Math 7330: Functional Analysis Fall 2002

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In the following, V is a *finite-dimensional* complex vector space with a Hermitian inner-product $(\cdot, \cdot)$, and $A : V \to V$ a linear map.

1. Let $e_1, ..., e_n$ be an *orthonormal* basis of V.
   (i) Show that the matrix for A relative to the basis $e_1, ..., e_n$ has $A_{ij} = (Ae_j, e_i)$ as the entry at the i–th row and j–th column.

   (ii) Show that for the matrix of A^*,

$$(A^*)_{ij} = \overline{A}_{ji}$$

2. Suppose that A is a *normal* operator, i.e. it commutes with its adjoint:

$$AA^* = A^*A$$

Show that

$$|Ax| = |A^*x|$$

for all $x \in V$.

3. Show that for a complex number $\lambda \in \mathbf{C}$ the following are equivalent:
   - $A - \lambda I$ is not invertible
   - there is a *non-zero* vector $x \in V$ for which $Ax = \lambda x$
   - $\det(A - \lambda I) = 0$

If $k \in \mathbf{C}$ and non-zero $y \in V$ satisfy $Ay = ky$ then k is an *eigenvalue* of A and y is an *eigenvector* corresponding to the eigenvalue k. In general, we shall use the notation

$$M_k = \{v \in V : Av = kv\} = \ker(A - kI)$$

The set of all $\lambda \in \mathbf{C}$ for which $A - \lambda I$ is not invertible is called the *spectrum* of A.

4. Determine the spectrum of A if its matrix $[A_{ij}]$ is diagonal

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

5. Prove that the spectrum $\sigma(A)$ of A is non-empty and contains at most n elements, where $n = \dim V$.

6. Suppose A is normal. Show that

$$Ax = \lambda x \quad \Leftrightarrow \quad A^* x = \overline{\lambda} x$$

7. Suppose A is normal. Show that M_λ and M_μ are orthogonal if $\lambda \neq \mu$. (Hint: Let $x \in M_\lambda$ and $y \in M_\mu$, and consider $(x, Ay) = (A^* x, y)$.)

8. Suppose $X \subset V$ a subspace such that $A(X) \subset X$. Show that

$$A^*(X^\perp) \subset X^\perp$$

where $X^\perp$ is the orthogonal complement of X in V.

9. Suppose A is normal, and let X be the subspace spanned by all the subspaces M_λ:

$$X = \sum_{\lambda \in \sigma(A)} M_\lambda$$

Show that $X = V$. [Hint: Use several Problems 8,5 and 6.]

10. **Spectral Theorem** in finite dimensions: Suppose that the operator $A : V \to V$ is normal. Let $P_\lambda : V \to V$ be the *orthogonal projection* onto M_λ. This is the linear operator which satisfies $P_\lambda x = x$ if $x \in M_\lambda$ and $P_\lambda x = 0$ if $x \in M_\lambda^\perp$. Show that

$$A = \sum_{\lambda \in \mathbf{C}} \lambda P_\lambda$$

Let $e_1, ..., e_n$ be any orthonormal basis of V made up of bases of the subspaces M_λ (for $\lambda \in \mathbf{C}$). Show that the matrix of A relative to such a basis is diagonal. Conversely, show that if there is an orthonormal basis relative to which the matrix of a certain operator is diagonal then that operator is normal.