Chapter 1

Sigma-Algebras

1.1 Definition

Consider a set X.

A <u>σ–algebra</u> $\mathcal{F}$ of subsets of X is a collection $\mathcal{F}$ of subsets of X satisfying the following conditions:

(a) $\emptyset \in \mathcal{F}$

(b) if $B \in \mathcal{F}$ then its complement B^c is also in $\mathcal{F}$

(c) if $B_1, B_2, ...$ is a *countable* collection of sets in $\mathcal{F}$ then their union $\cup_{n=1}^{\infty} B_n$

Sometimes we will just write "sigma-algebra" instead of "sigma-algebra of subsets of X."

There are two extreme examples of sigma-algebras:

- the collection $\{\emptyset, X\}$ is a sigma-algebra of subsets of X
- the set $\mathcal{P}(X)$ of all subsets of X is a sigma-algebra

Any sigma-algebra $\mathcal{F}$ of subsets of X lies between these two extremes:

$$\{\emptyset, X\} \subset \mathcal{F} \subset \mathcal{P}(X)$$

An <u>atom</u> of $\mathcal{F}$ is a set $A \in \mathcal{F}$ such that the only subsets of A which are also in $\mathcal{F}$ are the empty set $\emptyset$ and A itself.

A partition of X is a collection of *disjoint* subsets of X whose union is all of X. For simplicity, consider a partition consisting of a finite number of sets $A_1, ..., A_N$. Thus

$$A_i \cap A_j = \emptyset \quad \text{and} \quad A_1 \cup \cdots \cup A_N = X$$

Then the collect $\mathcal{F}$ consisting of all unions of the sets A_j forms a σ–algebra.

Here are a few simple observations:

Proposition 1 *Let $\mathcal{F}$ be a sigma-algebra of subsets of X.*

(i) $X \in \mathcal{F}$

(ii) *If $A_1, ..., A_n \in \mathcal{F}$ then $A_1 \cup \cdots \cup A_n \in \mathcal{F}$*

(iii) *If $A_1, ..., A_n \in \mathcal{F}$ then $A_1 \cap \cdots \cap A_n \in \mathcal{F}$*

(iv) *If $A_1, A_2, ...$ is a countable collection of sets in $\mathcal{F}$ then $\cap_{n=1}^{\infty} A_n \in \mathcal{F}$*

(v) *If $A, B \in \mathcal{F}$ then $A - B \in \mathcal{F}$.*

Proof Since $\emptyset \in \mathcal{F}$ and

$$X = \emptyset^c$$

it follows that $X \in \mathcal{F}$.

For (ii) we have

$$A_1 \cup \cdots \cup A_n = A_1 \cup \cdots \cup A_n \cup \emptyset \cup \emptyset \cup \cdots \in \mathcal{F}$$

Then (iii) follows by complementation:

$$A_1 \cap \cdots \cap A_n = (A_1^c \cup \cdots \cup A_n^c)^c$$

which is in $\mathcal{F}$ because each $A_i^c \in \mathcal{F}$ and, by (i), $\mathcal{F}$ is closed under finite unions. Similarly, (iv) follows by taking complements :

$$\cap_{n=1}^{\infty} A_n = [\cup_{n=1}^{\infty} A_n^c]^c$$

which belongs to $\mathcal{F}$ because $\mathcal{F}$ is closed under complements and countable unions.

Finally,

$$A - B = A \cap B^c$$

is in $\mathcal{F}$, because $A, B^c \in \mathcal{F}$. QED

1.2 Generated Sigma-algebra $\sigma(\mathcal{B})$

Let X be a set and $\mathcal{B}$ a non-empty collection of subsets of X. The *smallest* σ–algebra containing all the sets of $\mathcal{B}$ is denoted

$$\sigma(\mathcal{B})$$

and is called the sigma-algebra generated by the collection $\mathcal{B}$.

The term "smallest" here means that any sigma-algebra containing the sets of $\mathcal{B}$ would have to contain all the sets of $\sigma(\mathcal{B})$ as well.

We need to check that such a smalled sigma-algebra exists. To this end observe first the following fact:

- *If G is any non-empty collection of sigma-algebras of subsets of X then the intersection $\cap G$ is also a sigma-algebra of subsets of X.* Here

  $$\cap G = \{A \subset X | \, A \in \mathcal{F} \quad \text{for every } \mathcal{F} \in G\}$$

  consists of all sets A which belong to each sigma-algebra $\mathcal{F}$ of G.

The verification of this statement is left as an (easy) exercise.

Given a collection $\mathcal{B}$ of subsets of X, let $G_{\mathcal{B}}$ be the collection of all sigma-algebras containing all the sets of $\mathcal{B}$. Note that

$$\mathcal{P}(X) \in G_{\mathcal{B}}$$

and so $G_{\mathcal{B}}$ is not empty. Then

$$\cap G_{\mathcal{B}}$$

is a sigma-algebra, contains all the sets of $\mathcal{B}$, and is minimal among such sigma-algebras. Minimality here means that if $\mathcal{F}$ is a sigma-algebra such that

$$\mathcal{B} \subset \mathcal{F}$$

then

$$\cap G_{\mathcal{B}} \subset \mathcal{F}$$

Thus $\cap G_{\mathcal{B}}$ is the sigma-algebra generated by $\mathcal{B}$:

$$\sigma(\mathcal{B}) = \cap G_{\mathcal{B}}$$

If $\mathcal{B}$ is itself a sigma-algebra then of course $\sigma(\mathcal{B}) = \mathcal{B}$.

1.3 The Dynkin $\pi - \lambda$ Theorem

Let X be a set.

A collection P of subsets of X is a π-system if

(π) P is closed under finite intersections: if $A, B \in P$ then $A \cap B \in P$

Note that by the usual induction argument, this condition implies that if $A_1, ..., A_n$ are a *finite* number of sets in P then their intersection $A_1 \cap \cdots \cap A_n$ is also in P.

A collection L of subsets of X is called a λ–system if

(λ1) L contains the empty set $\emptyset$

(λ2) L is closed under complements: if $A \in L$ then $A^c \in L$

(λ3) L is closed under countable *disjoint* union: if $A_1, A_2, ... \in L$ and $A_i \cap A_j = \emptyset$ for every $i \neq j$, then $\cup_{n=1}^{\infty} A_n \in L$

Unlike a σ–algebra, the notions of π–system and λ–system are not in themselves fundamental. Their significance is contained in the following theorem which will be of great use later in proving uniqueness of measures:

Theorem 1 *The Dynkin $\pi - \lambda$ theorem If P is a π–system and L a λ–system of subsets of X then*

$$\sigma(P) \subset L,$$

i.e. the sigma-algebra generated by P is contained in L.

The proof of this result is long but can be broken up into simple little pieces.

As a first step, we have

Lemma 1 *A λ–system is closed under proper differences, i.e. if $A, B \in L$, where L is a λ–system, and $A \subset B$ then the difference $B - A$ is also in L.*

Proof. It is best to draw a little diagram illustrating the fact that $A \subset B$. From this you can see that $B - A$ is the complement of the set $A \cup B^c$, and the latter, being the disjoint union of $A \in L$ and $B^c \in L$, is in L; thus $B - A \in L$. More formally,

$$B - A = B \cap A^c = (B^c \cup A)^c$$

is in L because it is the complement of the set $B^c \cup A$ which is in L because it is the union of two disjoint sets A and B^c both of which are in L. QED

The next step is more substantial:

Lemma 2 *A family which is both a π–system and a λ–system is a σ–algebra.*

Proof. Let S be a collection of subsets of X which is both a π system and a λ system. To prove that S is a σ–algebra it will be enough to show that S is closed under *countable unions* (not just disjoint countable unions).

Let $A_1, A_2, ... \in S$. We have to show that their union $\cup_{n=1}^{\infty} A_n$ is in S. The trick (and it is a very useful trick) is to rewrite $\cup_{n=1}^{\infty} A_n$ as a countable union of *disjoint* sets:

$$\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} B_n$$

where $B_1 = A_1$ and, for $n \geq 1$,

$$B_n = A_n - (A_1 \cup A_2 \cup \cdots \cup A_{n-1}) = A_n \cap A_1^c \cap A_2^c \cap \cdots \cap A_n^c \qquad (1.1)$$

Thus B_n consists of all elements of A_n which do not appear in any "previous" A_i.

It is clear that the sets B_1, $B_2, ...$ are mutually disjoint. Since S is a λ–system, each complement A_i^c is in S, and since S is a π–system it follows then that B_n, which is a finite intersection of sets in S, is also in S. QED

As further preparation for the proof of the main theorem let us make one more observation, though its significance will only become clear later:

Lemma 3 *Suppose L' is a λ–system of subsets of X. For any set $A \in L'$, let S_A be the set of all $B \subset X$ for which $A \cap B \in L'$. Then S_A is a λ–system.*

Proof. First note that $\emptyset \in S_A$, because $A \cap \emptyset = \emptyset \in L'$.

It is also clear that S_A is closed under countable *disjoint* unions.

The last thing we have to show is that S_A is closed under complements. To this end, let $B \in S_A$ and observe that

$$A \cap B^c = A - B = A - (A \cap B)$$

The utility in writing the difference $A - B$ as the *proper difference* $A - (A \cap B)$ lies in the fact that $A \cap B \subset A$ and we can appeal to Lemma 1, along with the facts that A and $A \cap B$ are both in L', to conclude that $A - (A \cap B)$ is in L'. QED

Now we return to the proof of the main theorem. As before, P is a π–system and L a λ–system, with $P \subset L$. Our objective is to show that the sigma-algebra $\sigma(P)$ generated by P is contained in L. The strategy will be to produce a sigma-algebra which lies between P and L, i.e. contains P and is contained in L. This will imply that $\sigma(P)$, which is the *smallest* sigma–algebra containing P, is contained in L.

We look at

$$l(P),$$

the intersection of all λ–systems containing P. Clearing $l(P)$ is itself also a λ–system and contains P, and is thus the minimal λ–system containing P. This means that any λ system which contains P must also contain $l(P)$.

The objective will be to show that the λ–system $l(P)$ is also a π–system. This would imply that $l(P)$ *is a sigma-algebra*. It contains P and, being the minimal λ–system containing P, is a subset of L. This would provide our sigma-algebra lying between P and L. So the last piece of the argument is:

Lemma 4 *$l(P)$ is a π–system.*

The proof of this uses a "bootstrap" argument which is often useful in measure theory. We start with a set $A \in P$ and show that $A \cap B$ is in $l(P)$ for every B in $l(P)$; then we turn around and use this to show that if A and B are in $l(P)$ then so is their intersection.

Proof. Let $A \in P$, and let S_A be the set of all sets $B \subset X$ for which $A \cap B$ is in $l(P)$. We have already proven that S_A is a λ–system. Moreover, it is clear that every element of P is in S_A. Thus S_A is a λ–system with $P \subset S_A$. Therefore, $l(P) \subset S_A$. Which means that we have proven that for any $A \in P$ and any $B \in l(P)$ the intersection $A \cap B$ is in $l(P)$.

So now consider a $B \in l(P)$, and look at S_B. The preceding paragraph proves that $P \subset S_B$. On the other hand, by Lemma 3, S_B is a λ–system. Therefore, $l(P) \subset S_B$. Which means: for any $A \in l(P)$, the intersection $A \cap B$ is in $l(P)$. Thus, $l(P)$ is a π–system. QED

Putting all of the strands together, we have:

Proof of Dynkin's theorem. We have proven that the λ–system $l(P)$ is also a π–system, and is therefore a σ–algebra. On the other hand, we also know that

$$P \subset l(P) \subset L$$

because $l(P)$ is the intersection of *all* λ–systems containing P, and L is just one λ–system containing P. Thus we have produced a sigma-algebra $l(P)$ lying between P and L. Therefore,

$$P \subset \sigma(P) \subset l(P) \subset L$$

since $\sigma(P)$ is the intersection of *all* sigma-algebras which contain P. QED

There are several other similar results which can substitute for the Dynkin $\pi - \lambda$ theorem. The best known alternative is the *monotone class* lemma, but we shall not go into this.