Chapter 1

Sigma-Algebras

1.1 Definition

Consider a set $X$.

A σ-algebra $\mathcal{F}$ of subsets of $X$ is a collection $\mathcal{F}$ of subsets of $X$ satisfying the following conditions:

(a) $\emptyset \in \mathcal{F}$

(b) if $B \in \mathcal{F}$ then its complement $B^c$ is also in $\mathcal{F}$

(c) if $B_1, B_2, ...$ is a countable collection of sets in $\mathcal{F}$ then their union $\cup_{n=1}^{\infty} B_n$

Sometimes we will just write “sigma-algebra” instead of “sigma-algebra of subsets of $X$.”

There are two extreme examples of sigma-algebras:

• the collection $\{\emptyset, X\}$ is a sigma-algebra of subsets of $X$

• the set $\mathcal{P}(X)$ of all subsets of $X$ is a sigma-algebra

Any sigma-algebra $\mathcal{F}$ of subsets of $X$ lies between these two extremes:

$$\{\emptyset, X\} \subset \mathcal{F} \subset \mathcal{P}(X)$$

An atom of $\mathcal{F}$ is a set $A \in \mathcal{F}$ such that the only subsets of $A$ which are also in $\mathcal{F}$ are the empty set $\emptyset$ and $A$ itself.
A partition of $X$ is a collection of disjoint subsets of $X$ whose union is all of $X$. For simplicity, consider a partition consisting of a finite number of sets $A_1, \ldots, A_N$. Thus

$$A_i \cap A_j = \emptyset \quad \text{and} \quad A_1 \cup \cdots \cup A_N = X$$

Then the collection $\mathcal{F}$ consisting of all unions of the sets $A_j$ forms a $\sigma$-algebra.

Here are a few simple observations:

**Proposition 1** Let $\mathcal{F}$ be a sigma-algebra of subsets of $X$.

(i) $X \in \mathcal{F}$

(ii) If $A_1, \ldots, A_n \in \mathcal{F}$ then $A_1 \cup \cdots \cup A_n \in \mathcal{F}$

(iii) If $A_1, \ldots, A_n \in \mathcal{F}$ then $A_1 \cap \cdots \cap A_n \in \mathcal{F}$

(iv) If $A_1, A_2, \ldots$ is a countable collection of sets in $\mathcal{F}$ then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$

(v) If $A, B \in \mathcal{F}$ then $A - B \in \mathcal{F}$.

**Proof** Since $\emptyset \in \mathcal{F}$ and

$$X = \emptyset^c$$

it follows that $X \in \mathcal{F}$.

For (ii) we have

$$A_1 \cup \cdots \cup A_n = A_1 \cup \cdots \cup A_n \cup \emptyset \cup \emptyset \cup \cdots \in \mathcal{F}$$

Then (iii) follows by complementation:

$$A_1 \cap \cdots \cap A_n = (A_1^c \cup \cdots \cup A_n^c)^c$$

which is in $\mathcal{F}$ because each $A_i^c \in \mathcal{F}$ and, by (i), $\mathcal{F}$ is closed under finite unions. Similarly, (iv) follows by taking complements:

$$\bigcap_{n=1}^{\infty} A_n = [\bigcup_{n=1}^{\infty} A_n^c]^c$$

which belongs to $\mathcal{F}$ because $\mathcal{F}$ is closed under complements and countable unions.

Finally,

$$A - B = A \cap B^c$$

is in $\mathcal{F}$, because $A, B^c \in \mathcal{F}$. \[ \text{QED} \]
1.2 Generated Sigma-algebra $\sigma(\mathcal{B})$

Let $X$ be a set and $\mathcal{B}$ a non-empty collection of subsets of $X$. The smallest $\sigma$–algebra containing all the sets of $\mathcal{B}$ is denoted

$$\sigma(\mathcal{B})$$

and is called the sigma-algebra generated by the collection $\mathcal{B}$.

The term “smallest” here means that any sigma-algebra containing the sets of $\mathcal{B}$ would have to contain all the sets of $\sigma(\mathcal{B})$ as well.

We need to check that such a smallest sigma-algebra exists. To this end observe first the following fact:

- If $G$ is any non-empty collection of sigma-algebras of subsets of $X$ then the intersection $\cap G$ is also a sigma-algebra of subsets of $X$. Here

$$\cap G = \{ A \subset X | A \in \mathcal{F} \text{ for every } \mathcal{F} \in G \}$$

consists of all sets $A$ which belong to each sigma-algebra $\mathcal{F}$ of $G$.

The verification of this statement is left as an (easy) exercise.

Given a collection $\mathcal{B}$ of subsets of $X$, let $G_\mathcal{B}$ be the collection of all sigma-algebras containing all the sets of $\mathcal{B}$. Note that

$$\mathcal{P}(X) \in G_\mathcal{B}$$

and so $G_\mathcal{B}$ is not empty. Then

$$\cap G_\mathcal{B}$$

is a sigma-algebra, contains all the sets of $\mathcal{B}$, and is minimal among such sigma-algebras. Minimality here means that if $\mathcal{F}$ is a sigma-algebra such that

$$\mathcal{B} \subset \mathcal{F}$$

then

$$\cap G_\mathcal{B} \subset \mathcal{F}$$

Thus $\cap G_\mathcal{B}$ is the sigma-algebra generated by $\mathcal{B}$:

$$\sigma(\mathcal{B}) = \cap G_\mathcal{B}$$

If $\mathcal{B}$ is itself a sigma-algebra then of course $\sigma(\mathcal{B}) = \mathcal{B}$. 
1.3 The Dynkin $\pi - \lambda$ Theorem

Let $X$ be a set.

A collection $P$ of subsets of $X$ is a $\pi$-system if

$$(\pi) \ P \text{ is closed under finite intersections: if } A, B \in P \text{ then } A \cap B \in P$$

Note that by the usual induction argument, this condition implies that if $A_1, ..., A_n$ are a finite number of sets in $P$ then their intersection $A_1 \cap \cdots \cap A_n$ is also in $P$.

A collection $L$ of subsets of $X$ is called a $\lambda$-system if

$$(\lambda_1) \ L \text{ contains the empty set } \emptyset$$

$$(\lambda_2) \ L \text{ is closed under complements: if } A \in L \text{ then } A^c \in L$$

$$(\lambda_3) \ L \text{ is closed under countable disjoint union: if } A_1, A_2, ... \in L \text{ and } A_i \cap A_j = \emptyset \text{ for every } i \neq j, \text{ then } \bigcup_{n=1}^{\infty} A_n \in L$$

Unlike a $\sigma$-algebra, the notions of $\pi$-system and $\lambda$-system are not in themselves fundamental. Their significance is contained in the following theorem which will be of great use later in proving uniqueness of measures:

**Theorem 1** The Dynkin $\pi - \lambda$ theorem If $P$ is a $\pi$-system and $L$ a $\lambda$-system of subsets of $X$ then

$$\sigma(P) \subset L,$$

i.e. the sigma-algebra generated by $P$ is contained in $L$.

The proof of this result is long but can be broken up into simple little pieces.

As a first step, we have

**Lemma 1** A $\lambda$-system is closed under proper differences, i.e. if $A, B \in L$, where $L$ is a $\lambda$-system, and $A \subset B$ then the difference $B - A$ is also in $L$.

**Proof.** It is best to draw a little diagram illustrating the fact that $A \subset B$. From this you can see that $B - A$ is the complement of the set $A \cup B^c$, and the latter, being the disjoint union of $A \in L$ and $B^c \in L$, is in $L$; thus $B - A \in L$. More formally,

$$B - A = B \cap A^c = (B^c \cup A)^c$$
is in $L$ because it is the complement of the set $B^c \cup A$ which is in $L$ because it is the union of two disjoint sets $A$ and $B^c$ both of which are in $L$. \[QED\]

The next step is more substantial:

**Lemma 2** A family which is both a $\pi$–system and a $\lambda$–system is a $\sigma$–algebra.

**Proof.** Let $S$ be a collection of subsets of $X$ which is both a $\pi$ system and a $\lambda$ system. To prove that $S$ is a $\sigma$–algebra it will be enough to show that $S$ is closed under countable unions (not just disjoint countable unions).

Let $A_1, A_2, ... \in S$. We have to show that their union $\bigcup_{n=1}^{\infty} A_n$ is in $S$. The trick (and it is a very useful trick) is to rewrite $\bigcup_{n=1}^{\infty} A_n$ as a countable union of disjoint sets:

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$$

where $B_1 = A_1$ and, for $n \geq 1$,

$$B_n = A_n - (A_1 \cup A_2 \cup \cdots \cup A_{n-1}) = A_n \cap A_1^c \cap A_2^c \cap \cdots \cap A_n^c \quad (1.1)$$

Thus $B_n$ consists of all elements of $A_n$ which do not appear in any “previous” $A_i$.

It is clear that the sets $B_1, B_2, ...$ are mutually disjoint. Since $S$ is a $\lambda$–system, each complement $A_i^c$ is in $S$, and since $S$ is a $\pi$–system it follows then that $B_n$, which is a finite intersection of sets in $S$, is also in $S$. \[QED\]

As further preparation for the proof of the main theorem let us make one more observation, though its significance will only become clear later:

**Lemma 3** Suppose $L'$ is a $\lambda$–system of subsets of $X$. For any set $A \in L'$, let $S_A$ be the set of all $B \subseteq X$ for which $A \cap B \in L'$. Then $S_A$ is a $\lambda$–system.

**Proof.** First note that $\emptyset \in S_A$, because $A \cap \emptyset = \emptyset \in L'$.

It is also clear that $S_A$ is closed under countable disjoint unions.

The last thing we have to show is that $S_A$ is closed under complements. To this end, let $B \in S_A$ and observe that

$$A \cap B^c = A - B = A - (A \cap B)$$

The utility in writing the difference $A - B$ as the proper difference $A - (A \cap B)$ lies in the fact that $A \cap B \subseteq A$ and we can appeal to Lemma 1, along with the facts that $A$ and $A \cap B$ are both in $L'$, to conclude that $A - (A \cap B)$ is in $L'$. \[QED\]
Now we return to the proof of the main theorem. As before, $P$ is a $\pi$-system and $L$ a $\lambda$-system, with $P \subset L$. Our objective is to show that the sigma-algebra $\sigma(P)$ generated by $P$ is contained in $L$. The strategy will be to produce a sigma-algebra which lies between $P$ and $L$, i.e. contains $P$ and is contained in $L$. This will imply that $\sigma(P)$, which is the smallest sigma-algebra containing $P$, is contained in $L$.

We look at $l(P)$, the intersection of all $\lambda$-systems containing $P$. Clearing $l(P)$ is itself also a $\lambda$-system and contains $P$, and is thus the minimal $\lambda$-system containing $P$. This means that any $\lambda$ system which contains $P$ must also contain $l(P)$.

The objective will be to show that the $\lambda$-system $l(P)$ is also a $\pi$-system. This would imply that $l(P)$ is a sigma-algebra. It contains $P$ and, being the minimal $\lambda$-system containing $P$, is a subset of $L$. This would provide our sigma-algebra lying between $P$ and $L$. So the last piece of the argument is:

**Lemma 4** $l(P)$ is a $\pi$-system.

The proof of this uses a “bootstrap” argument which is often useful in measure theory. We start with a set $A \in P$ and show that $A \cap B$ is in $l(P)$ for every $B \in l(P)$; then we turn around and use this to show that if $A$ and $B$ are in $l(P)$ then so is their intersection.

**Proof.** Let $A \in P$, and let $S_A$ be the set of all sets $B \subset X$ for which $A \cap B$ is in $l(P)$. We have already proven that $S_A$ is a $\lambda$-system. Moreover, it is clear that every element of $P$ is in $S_A$. Thus $S_A$ is a $\lambda$-system with $P \subset S_A$. Therefore, $l(P) \subset S_A$. Which means that we have proven that for any $A \in P$ and any $B \in l(P)$ the intersection $A \cap B$ is in $l(P)$.

So now consider a $B \in l(P)$, and look at $S_B$. The preceding paragraph proves that $P \subset S_B$. On the other hand, by Lemma 3, $S_B$ is a $\lambda$-system. Therefore, $l(P) \subset S_B$. Which means: for any $A \in l(P)$, the intersection $A \cap B$ is in $l(P)$. Thus, $l(P)$ is a $\pi$-system. \[\text{QED}\]

Putting all of the strands together, we have:

**Proof of Dynkin’s theorem.** We have proven that the $\lambda$-system $l(P)$ is also a $\pi$-system, and is therefore a $\sigma$-algebra. On the other hand, we also know that

$$P \subset l(P) \subset L$$
1.3. THE DYNKIN $\pi - \lambda$ THEOREM

because $l(P)$ is the intersection of all $\lambda$–systems containing $P$, and $L$ is just one $\lambda$–system containing $P$. Thus we have produced a sigma-algebra $l(P)$ lying between $P$ and $L$. Therefore,

\[ P \subset \sigma(P) \subset l(P) \subset L \]

since $\sigma(P)$ is the intersection of all sigma-algebras which contain $P$. $\blacksquare$

There are several other similar results which can substitute for the Dynkin $\pi - \lambda$ theorem. The best known alternative is the monotone class lemma, but we shall not go into this.