$(x_1, y_1) \cdot (x_2, y_2) = x_1 x_2 + y_1 y_2$$

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For example,

$$(1, -4) \cdot (5, 3) = 1 * 5 + (-4) * 3 = -7$$

Scalar product in $\mathbb{R}^3$

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = x_1 x_2 + y_1 y_2 + z_1 z_2$$

Scalar product in $\mathbb{R}^3$

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = x_1 x_2 + y_1 y_2 + z_1 z_2$$

For example,

$$(1, -4, 2) \cdot (5, 3, 4) = 1 * 5 + (-4) * 3 + 2 * 4 = 1$$

Scalar product and lengths and angles

For the vector

$$\mathbf{v} = (a, b, c)$$

the scalar product with itself is

$$\mathbf{v} \cdot \mathbf{v} = a * a + b * b + c * c = a^2 + b^2 + c^2$$

Geometrically it is, by Pythagoras, the square of the length of **v**.

Magnitude or Norm

The *length* or *magnitude* or *norm* of a general vector **v** is taken to be

$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \tag{13}$$

Scalar product, lengths, and angles

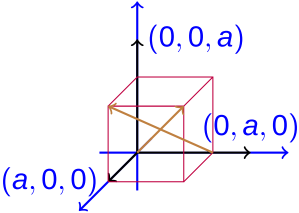
The angle θ between vectors **v** and **w** can be worked out from the formula

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos\theta$$

The vectors are perpendicular if their scalar product is 0, but neither vector is **0**.

Diagonals of a Cube

Exercise. Find the angle between the diagonals of a cube.



Diagonals of a Cube: solution

Sol: For convenience of calculation, take a coordinate system with origin at one corner, and axes along the edges. Say each side has length a. Then the two diagonal vectors are

$$\mathbf{d}_1 = (a, a, a) \qquad \text{and} \qquad \mathbf{d}_2 = (a, -a, a)$$

Work out the lengths of these two vectors, and their scalar product. Then work out

$$\cos\theta = \frac{\mathbf{d}_1 \cdot \mathbf{d}_2}{|\mathbf{d}_1|\,|\mathbf{d}_2|}$$

where θ is the angle between the diagonals.

Angle between Diagonals of a Cube

Now

$$|\mathbf{d}_1| = \sqrt{a^2 + a^2 + a^2} = \sqrt{3a^2} = a\sqrt{3}$$

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$$\mathbf{d}_1 \cdot \mathbf{d}_2 = a * a + a * (-a) + a * a = a^2$$

Then

$$\cos\theta = \frac{a^2}{\sqrt{3a^2}\,\sqrt{3a^2}} = \frac{a^2}{3a^2} = \frac{1}{3}$$

and so

$$\theta = \arccos\frac{1}{3}$$

Orthonormal Basis

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In a vector space, a basis is said to be *orthonormal* if the vectors in the basis are each unit vectors and they are all perpendicular to each other.

Thus the standard basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is an orthonormal basis of $\mathbb{R}^3$:

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$$

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$$

Wedge Product

To model a parallelogram with sides given by vectors v and w, and with a chosen orientation, we consider a new object, the *wedge product*

$$v \wedge w$$

One can form a new vector space by using wedge products of pairs of vectors in a vector space V; this space is

$$\Lambda^2 V$$

Wedge Product Rules: Alternating and Bilinear

The wedge product is *alternating*, i.e. the wedge of a vector with itself is zero:

$$v \wedge v = 0 \tag{14}$$

(Okay, to be sure the 0 here is the zero vector in $\Lambda^2 V$.)

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The wedge product is *alternating*, i.e. the wedge of a vector with itself is zero:

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The wedge product is *bilinear*:

$$\begin{aligned} u \wedge (v + w) &= u \wedge v \ + u \wedge w \\ (u + v) \wedge w &= u \wedge w \ + v \wedge w \\ u \wedge kv = k(u \wedge v) &= (ku) \wedge v \end{aligned} \tag{15}$$

for all vectors $u, v, w \in V$ and all scalars $k \in \mathbb{R}$.

Wedge Product Rules: Basis behavior

Dont worry about this too much at this stage ...

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Dont worry about this too much at this stage ...
If

$$e_1, e_2, ..., e_N$$

is a basis of V then the wedge products

$$e_1 \wedge e_2, e_1 \wedge e_3, ..., e_1 \wedge e_N, e_2 \wedge e_3, ..., e_{N-1} \wedge e_N$$

form a basis of $\Lambda^2 V$.

Wedge Product for $\mathbb{R}^3$: working it out

Consider

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}, \qquad \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

Wedge Product for $\mathbb{R}^3$: working it out

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Then

$$\begin{aligned}
\mathbf{u}\wedge\mathbf{v} = {} & u_1v_1 \underbrace{\mathbf{i}\wedge\mathbf{i}}_{\mathbf{0}} + u_1v_2\mathbf{i}\wedge\mathbf{j} + u_1v_3\underbrace{\mathbf{i}\wedge\mathbf{k}}_{-\mathbf{k}\wedge\mathbf{i}} \\
& + u_2v_1\underbrace{\mathbf{j}\wedge\mathbf{i}}_{-\mathbf{i}\wedge\mathbf{j}} + u_2v_2\mathbf{j}\wedge\mathbf{j} + u_2v_3\mathbf{j}\wedge\mathbf{k} \\
& + u_3v_1\mathbf{k}\wedge\mathbf{i} + u_3v_2\underbrace{\mathbf{k}\wedge\mathbf{j}}_{-\mathbf{j}\wedge\mathbf{k}} + u_3v_3\mathbf{k}\wedge\mathbf{k} \\
= {} & (u_2v_3 - u_3v_2)\mathbf{j}\wedge\mathbf{k} + (u_3v_1 - u_1v_3)\mathbf{k}\wedge\mathbf{i} + (u_1v_2 - u_2v_1)\mathbf{i}\wedge\mathbf{j}
\end{aligned} \tag{16}$$

Wedge Product for $\mathbb{R}^3$: the formula

$$\mathbf{u} \wedge \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{j} \wedge \mathbf{k} + (u_3v_1 - u_1v_3)\mathbf{k} \wedge \mathbf{i} + (u_1v_2 - u_2v_1)\mathbf{i} \wedge \mathbf{j} \tag{17}$$

Hodge Star in $\mathbb{R}^3$

The *Hodge star* operator in $\mathbb{R}^3$ associates two a wedge $u \wedge v$ a certain vector in $\mathbb{R}^3$ using the following scheme for the basis vectors:

$$\begin{aligned} *(\mathbf{j} \wedge \mathbf{k}) &= \mathbf{i} \\ *(\mathbf{k} \wedge \mathbf{i}) &= \mathbf{j} \\ *(\mathbf{i} \wedge \mathbf{j}) &= \mathbf{k} \end{aligned} \tag{18}$$

Cross Product in $\mathbb{R}^3$: Definition

The *cross product* of vectors in $\mathbb{R}^3$ is given by

$$\mathbf{u} \times \mathbf{v} = *(\mathbf{u} \wedge \mathbf{v}) \tag{19}$$

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Thus,

$$\begin{aligned} \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j} \\ \mathbf{i} \times \mathbf{j} &= \mathbf{k} \end{aligned} \tag{20}$$

Cross Product in $\mathbb{R}^3$: formula

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k} \quad (21)$$

Triple Wedge

Just as $\Lambda^2 V$ we can also form $\Lambda^3 V$. The elements are sums of triple wedge products

$$u \wedge v \wedge w$$

Triple Wedge rules

$$u \wedge v \wedge w$$

is *multilinear*, i.e. it is *linear* in each of the vectors u, v, w; for example,

$$u \wedge (3v + 4v') \wedge w = 3u \wedge v \wedge w + 4u \wedge v' \wedge w$$

and it is *alternating*, i.e. it is 0 whenever two of u, v, w are equal; for instance,

$$u \wedge v \wedge u = 0$$

and

$$u \wedge u \wedge w = 0$$

Skew-symmetry

From the multilinearity it follows that the triple wedge is 0 if at least one of the vectors is 0.

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From the multilinearity it follows that the triple wedge is 0 if at least one of the vectors is 0.

One other interesting fact we proved in class is *skew-symmetry*: if you switch any two of the vectors then the triple product changes sign:

$$u \wedge v \wedge w = -v \wedge u \wedge w$$

and

$$u \wedge v \wedge w = -w \wedge v \wedge u$$

and

$$u \wedge v \wedge w = -u \wedge w \wedge v$$

A triple product exercise

$$2\mathbf{j} \wedge (3\mathbf{j} \wedge \mathbf{k} - 5\mathbf{k} \wedge \mathbf{i} + 4\mathbf{i} \wedge \mathbf{j}) = 6\mathbf{j} \wedge \mathbf{j} \wedge \mathbf{k} - 10 \underbrace{\mathbf{j} \wedge \mathbf{k} \wedge \mathbf{i}}_{-\underbrace{\mathbf{j} \wedge \mathbf{i} \wedge \mathbf{k}}_{\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}}} + 8\mathbf{j} \wedge \mathbf{i} \wedge \mathbf{j}$$
$$= 0 + 10\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k} + 0 \tag{22}$$

A triple product exercise

$$2\mathbf{j} \wedge (3\mathbf{j} \wedge \mathbf{k} - 5\mathbf{k} \wedge \mathbf{i} + 4\mathbf{i} \wedge \mathbf{j}) = 6\mathbf{j} \wedge \mathbf{j} \wedge \mathbf{k} - 10 \underbrace{\mathbf{j} \wedge \mathbf{k} \wedge \mathbf{i}}_{-\underbrace{\mathbf{j} \wedge \mathbf{i} \wedge \mathbf{k}}_{\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}}} + 8\mathbf{j} \wedge \mathbf{i} \wedge \mathbf{j}$$
$$= 0 + 10\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k} + 0 \tag{22}$$

Check that

$$\mathbf{k} \wedge \mathbf{i} \wedge \mathbf{j} = \mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}$$

Triple product worked out

Let's work out the triple wedge of vectors

$$\mathbf{a} = (a_1, a_2, a_3), \quad \mathbf{b} = (b_1, b_2, b_3), \quad \mathbf{c} = (c_1, c_2, c_3)$$

$$\begin{aligned}
&\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \\
&= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\wedge \\
&\qquad [(b_2c_3 - b_3c_2)\mathbf{j} \wedge \mathbf{k} - (b_1c_3 - b_3c_1)\mathbf{k} \wedge \mathbf{i} + (b_1c_2 - b_2c_1)\mathbf{i} \wedge \mathbf{j}] \\
&= a_1(b_2c_3 - b_3c_2)\mathbf{i}\mathbf{j} \wedge \mathbf{k} - a_2(b_1c_3 - b_3c_1)\mathbf{j} \wedge \mathbf{k} \wedge \mathbf{i} \\
&\qquad + a_3(b_1c_2 - b_2c_1)\mathbf{k} \wedge \mathbf{i} \wedge \mathbf{j} \\
&= [a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + +a_3(b_1c_2 - b_2c_1)]\, \mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}
\end{aligned}$$

Triple product and Determinant

Thus

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \det(\mathbf{a}, \mathbf{b}, \mathbf{c})\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k} \tag{23}$$

where the quantity $\det[\cdots]$ on the right is the *determinant*:

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \tag{24}$$

Properties of the Determinant

From the properties of the triple wedge propduct we see that the determinant

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

- is equal to 0 if two of the columns are the same (i.e. if two of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are equal);
- switched sign if two columns are interchanged (i.e., for instance, $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ flips to its negative when two of the vectors are interchanged).

Scalar Triple Product and the Determinant

Recall that

$$\mathbf{b} \times \mathbf{c} = (b_2c_3 - b_3c_2)\mathbf{i} - (b_1c_3 - b_3c_1)\mathbf{j} + (b_1c_2 - b_2c_1)\mathbf{k}$$

Taking the scalar product of this with the vector

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

gives

$$\begin{aligned} &\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \\ &\quad = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \end{aligned} \tag{25}$$

which is exactly the determinant $\det[\mathbf{a}, \mathbf{b}, \mathbf{c}]$.

Scalar Triple Product and the Determinant

Thus,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \qquad (26)$$

Properties of the scalar triple product

Using the triple product's relationship with the determinant see that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

is 0 if any pair of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are equal to each other.

Properties of the scalar triple product

Using the triple product's relationship with the determinant see that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

is 0 if any pair of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are equal to each other. Also it flips sign if two of the vectors are interchanged.

Direction of the cross product

Now

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0 \qquad \text{and} \qquad \mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$$

Thus, **a** and **b** are both perpendicular to $\mathbf{a} \times \mathbf{b}$.

Direction of the cross product

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$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0 \qquad \text{and} \qquad \mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$$

Thus, $\mathbf{a}$ and $\mathbf{b}$ are both perpendicular to $\mathbf{a} \times \mathbf{b}$.
Thus, $\mathbf{a} \times \mathbf{b}$ points perpendicularly to the plane containing $\mathbf{a}$ and $\mathbf{b}$.

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Thus, $\mathbf{a} \times \mathbf{b}$ points perpendicularly to the plane containing $\mathbf{a}$ and $\mathbf{b}$.
Of course, if $\mathbf{a}$ equals $\mathbf{b}$, or if either is $\mathbf{0}$, then $\mathbf{a} \times \mathbf{b}$ is also $\mathbf{0}$.

Cross and Scalar

$$(\mathbf{a} \cdot \mathbf{b})^2 + |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 \, |\mathbf{b}|^2 \qquad (27)$$

This can be verified by longhand calculation!

The cross product again

$$\mathbf{a} \times \mathbf{b}$$

is a vector which is perpendicular to the plane containing **a** and **b**. Its magnitude is

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta$$

where θ is the angle between **a** and **b** (taken between 0 and π).

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The eaxct direction of $\mathbf{a} \times \mathbf{b}$ is obtained by the “right hand rule”.

Cross product and area

The magnitude of the cross product of **a** and **b** is

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta$$

which is the *area of the parallelogram formed by* **a** and **b**.

Summary of some properties of the Scalar Triple Product

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$$

Scalar triple product and volume

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det[\mathbf{a}, \mathbf{b}, \mathbf{c}]$$

is the *volume* of the parallelopiped formed by the three vectors **a**, **b**, **c**.

Scalar triple product and volume

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det[\mathbf{a}, \mathbf{b}, \mathbf{c}]$$

is the *volume* of the parallelopiped formed by the three vectors **a**, **b**, **c**.

In particular, this is 0 if the solid body collapses to something lower dimensional, for instance if **a** lies in the plane of **b** and **c**.

A vector triple product identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$