In the following, $H$ is a complex Hilbert space with a Hermitian inner-product $(\cdot, \cdot)$. All operators are operators on $H$.

1. Suppose $P$ and $Q$ are orthogonal projections.
   (i) Show that if $PQ = QP$ then $PQ$ is an orthogonal projection.
   (ii) Show that, conversely, if $PQ$ is an orthogonal projection then $PQ = QP$. 

2. Let $P$ and $Q$ be orthogonal projections.

(i) Show that if $PQ = P$ then $PQ = QP$ and $\text{Im}(P) \subset \text{Im}(Q)$. Show that the same conclusions hold if $QP = P$.

(ii) Show that if $\text{Im}(P) \subset \text{Im}(Q)$ then $QP = P$. 

3. Suppose $A, B, C$ are mutually orthogonal closed subspaces of $H$, and let $P_A, P_B, P_C$ be the orthogonal projections onto $A, B, C$, respectively. Let $X = A + B$ and $Y = C + B$, and let $P_X$ and $P_Y$ be the orthogonal projections onto $X$ and $Y$, respectively.

(i) Show that $P_X P_Y = P_Y P_X$.

(ii) Express $P_X$ and $P_Y$ in terms of $P_A$, $P_B$ and $P_C$.

(iii) Express $P_A$, $P_B$ and $P_C$ in terms of $P_X$ and $P_Y$. 
4. Suppose $P$ and $Q$ are orthogonal projections which commute, i.e. $PQ = QP$. The goal is to show that then the geometric situation of the preceding problem holds, i.e. there are mutually orthogonal closes subspaces $A, B, C$ such that $P$ is the orthogonal projection onto $A + B$ and $Q$ is the orthogonal projection onto $C + B$. Let

$$R = PQ, \quad S = P(I - Q), \quad T = Q(I - P)$$

Observe that

$$P = S + R \quad \text{and} \quad Q = T + R$$

(i) Show that $R, S, \text{and } T$ are orthogonal projections. [Note that if $A$ is an orthogonal projection then so is $I - A$, and $B$ commutes with $A$ then it also commutes with $I - A$.]

(ii) Show that $RS = SR = 0$, $RT = TR = 0$, and $ST = TS = 0$.

(iii) Show that $\text{Im}(R), \text{Im}(S), \text{and } \text{Im}(T)$ are mutually orthogonal. Thus $R, S, T$ are orthogonal projections onto mutually orthogonal closed subspaces.
5. Let $x_1, x_2, x_3, \ldots$ be a sequence of mutually orthogonal vectors in the Hilbert space $H$. Let $S_n = x_1 + \cdots + x_n$. Let $S'_n = |x_1|^2 + \cdots + |x_n|^2$.

(i) Show that for any integers $m \geq n$, 
\[ |S_m - S_n|^2 = S'_m - S'_n \]

(ii) Show that the series $\sum_{n=1}^{\infty} x_n$ to converge in $H$ if and only if the series $\sum_{n=1}^{\infty} |x_n|^2$ converges.
Spectral Measures

In the following, $\Omega$ is a non-empty set, $\mathcal{B}$ is a $\sigma$–algebra of subsets of $\Omega$. A spectral measure is a mapping $E$ from $\mathcal{B}$ to the set of all orthogonal projections on $H$ satisfying the following conditions:

(i) $E(\emptyset) = 0$

(ii) $E(\Omega) = I$

(iii) if $A_1, A_2, \ldots \in \mathcal{B}$ are mutually disjoint and their union is the set $A$ then

$$(E(A)x, y) = \sum_{n=1}^\infty (E(A_n)x, y)$$

for every $x, y \in H$

(iv) if $A, B \in \mathcal{B}$ then

$E(A)E(B) = E(B)E(A) = E(A \cap B)$

For $x, y \in H$ define $E_{x,y} : \mathcal{B} \to \mathbb{C}$ by

$$E_{x,y}(A) \overset{\text{def}}{=} \langle E(A)x, y \rangle$$

Conditions (i) and (iii) say that $E_{x,y}$ is a complex measure. If $x = y$ we have

$$E_{x,x}(A) = \langle E(A)x, x \rangle = |E(A)x|^2 \geq 0$$

where we used the fact if $P$ is any orthogonal projection then any $x \in H$ decomposes as $Px + x - Px$ with $Px$ being perpendicular to $x - Px$ and so

$$\langle Px, x \rangle = \langle Px, Px + x - Px \rangle = \langle Px, Px \rangle + 0 = |Px|^2$$

The non-negativity in (2) shows that

$E_{x,x}$ is an (ordinary) measure on $(\Omega, \mathcal{B})$

Recall that on the complex Hilbert space $H$ any bounded linear operator $A$ is determined uniquely by the “diagonal values” $\langle Ax, x \rangle$. It follows that if $E$ and $E'$ are spectral measures for which $E_{x,x} = E'_{x,x}$ for all $x \in H$ then $E = E'$. 
6. Let $E$ be a spectral measure on $(\Omega, \mathcal{B})$ with values being orthogonal projections in the complex Hilbert space $H$. By a “measurable subset of $\Omega$” we mean, of course, a subset of $\Omega$ which belongs to the $\sigma$–algebra $\mathcal{B}$.

(i) Show that if $A$ and $B$ are disjoint measurable subsets of $\Omega$ then $E(A)$ and $E(B)$ are projections onto orthogonal subspaces, i.e. $\text{Im}(E(A))$ and $\text{Im}(E(B))$ are orthogonal to each other.

(ii) Let $A_1, A_2, \ldots$ be a sequence of disjoint measurable subsets of $\Omega$ (i.e. each $A_j$ is in $\mathcal{B}$). Let $A = \bigcup_{j=1}^{\infty} A_j$. Show that for every $x \in H$ the series

$$\sum_{n=1}^{\infty} E(A_n)x$$

is convergent in $H$. 
(iii) With notation and hypotheses as before, show that

\[ E(A)x = \sum_{n=1}^{\infty} E(A_n)x \]

for every \( x \in H \). [Hint: Take inner-product with any \( y \in H \)]

(iv) Suppose \( A_1, A_2, \ldots \) are as above but assume now also that infinitely many of the projections \( E(A_n) \) are non-zero. Prove that the series \( \sum_{n=1}^{\infty} E(A_n) \) does not converge in operator norm. [Hint: Let \( s_n = E(A_1) + \cdots + E(A_n) \), and suppose \( s = \lim_{n \to \infty} s_n \) exists. Then \( \lim_{n \to \infty}(s_n - s_{n-1}) = s - s = 0 \). What is \( s_n - s_{n-1} \) and what is the norm of a non-zero projection?]
Measure Theory and Integration

We recall a few facts from measure theory and integration. In the following, \( \Omega \) is a non-empty set, \( \mathcal{B} \) is a \( \sigma \)-algebra of subsets of \( \Omega \), and \( \mu \) a measure on \( \mathcal{B} \).

(a) A function \( f: \Omega \to \mathbb{C} \) is said to be measurable if \( f^{-1}(U) \) is in \( \mathcal{B} \) for every open set \( U \subset \mathbb{C} \). Write \( f = u + iv \), where \( u \) and \( v \) are real-valued. Then \( f \) is measurable if and only if \( u \) and \( v \) are measurable. Write \( u \) as \( u^+ - u^- \), where \( u^+ = \max\{u, 0\} \) and \( u^- = -\min\{u, 0\} \). Then \( u \) is measurable if and only if \( u^+ \) and \( u^- \) are measurable.

(b) A function \( s: \Omega \to \mathbb{C} \) is a simple function if it has only finitely many values, i.e. \( s(\Omega) \) is a finite subset of \( \Omega \). If \( c_1, \ldots, c_n \) are all the distinct values of \( s \) and \( A_i = s^{-1}(c_i) \) the set on which \( s \) has value \( c_i \), then

\[
s = \sum_{j=1}^{n} c_j 1_{A_j}
\]

Here \( 1_B \) denotes the indicator function of \( B \), equal to 1 on \( B \) and 0 outside \( B \). The simple function \( s \) is measurable if and only if each of the sets \( A_i \) is measurable.

(c) Let \( F: \Omega \to [0, \infty] \) be a non-negative function. For each positive integer \( n \), divide \( [0, \infty] \) into intervals of length \( 1/2^n \), i.e. into the intervals \( [(k-1)2^{-n}, k2^{-n}) \). Define a function \( s_n \) which is equal to the lower value \( (k-1)2^{-n} \) on the set \( A_{nk} = F^{-1}[(k-1)2^{-n}, k2^{-n}) \), for \( k = 1, \ldots, n2^n \), but cut off the value of \( s_n \) at the maximum value \( n \) at all points in the set \( A'_n \) where \( F > n \). The construction ensures that \( 0 \leq s_n \leq F \), \( s_n \leq n \), and that \( |F - s_n| \leq 2^{-n} \) at all points where \( F \leq n \). Thus if the function \( F \) is bounded then \( |F - s_n| < 2^{-n} \) holds for all \( n \) large enough and so, in particular, \( s_n(x) \to F(x) \) uniformly in \( x \in \Omega \). If \( F \) is measurable so is each of the sets \( A_{nk} \) and \( A'_n \) and so the function \( s_n \) is then also measurable. Now consider a function \( f: \Omega \to \mathbb{C} \). Writing \( f = u + iv \), with \( u \) and \( v \) real-valued, and then splitting \( u = u^+ - u^- \) and \( v = v^+ - v^- \), it follows that we can construct a sequence of simple functions \( s_n \) such that \( |s_n(x)| \leq |f(x)| \) for all \( x \in \Omega \), \( s_n(x) \to f(x) \) uniformly if \( f \) is bounded, and each \( s_n \) is measurable if \( f \) is measurable.

(d) If \( s \) is a measurable simple function and \( c_1, \ldots, c_n \) are all the distinct values of \( s \) then

\[
\int s \, d\mu \overset{\text{def}}{=} \sum_{j=1}^{n} c_j \mu([s = c_j])
\]

where \( [s = c_j] \) is the set \( s^{-1}(c_j) \) of all points where \( s \) has value \( c_j \).

(e) If \( s \) and \( t \) are measurable simple functions then considering the number of ways \( s + t \) can take a particular value, it follows that \( \int (s + t) \, d\mu = \int s \, d\mu + \int t \, d\mu \). Also, \( \int \alpha s \, d\mu = \alpha \int s \, d\mu \) for every \( \alpha \in \mathbb{C} \). The additivity property has the following consequence: if \( s = a_1 1_{A_1} + \cdots + a_m 1_{A_m} \), where \( A_1, \ldots, A_m \) are measurable but may overlap then \( \int s \, d\mu = \sum_{j=1}^{m} a_j \mu(A_j) \) still holds.
7. Let $E$ be a spectral measure on $(\Omega, \mathcal{B})$ with values being orthogonal projections in the complex Hilbert space $H$. Let $\mathcal{N}$ be the set of all sets $A \in \mathcal{B}$ for which $E(A) = 0$. Thus $\mathcal{N}$ consists of sets of $E$–measure 0.

(i) Show that if $A$ and $B$ are measurable sets and $A \subset B$ and $E(B) = 0$ then $E(A) = 0$.

(ii) Show that $\mathcal{N}$ is closed under countable unions.

(ii) Let $f : \Omega \to \mathbb{C}$ be a measurable function. Show that there is a largest open subset $U$ of $\mathbb{C}$ such that $f^{-1}(U)$ is in $\mathcal{N}$. 
(iii) The essential range $\sigma_f$ of $f$ is the closed set given by the complement of the open set $U$ of (ii). The essential supremum of $f$, denoted $|f|_\infty$, is the radius of the smallest closed ball (center 0) containing $\sigma_f$. Thus

$$|f|_\infty = \inf\{r \geq 0 : E[|f| > r] = 0\}$$

Suppose $f$ and $g$ are measurable functions which are essentially bounded, i.e. $|f|_\infty$ and $|g|_\infty$ are finite. Then show

$$|f + g|_\infty \leq |f|_\infty + |g|_\infty$$

and for every complex number $\alpha$:

$$|\alpha f|_\infty = |\alpha||f|_\infty$$
8. Let $E$ be a spectral measure on $(\Omega, \mathcal{B})$ with values being orthogonal projections in the complex Hilbert space $H$.

(i) Let $A_1, ..., A_n, B_1, ..., B_m \in \mathcal{B}$ and $a_1, ..., a_n, b_1, ..., b_m \in \mathbb{C}$, and suppose

$$\sum_{j=1}^{n} a_j 1_{A_j} = \sum_{j=m}^{n} b_j 1_{B_j}$$

Show that

$$\sum_{j=1}^{n} a_j E(A_j) = \sum_{j=m}^{n} b_j E(B_j)$$

[Hint: Let $s = \sum_{j=1}^{n} a_j 1_{A_j} = \sum_{j=m}^{n} b_j 1_{B_j}$, and consider the operators $T = \sum_{j=1}^{n} a_j E(A_j)$ and $R = \sum_{j=m}^{n} b_j E(B_j)$. Take any $x \in H$ and show that both $(Tx, x)$ and $(Rx, x)$ equal $\int s \, dE_{x,x}$.] The common value in (4) will be denote

$$\int s \, dE$$

(ii) Check that for any measurable simple function $s$ on $\Omega$:

$$\left( \left( \int s \, dE \right)x, x \right) = \int s \, dE_{x,x}$$

holds for every $x \in H$. 

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(iii) Let $s, t$ be measurable simple functions on $\Omega$ and $\alpha, \beta \in \mathbb{C}$. Show that

$$\int (\alpha s + \beta t) \, dE = \alpha \int s \, dE + \beta \int t \, dE$$

(iv) Let $s, t$ be measurable simple functions on $\Omega$. Show that

$$\left( \int s \, dE \right) \left( \int t \, dE \right) = \int st \, dE$$

[Hint: Write out $s$ and $t$ in the usual forms $\sum_{j} a_{j}1_{A_{j}}$ and $\sum_{k} b_{j}1_{B_{k}}$ and then work out $st$ and write out both sides of the above equation.]

(v) Let $s$ be a measurable simple function on $\Omega$. Show that

$$\left( \int s \, dE \right)^{*} = \int \overline{s} \, dE$$
(vi) Let \( s \) be a measurable simple function on \( \Omega \). Show that

\[
\left| \int s \, dE \right| \leq |s|_\infty
\]

[Hint: Let \( T \) be the operator \( \int s \, dE \). Then \( |T| = \sup_{|x| \leq 1} |Tx| \). Now \( |Tx|^2 = (Tx, Tx) = (T^*Tx, x) \). Show that \( (T^*Tx, x) \) equals \( \int |s|^2 \, dE_{x,x} \). Next use \( |s| \leq |s|_\infty \) almost-everywhere for the measure \( E_x \).

(vii) Let \( f : \Omega \to \mathbb{C} \) be a bounded measurable function. We know that there exists a sequence of measurable simple functions \( s_n \) on \( \Omega \) such that \( s_n(x) \to f(x) \), as \( n \to \infty \), uniformly for \( x \in \Omega \) and \( |s_n(x)| \leq |f(x)| \) for all \( x \in \Omega \). Part (vi) above shows then that the sequence of operators \( \int s_n \, dE \) is Cauchy in operator norm and therefore converges in operator norm to a limit which we denote by \( \int f \, dE \):

\[
\int f \, dE \overset{\text{def}}{=} \lim_{n \to \infty} \int s_n \, dE
\]

where the limit is in operator norm. Now suppose \( s'_n \) is another sequence of measurable functions on \( \Omega \) which converge to \( f \) in the sense that \( |s'_n - f|_\infty \to 0 \) as \( n \to \infty \). Show that \( \int s'_n \, dE \) also converges to \( \int f \, dE \) as \( n \to \infty \). [Hint: Use (vi) for \( s_n - s'_n \).] Thus the definition of \( \int f \, dE \) does not depend on the choice of the sequence \( s_n \) converging to \( f \).
(viii) Show that
\[ \left( \int f \, dE \right)_x, x = \int f \, dE_{x,x} \]
for every bounded measurable function \( f \) and every \( x \in H \).

(ix) Prove the analogs of (iii)-(vi) for bounded measurable functions.
9. Let \((\Omega, \mathcal{B}, \mu)\) be a measure space. For any measurable functions \(f\) and \(g\) on \(\Omega\) let \(M_{fg}\) denote the function \(fg\). If \(f\) is bounded and \(g \in L^2(\mu)\) then clearly \(M_{fg}\) is also in \(L^2(\mu)\) and indeed \(M_f : L^2(\mu) \to L^2(\mu)\) is a bounded linear operator with norm \(|M_f| \leq |f|_{\infty}\) (in all practical cases \(|M_f|\) is actually equal to \(|f|_{\infty}\)). It is clear that \(f \mapsto M_f\) is linear and, moreover, \(M_{fh} = M_f M_h\).

(i) Show that \(M_f^* = M_{T_f}\). (Hint: Let \(g, h \in L^2(\mu)\) and work out \((M_{fg}, h)_{L^2}\).)

(ii) Show that for any measurable set \(A\), the operator \(M_{1_A}\) is an orthogonal projection operator.

(iii) Show that \(E : A \mapsto M_{1_A}\) is a spectral measure. [Hint: The only non-trivial thing to check is that for any \(g \in L^2(\mu)\) and disjoint measurable sets \(A_n\) whose union is \(A\) we have \(\sum_n E(A_n)g = E(A)g\) with the sum \(\sum_n\) being \(L^2\)-convergent. To this end, let \(G_n = \sum_{j=1}^n E(A_j)g\) and look at what happens to \(\int |G_n - 1_A g|^2 \, d\mu\) as \(n \to \infty\).]
(iv) For any measurable simple function $s$ show that $\int s \, dE = M_s$, where $E$ is as in (iii).

(v) For any bounded measurable function $f$ show that $\int f \, dE = M_f$, where $E$ is as in (iii). [Hint: Choose measurable simple $s_n$ converging uniformly to $f$, and with $|s_n(x)| \leq |f(x)|$ for all $x \in \Omega$. Consider the norms of $\int f \, dE - \int s_n \, dE$ and $M_f - M_{s_n}$.]
10. Let $E$ be a spectral measure for a measurable space $(\Omega, \mathcal{B})$ with values being orthogonal projection operators in a complex Hilbert space $H$. Let $f : \Omega \to \mathbb{C}$ be a measurable function not necessarily bounded. Let

$$D_f = \{ x \in H : \int |f|^2 \, dE_{x,x} < \infty \}$$

(i) For any $x, y \in H$ and any measurable set $A$, show that

$$E_{x+y,x+y}(A) \leq 2E_{x,x}(A) + 2E_{y,y}(A)$$

[Hint: First recall that $E_{v,v}(B) = |E(B)v|^2$. Next, for any vectors $a, b \in H$ we have the Cauchy-Schwarz inequality $|\langle a, b \rangle| \leq |a||b|$ which leads to the inequality $|a + b|^2 \leq |a|^2 + |b|^2 + 2|a||b|$. This, together with $(|a| - |b|)^2 \geq 0$ implies that $|a + b|^2 \leq 2|a|^2 + 2|b|^2$.]

(ii) Show that $D_f$ is a linear subspace of $H$, i.e. if $x, y \in D_f$ then $x + y \in D_f$ and $ax \in D_f$ for every $a \in \mathbb{C}$.

(iii) Let $A_n = \{ p \in \Omega : |f(p)| \leq n \}$. Consider any vector $x$ in the range of the projection $E(A_n)$. Show that

$$E_{x,x}(A) = E_{x,x}(A \cap A_n)$$

for every $A \in \mathcal{B}$. [Hint: What is $E(A_n)x$?]
(iv) With notation as above, show that

\[ \int s \, dE_{x,x} = \int_{A_n} s \, dE_{x,x} \]

for every measurable simple function \( s \) on \( \Omega \).

(v) With notation as above, show that

\[ \int |f|^2 \, dE_{x,x} = \int_{A_n} |f|^2 \, dE_{x,x} \]

Note that the right side is \( \leq n^2 E_{x,x}(\Omega) = n^2 |x|^2 < \infty \), and so \( x \in D_f \).