Math 7330: Functional Analysis Fall 2002

Homework 2 A. Sengupta

In the following, H is a complex Hilbert space with a Hermitian inner-product $(\cdot,\cdot)$. All operators are operators on H.

1. Suppose P and Q are orthogonal projections.

   (i) Show that if $PQ = QP$ then PQ is an orthogonal projection.

   (ii) Show that, conversely, if PQ is an orthogonal projection then $PQ = QP$.

2. Let P and Q be orthogonal projections.

(i) Show that if $PQ = P$ then $PQ = QP$ and $\mathrm{Im}(P) \subset \mathrm{Im}(Q)$. Show that the same conclusions hold if $QP = P$.

(ii) Show that if $\mathrm{Im}(P) \subset \mathrm{Im}(Q)$ then $QP = P$.

3. Suppose A, B, C are mutually orthogonal closed subspaces of H, and let P_A, P_B, P_C be the orthogonal projections onto A, B, C, respectively. Let $X = A+B$ and $Y = C+B$, and let P_X and P_Y be the orthogonal projections onto X and Y, respectively.
(i) Show that $P_X P_Y = P_Y P_X$.

(ii) Express P_X and P_Y in terms of P_A, P_B and P_C.

(iii) Express P_A, P_B and P_C in terms of P_X and P_Y.

4. Suppose P and Q are orthogonal projections which commute, i.e. $PQ = QP$. The goal is to show that then the geometric situation of the preceding problem holds, i.e. there are mutually orthogonal closes subspaces A, B, C such that P is the orthogonal projection onto $A + B$ and Q is the orthogonal projection onto $C + B$. Let

$$R = PQ, \qquad S = P(I - Q), \qquad T = Q(I - P)$$

Observe that

$$P = S + R \qquad \text{and} \qquad Q = T + R$$

(i) Show that R, S, and T are orthogonal projections. [Note that if A is an orthogonal projection then so is $I - A$, and B commutes with A then it also commutes with $I - A$.]

(ii) Show that $RS = SR = 0$, $RT = TR = 0$, and $ST = TS = 0$.

(iii) Show that $\mathrm{Im}(R)$, $\mathrm{Im}(S)$, and $\mathrm{Im}(T)$ are mutually orthogonal. Thus R, S, T are orthogonal projections onto mutually orthogonal closed subspaces.

5. Let $x_1, x_2, x_3, ...$ be a sequence of mutually orthogonal vectors in the Hilbert space H. Let $S_n = x_1 + \cdots + x_n$. Let $S'_n = |x_1|^2 + \cdots + |x_n|^2$.

(i) Show that for any integers $m \geq n$,

$$|S_m - S_n|^2 = S'_m - S'_n$$

(ii) Show that the series $\sum_{n=1}^{\infty} x_n$ to converge in H if and only if the series $\sum_n |x_n|^2$ converges.

Spectral Measures

In the following, Ω is a non-empty set, $\mathcal{B}$ is a σ–algebra of subsets of Ω. A *spectral measure* is a mapping E from $\mathcal{B}$ to the set of all orthogonal projections on H satisfying the following conditions:

(i) $E(\emptyset) = 0$

(ii) $E(\Omega) = I$

(iii) if $A_1, A_2, ... \in \mathcal{B}$ are mutually disjoint and their union is the set A then

$$(E(A)x, y) = \sum_{n=1} \big(E(A_n)x, y\big) \tag{1}$$

for every $x, y \in H$

(iv) if $A, B \in \mathcal{B}$ then

$$E(A)E(B) = E(B)E(A) = E(A \cap B)$$

For $x, y \in H$ define $E_{x,y} : \mathcal{B} \to \mathbf{C}$ by

$$E_{x,y}(A) \stackrel{\text{def}}{=} (E(A)x, y)$$

Conditions (i) and (iii) say that $E_{x,y}$ is a complex measure. If $x = y$ we have

$$E_{x,x}(A) = (E(A)x, x) = |E(A)x|^2 \geq 0 \tag{2}$$

where we used the fact if P is any orthogonal projection then any $x \in H$ decomposes as $Px + x - Px$ with Px being perpendicular to $x - Px$ and so

$$(Px, x) = (Px, Px + x - Px) = (Px, Px) + 0 = |Px|^2 \tag{3}$$

The non-negativity in (2) shows that

$$E_{x,x} \textit{ is an (ordinary) measure on } (\Omega, \mathcal{B})$$

Recall that on the complex Hilbert space H any bounded linear operator A is determined uniquely by the "diagonal values" (Ax, x). It follows that if E and E' are spectral measures for which $E_{x,x} = E'_{x,x}$ for all $x \in H$ then $E = E'$.

6. Let E be a spectral measure on $(\Omega, \mathcal{B})$ with values being orthogonal projections in the complex Hilbert space H. By a "measurable subset of Ω" we mean, of course, a subset of Ω which belongs to the σ–algebra $\mathcal{B}$.

   (i) Show that if A and B are disjoint measurable subsets of Ω then $E(A)$ and $E(B)$ are projections onto orthogonal subspaces, i.e. $\mathrm{Im}\big(E(A)\big)$ and $\mathrm{Im}\big(E(B)\big)$ are orthogonal to each other.

   (ii) Let $A_1, A_2, ...$ be a sequence of disjoint measurable subsets of Ω (i.e. each A_j is in $\mathcal{B}$). Let $A = \cup_{j=1}^{\infty} A_j$. Show that for every $x \in H$ the series

$$\sum_{n=1}^{\infty} E(A_n)x$$

   is convergent in H.

(iii) With notation and hypotheses as before, show that

$$E(A)x = \sum_{n=1}^{\infty} E(A_n)x$$

for every $x \in H$. [Hint: Take inner-product with any $y \in H$]

(iv) Suppose $A_1, A_2, \ldots$ are as above but assume now also that infinitely many of the projections $E(A_n)$ are non-zero. Prove that the series $\sum_{n=1}^{\infty} E(A_n)$ does not converge in operator norm. [Hint: Let $s_n = E(A_1) + \cdots + E(A_n)$, and suppose $s = \lim_{n\to\infty} s_n$ exists. Then $\lim_{n\to\infty}(s_n - s_{n-1}) = s - s = 0$. What is $s_n - s_{n-1}$ and what is the norm of a non-zero projection?]

Measure Theory and Integration

We recall a few facts from measure theory and integration. In the following, Ω is a non-empty set, $\mathcal{B}$ is a σ–algebra of subsets of Ω, and μ a measure on $\mathcal{B}$.

(a) A function $f : \Omega \to \mathbf{C}$ is said to be *measurable* if $f^{-1}(U)$ is in $\mathcal{B}$ for every open set $U \subset \mathbf{C}$. Write $f = u + iv$, where u and v are real-valued. Then f is measurable if and only if u and v are measurable. Write u as $u^+ - u^-$, where $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$. Then u is measurable if and only if u^+ and u^- are measurable.

(b) A function $s : \Omega \to \mathbf{C}$ is a *simple function* if it has only finitely many values, i.e. $s(\Omega)$ is a finite subset of Ω. If $c_1, ..., c_n$ are all the distinct values of s and $A_i = s^{-1}(c_i)$ the set on which s has value c_i, then

$$s = \sum_{j=1}^{n} c_j 1_{A_j}$$

Here 1_B denotes the *indicator function* of B, equal to 1 on B and 0 outside B. The simple function s is measurable if and only if each of the sets A_i is measurable.

(c) Let $F : \Omega \to [0, \infty]$ be a non-negative function. For each positive integer n, divide $[0, \infty]$ into intervals of length $1/2^n$, i.e. into the intervals $[(k-1)2^{-n}, k2^{-n})$. Define a function s_n which is equal to the lower value $(k-1)2^{-n}$ on the set $A_{nk} = F^{-1}[(k-1)2^{-n}, k2^{-n})$, for $k = 1, ..., n2^n$, but cut off the value of s_n at the maximum value n at all points in the set A'_n where $F > n$. The construction ensures that $0 \le s_n \le F$, $s_n \le n$, and that $|F - s_n| \le 2^{-n}$ at all points where $F \le n$. Thus if the function F is *bounded* then $|F - s_n| < 2^{-n}$ holds for all n large enough and so, in particular, $s_n(x) \to F(x)$ *uniformly* in $x \in \Omega$. If F is measurable so is each of the sets A_{nk} and A'_n and so the function s_n is then also measurable. Now consider a function $f : \Omega \to \mathbf{C}$. Writing $f = u + iv$, with u and v real–valued, and then splitting $u = u^+ - u^-$ and $v = v^+ - v^-$, it follows that we can construct a sequence of simple functions s_n such that $|s_n(x)| \le |f(x)|$ for all $x \in \Omega$, $s_n(x) \to f(x)$ *uniformly* if f is *bounded*, and each s_n is measurable if f is measurable.

(d) If s is a measurable simple function and $c_1, ..., c_n$ are all the distinct values of s then

$$\int s \, d\mu \stackrel{\text{def}}{=} \sum_{j=1}^{n} c_j \mu([s = c_j])$$

where $[s = c_j]$ is the set $s^{-1}(c_j)$ of all points where s has value c_j.

(e) If s and t are measurable simple functions then considering the number of ways $s+t$ can take a particular value, it follows that $\int (s+t)\, d\mu = \int s\, d\mu + \int t\, d\mu$. Also, $\int \alpha s\, d\mu = \alpha \int s\, d\mu$ for every $\alpha \in \mathbf{C}$. The additivity property has the following consequence: if $s = a_1 1_{A_1} + \cdots + a_m 1_{A_m}$, where $A_1, ..., A_m$ are measurable but *may overlap* then $\int s\, d\mu = \sum_{j=1}^{m} a_j \mu(A_j)$ still holds.

7. Let E be a spectral measure on $(\Omega, \mathcal{B})$ with values being orthogonal projections in the complex Hilbert space H. Let $\mathcal{N}$ be the set of all sets $A \in \mathcal{B}$ for which $E(A) = 0$. Thus $\mathcal{N}$ consists of sets of E–measure 0.

   (i) Show that if A and B are measurable sets and $A \subset B$ and $E(B) = 0$ then $E(A) = 0$.

   (ii) Show that $\mathcal{N}$ is closed under countable unions.

   (ii) Let $f : \Omega \to \mathbf{C}$ be a measurable function. Show that there is a largest open subset U of $\mathbf{C}$ such that $f^{-1}(U)$ is in $\mathcal{N}$.

(iii) The *essential range* σ_f of f is the closed set given by the complement of the open set U of (ii). The *essential supremum* of f, denoted $|f|_\infty$, is the radius of the smallest closed ball (center 0) containing σ_f. Thus

$$|f|_\infty = \inf\{r \geq 0 : E[|f| > r] = 0\}$$

Suppose f and g are measurable functions which are *essentially bounded*, i.e. $|f|_\infty$ and $|g|_\infty$ are finite. Then show

$$|f+g|_\infty \leq |f|_\infty + |g|_\infty$$

and for every complex number α:

$$|\alpha f|_\infty = |\alpha||f|_\infty$$

8. Let E be a spectral measure on $(\Omega, \mathcal{B})$ with values being orthogonal projections in the complex Hilbert space H.

(i) Let $A_1, ..., A_n, B_1, ..., B_m \in \mathcal{B}$ and $a_1, ..., a_n, b_1, ..., b_m \in \mathbf{C}$, and suppose

$$\sum_{j=1}^{n} a_j 1_{A_j} = \sum_{j=m}^{n} b_j 1_{B_j}$$

Show that

$$\sum_{j=1}^{n} a_j E(A_j) = \sum_{j=m}^{n} b_j E(B_j) \tag{4}$$

[Hint: Let $s = \sum_{j=1}^{n} a_j 1_{A_j} = \sum_{j=m}^{n} b_j 1_{B_j}$, and consider the operators $T = \sum_{j=1}^{n} a_j E(A_j)$ and $R = \sum_{j=m}^{n} b_j E(B_j)$. Take any $x \in H$ and show that both (Tx, x) and (Rx, x) equal $\int s\, dE_{x,x}$.] The common value in (4) will be denote

$$\int s\, dE$$

(ii) Check that for any measurable simple function s on Ω:

$$\left(\left(\int s\, dE \right) x, x \right) = \int s\, dE_{x,x}$$

holds for every $x \in H$.

(iii) Let s, t be measurable simple functions on Ω and $\alpha, \beta \in \mathbf{C}$. Show that

$$\int (\alpha s + \beta t)\, dE = \alpha \int s\, dE + \beta \int t\, dE$$

(iv) Let s, t be measurable simple functions on Ω. Show that

$$\left(\int s\, dE\right)\left(\int t\, dE\right) = \int st\, dE$$

[Hint: Write out s and t in the usual forms $\sum_j a_j 1_{A_j}$ and $\sum_k b_j 1_{B_k}$ and then work out st and write out both sides of the above equation.]

(v) Let s be a measurable simple function on Ω. Show that

$$\left(\int s\, dE\right)^* = \int \overline{s}\, dE$$

(vi) Let s be a measurable simple function on Ω. Show that

$$\left|\int s\,dE\right| \leq |s|_\infty$$

[Hint: Let T be the operator $\int s\,dE$. Then $|T| = \sup_{|x|\leq 1} |Tx|$. Now $|Tx|^2 = (Tx, Tx) = (T^*Tx, x)$. Show that (T^*Tx, x) equals $\int |s|^2\,dE_{x,x}$. Next use $|s| \leq |s|_\infty$ almost-everywhere for the measure E_x.]

(vii) Let $f : \Omega \to \mathbf{C}$ be a bounded measurable function. We know that there exists a sequence of measurable simple functions s_n on Ω such that $s_n(x) \to f(x)$, as $n \to \infty$, *uniformly* for $x \in \Omega$ and $|s_n(x)| \leq |f(x)|$ for all $x \in \Omega$. Part (vi) above shows then that the sequence of operators $\int s_n\,dE$ is Cauchy in operator norm and therefore *converges in operator norm* to a limit which we denote by $\int f\,dE$:

$$\int f\,dE \stackrel{\text{def}}{=} \lim_{n\to\infty} \int s_n\,dE$$

where the limit is in operator norm. Now suppose s'_n is another sequence of measurable functions on Ω which converge to f in the sense that $|s'_n - f|_\infty \to 0$ as $n \to \infty$. Show that $\int s'_n\,dE$ also converges to $\int f\,dE$ as $n \to \infty$. [Hint: Use (vi) for $s_n - s'_n$.] Thus the definition of $\int f\,dE$ does not depend on the choice of the sequence s_n converging to f.

(viii) Show that

$$\left((\int f\, dE)x, x \right) = \int f\, dE_{x,x}$$

for every bounded measurable function f and every $x \in H$.

(ix) Prove the analogs of (iii)-(vi) for bounded measurable functions.

9. Let $(\Omega, \mathcal{B}, \mu)$ be a measure space. For any measurable functions f and g on Ω let $M_f g$ denote the function fg. If f is bounded and $g \in L^2(\mu)$ then clearly $M_f g$ is also in $L^2(\mu)$ and indeed $M_f : L^2(\mu) \to L^2(\mu)$ is a bounded linear operator with norm $|M_f| \leq |f|_\infty$ (in all practical cases $|M_f|$ is actually equal to $|f|_\infty$). It is clear that $f \mapsto M_f$ is linear and, moreover, $M_{fh} = M_f M_h$.

(i) Show that $M_f^* = M_{\overline{f}}$. (Hint: Let $g, h \in L^2(\mu)$ and work out $(M_f g, h)_{L^2}$.)

(ii) Show that for any measurable set A, the operator M_{1_A} is an orthogonal projection operator.

(iii) Show that $E : A \mapsto M_{1_A}$ is a spectral measure. [Hint: The only non-trivial thing to check is that for any $g \in L^2(\mu)$ and disjoint measurable sets A_n whose union is A we have $\sum_n E(A_n)g = E(A)g$ with the sum $\sum_n$ being L^2–convergent. To this end, let $G_n = \sum_{j=1}^n E(A_j)g$ and look at what happens to $\int |G_n - 1_A g|^2 \, d\mu$ a $n \to \infty$.]

(iv) For any measurable simple function s show that $\int s\,dE = M_s$, where E is as in (iii).

(v) For any bounded measurable function f show that $\int f\,dE = M_f$, where E is as in (iii). [Hint: Choose measurable simple s_n converging uniformly to f, and with $|s_n(x)| \leq |f(x)|$ for all $x \in \Omega$. Consider the norms of $\int f\,dE - \int s_n\,dE$ and $M_f - M_{s_n}$.]

10. Let E be a spectral measure for a measurable space $(\Omega, \mathcal{B})$ with values being orthogonal projection operators in a complex Hilbert space H. Let $f : \Omega \to \mathbf{C}$ be a measurable function not necessarily bounded). Let

$$D_f = \{x \in H : \int |f|^2 \, dE_{x,x} < \infty\}$$

(i) For any $x, y \in H$ and any measurable set A, show that

$$E_{x+y,x+y}(A) \leq 2E_{x,x}(A) + 2E_{y,y}(A)$$

[Hint: First recall that $E_{v,v}(B) = |E(B)v|^2$. Next, for any vectors $a, b \in H$ we have the Cauchy-Schwarz inequality $|(a, b)| \leq |a||b|$ which leads to the inequality $|a + b|^2 \leq |a|^2 + |b|^2 + 2|a||b|$. This, together with $(|a| - |b|)^2 \geq 0$ implies that $|a + b|^2 \leq 2|a|^2 + 2|b|^2$.]

(ii) Show that D_f is a *linear* subspace of H, i.e. if $x, y \in D_f$ then $x + y \in D_f$ and $ax \in D_f$ for every $a \in \mathbf{C}$.

(iii) Let $A_n = \{p \in \Omega : |f(p)| \leq n\}$. Consider any vector x in the range of the projection $E(A_n)$. Show that

$$E_{x,x}(A) = E_{x,x}(A \cap A_n)$$

for every $A \in \mathcal{B}$. [Hint: What is $E(A_n)x$?]

(iv) With notation as above, show that

$$\int s\, dE_{x,x} = \int_{A_n} s\, dE_{x,x}$$

for every measurable simple function s on Ω.

(v) With notation as above, show that

$$\int |f|^2\, dE_{x,x} = \int_{A_n} |f|^2\, dE_{x,x}$$

Note that the right side is $\leq n^2 E_{x,x}(\Omega) = n^2|x|^2 < \infty$, and so $x \in D_f$.