Chapter 5

Topological Vector Spaces

In this chapter V is a real or complex vector space.

5.1 Topological Vector Spaces

A complex vector space V equipped with a topology is a *broad-sense topological vector space* if the mappings

$$V \times V \to V : (x, y) \mapsto x + y \qquad \mathbf{C} \times V \to V : (\lambda, x) \mapsto \lambda x$$

are continuous. Observe that then, for each $x \in V$, the translation map

$$\tau_x : V \to V : y \mapsto y + x$$

is continuous. Since $\tau_x^{-1} = \tau_{-x}$, it follows that τ_x is a homeomorphism. The simple but important consequence of this is that V "looks the same everywhere", i.e. *if $a, b \in V$ then there is a homeomorphism*, specifically $\tau_{b-a} : V \to V$, *which maps a to b.* In particular, every neighborhood of $x \in V$ is a translate of a neighborhood of 0, i.e. of the form $x + U$ for some neighborhood U of 0. For this reason, we shall prove most of our results in a neighborhood of 0.

By a *topological vector space* we shall mean a broad-sense topological vector space which is *Hausdorff*, i.e. distinct points of disjoint neighborhoods.

Lemma 1 *Let V be a broad-sense topological vector space, and W an open set with $0 \in W$. Then there is an open set U with $0 \in U$, $U = -U$, and*

$$U + U \subset W$$

Proof. Since $V \times V \to V : (x, y) \mapsto x + y$ is continuous at 0, and W is a neighborhood of 0, there is a neighborhood U_1 of 0 such that

$$U_1 + U_1 \subset W$$

To get symmetry take

$$U = U_1 \cap (-U_1)$$

Note that continuity of multiplication by scalars implies that $x \mapsto -x$ is a homeomorphism and so $-U_1$ is open when U_1 is open. $\boxed{\text{QED}}$

Here is a simple but useful observation: if A is any subset of the broad-sense topological vector space V and U an open subset of V then

the set $A + U = \{a + x : a \in A, x \in U\}$ *is open.*

The reason is that $A+U$ is the union of the translates $a+U$, with a running over A, and each translate $a + U$ is an open set.

If A and B are subsets of a vector space V then we shall write $A - B$ to mean the set of all $a - b$, with a running over a and b running over B. The set difference of A and B, i.e. the set of all elements of A not in B, will be denoted $A \setminus B$.

The following observation shows that even without assuming Hausdorffness one can still separate closed sets and points by open neighborhoods.

Lemma 2 *Let V be a broad-sense topological vector space and W a neighborhood of* 0*. Then there is a neighborhood U of* 0 *such that $\overline{U} \subset W$. Equivalently, if C is a closed subset if V and x a point of V outside C then there are disjoint open sets U_1 and U_2 with $x \in U_1$ and $C \subset U_2$.*

Proof. Let x be a point outside a closed set $C \subset V$. We we will produce an open set U containing x with closure $\overline{U}$ disjoint from C; then $U_1 = U$ and $U_2 = (\overline{U})^c$, the complement of the closure of U, are disjoint open sets with $x \in U_1$ and $C \subset U_2$, as desired.

Since V looks the same everywhere, we may work with $x = 0$. Let W be the complement of C. This is an open set with $0 \in W$. By continuity of addition $V \times V \to V$ at 0, there is an open neighborhood U of 0 such that $U + U \subset W$. This means that $U + U$ is disjoint from C. Equivalently, U is disjoint from $C - U$ (for otherwise there would be an $x \in U$ which could be expressed as $c - y$ with $c \in C$ and $y \in U$, which would imply that $c = x + y \in U + U \subset W$ is in W). Now the set $-U$ is open because the map

$V \to V : x \mapsto (-1)x = -x$ is a homeomorphism, and hence so are all its translates $x - U$. So the set

$$C - U = \cup_{c \in C}(c - U)$$

is open, being the union of open sets.

Thus we have found an open set U containing 0 and an open set $C - U$, disjoint from U, with $C \subset C - U$. $\boxed{\text{QED}}$

An immediate consequence of this lemma shows how close a broad-sense topological vector space is to being Hausdorff:

Proposition 1 *Suppose that V is a broad-sense topological vector space and suppose that there is a point $x \in V$ such that the one-point set $\{x\}$ is a closed set. Then V is Hausdorff.*

Proof. Suppose $x \in V$ is such that $\{x\}$ is a closed set. Since V looks the same everywhere, it follows that *each one-point set is closed* in V. Let x and y be distinct points of V. Since $\{y\}$ is closed, the preceding result provides disjoint open sets U_1 and U_2 with $x \in U_1$ and $\{y\} \subset U_2$, i.e. x and y have disjoint open neighborhoods. $\boxed{\text{QED}}$

5.2 Continuous linear functionals

If X and Y are broad-sense topological vector spaces then linear mappings $X \to Y$ which are *continuous* are of great interest. Because these spaces look the same everywhere, continuity of linear mappings has interesting properties:

Proposition 2 *Let X and Y be broad-sense topological vector spaces. Let $L : X \to Y$ be a linear mapping. The following are equivalent:*

(i) *L is continuous at some point*

(ii) *L is continuous*

(iii) *L is uniformly continuous, i.e. if W is a neighborhood of 0 in Y then there is a neighborhood U of 0 in X such that for any $x_1, x_2 \in X$, if $x_1 - x_2 \in U$ then $L(x_1) - L(x_2) \in W$.*

Proof. Let $u \in X$, and write $v = Lu$. The translations $\tau_u : X \to X : x \mapsto x + u$ and $\tau_{-v} : Y \to Y : y \mapsto y - v$ are homeomorphisms, with $\tau_u(0) = u$ and $\tau_{-v}(v) = 0$. For any $x \in X$:

$$(\tau_{-v} \circ L \circ \tau_u)(x) = L(x + u) - v = Lx + Lu - v = Lx$$

which says that

$$\tau_{-v} \circ L \circ \tau_u = L$$

Thus L is continuous at 0 if and only if it is continuous at u. Thus, for any $x, y \in X$, the mapping L is continuous at x if and only if it is continuous at y.

Equivalence of continuity and uniform continuity is an immediate consequence of linearity of L. $\boxed{\text{QED}}$

A subset B of the topological vector space V is said to be *bounded* if for any neighborhood U of 0 in V the set B lies in some dilated version of U, i.e. there is a real number t such that

$$B \subset tU$$

There is a close relationship between continuity and boundedness for linear mappings:

Proposition 3 *Suppose $L : X \to Y$ is a linear mapping, where X and Y are broad-sense topological vector spaces. Then:*

(i) *If L maps some open neighborhood of 0 in X to a bounded subset of Y then L is continuous.*

(ii) *If L is continuous then it maps bounded sets to bounded sets.*

Proof. Suppose U is an open neighborhood of 0 in X and $L(U)$ is bounded. Let W be any neighborhood of 0 in Y. We wish to show that there is an open neighborhood of 0 in X which is mapped by L into a subset of W. Since $L(U)$ is bounded, there is a real number t such that $L(U) \subset tW$. If $t = 0$ then $L(U) = \{0\} \subset W$. If $t \neq 0$ then $t^{-1}U$ is an open neighborhood of 0 in X and $L(t^{-1}U) \subset W$. Thus, in either case, L maps a neighborhood of 0 in X into the given neighborhood W of 0 in Y, i.e. L is continuous at 0. Therefore, L is continuous. This proves (i).

Now assume that L is continuous at 0. Let B be a bounded subset of X. Our objective is to show that the image $L(B)$ is a bounded subset of Y. Let

W be any open neighborhood of 0 in Y. Since L is continuous at 0, and of course $L(0) = 0$ since L is linear, there is an open neighborhood U of 0 in X with $L(U) \subset W$. Since B is bounded there is a $t \in \mathbf{R}$ such that $B \subset tU$. Then

$$L(B) \subset L(tU) = tL(U) \subset tW$$

Thus $L(B)$ is bounded. This proves (ii). $\boxed{\text{QED}}$

From the definition of boundedness it is not apparent that the one-point set $\{x\}$ is bounded:

Proposition 4 *Let X be a broad-sense topological vector space. Then:*

(i) *If U is any neighborhood of 0 then $\cup_{t>0} tU = X$*

(ii) *every one-point set $\{x\}$ in V is bounded.*

Proof. For (i), we have to show that for any $x \in X$ there is a $t > 0$ such that $t^{-1}x$ belongs to the given neighborhood U of 0. Now the multiplication map

$$\mathbf{R} \times X \to X : (\lambda, v) \mapsto \lambda v$$

is continuous, and so there is a neighboorhood N of 0 in $\mathbf{R}$ and a neighborhood W of x such that $sy \in U$ for every $s \in N$ and $y \in W$. In particular, picking a positive s in N and writing t for s^{-1} we have $t^{-1}x \in U$, i.e. $x \in tU$.

(ii) follows directly from (i). $\boxed{\text{QED}}$