A **complex algebra** is a complex vector space $B$ on which there is a bilinear multiplication map

$$B \times B \to B : (x, y) \mapsto xy$$

which is associative. Bilinearity of multiplication means the distributive law

$$x(y + z) = xy + xz, \quad (y + z)x = yz + zx$$

for all $x, y, z \in B$, and

$$(\lambda a)b = \lambda(ab) = a(\lambda b)$$

for all $a, b \in B$ and $\lambda \in \mathbb{C}$. In particular, a complex algebra is automatically a ring. An element $e \in B$ is a multiplicative identity (or **unit element**) if

$$xe = ex = x$$

for all $x \in B$. If $e'$ is also a multiplicative identity then

$$e = ee' = e'$$

Thus the multiplicative identity, if it exists, is unique.

Suppose $B$ is a complex algebra with unit $e$. An element $x \in B$ is **invertible** if there exists an element $y \in B$, called an inverse of $x$, such that

$$yx = xy = e$$

If $y'$ is another element for which both $xy'$ and $y'x$ equal $e$ then

$$y = ey = (y'x)y = y'(xy) = y'e = y'$$

Thus if $x$ is invertible then it has a unique inverse, which is denoted $x^{-1}$.

The set of all invertible elements in $B$ will be denoted $G(B)$. It is clearly a group.

Assume, moreover, that there is a norm on the complex algebra $B$ which makes it a Banach space, the identity $e$ has norm 1:

$$|e| = 1,$$

and that

$$|xy| \leq |x||y|$$

for all $x, y \in B$. Then $B$ is called a **complex Banach algebra**.

In all that follows $B$ is a complex Banach algebra.

1. Let $B$ be a complex Banach algebra. Let $x \in B$, and let

$$s_N = \sum_{n=0}^{N} x^n = e + x + x^2 + \cdots + x^N$$
(i) Show that
\[(e - x)s_N = s_N(e - x) = e - x^{N+1}\]

(ii) Show that if \(|x| \neq 1\) then for any integers \(N \geq M \geq 0,\)
\[|x^M + x^{M+1} + \cdots + x^N| \leq \frac{|x|^M - |x|^{N+1}}{1 - |x|}\]

(iii) Show that if \(|x| < 1\) then the limit
\[s = \sum_{n=0}^{\infty} x^n \overset{\text{def}}{=} \lim_{N \to \infty} s_N\]
exists.
(iv) Show that if $|x| < 1$ then

$$s = (e - x)^{-1}$$

Thus for any $x \in B$ with $|x| < 1$ the element $e - x$ is invertible. Note that this conclusion is an algebraic property.

The spectrum $\sigma(x)$ of an element $x$ in a complex Banach algebra $B$ is the set of all complex numbers $\lambda \in \mathbb{C}$ for which $\lambda e - x$ does not have an inverse.

2. Show that for any $x \in B$, the spectrum $\sigma(x)$ is contained in the closed ball $\{\lambda \in \mathbb{C} : |\lambda| \leq |x|\}$:

$$\sigma(x) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq |x|\}$$
3. Let $G(B)$ be the set of all invertible elements of $B$. Show that $G(B)$ is open by going through the following argument. Let $x, h \in B$ be such that $x$ is invertible and $|h| < 1/|x^{-1}|$. Observe that $x + h = (e + hx^{-1})x$. So, since $x$ is invertible, invertibility of $x + h$ will be established if we can show that $e + hx^{-1}$ is invertible. For this use the result from the previous problem.

4. Show that the map $G(B) \to G(B) : x \mapsto x^{-1}$ is differentiable. Hint: Let $x \in G(B)$ and $h \in B$ be such that $|h| < 1/|x^{-1}|$. Look at

$$(x + h)^{-1} - x^{-1} = x^{-1}(e + hx^{-1})^{-1} - e$$

Set $y = -hx^{-1}$ and show that

$$(x + h)^{-1} - x^{-1} = x^{-1}[y + r]$$

where the remainder $r = y^2 + y^3 + \cdots$ has norm $\leq |y|^2 + |y|^3 + \cdots < |y|^2/(1 - |y|)$.

Now show that

$$\lim_{h \to 0} \frac{|(x + h)^{-1} - x^{-1} - L_x h|}{|h|} = 0$$

where $L_x : B \to B$ is the linear map given by

$L_x : B \to B : h \mapsto L_x h \overset{\text{def}}{=} -x^{-1}hx^{-1}$
5. The spectrum $\sigma(x)$ is not empty for every $x \in B$.

Suppose $\sigma(x) = \emptyset$. Then for every $\lambda \in \mathbb{C}$ the element $\lambda e - x$ is invertible. Let $f : B \to \mathbb{C}$ be any bounded linear functional. Then the function $h$ on $\mathbb{C}$ given by

$$h(\lambda) = f ((\lambda e - x)^{-1})$$

is complex differentiable (i.e. holomorphic) everywhere. We have

$$(\lambda e - x)^{-1} = \frac{1}{\lambda} \left( e + (\lambda^{-1} x) + (\lambda^{-1} x)^2 + \cdots \right)$$

whenever $|\lambda^{-1} x| < 1$, i.e. for all complex $\lambda$ for which $|\lambda| > |x|$. Moreover, for such $\lambda$, we have

$$|(\lambda e - x)^{-1}| \leq \frac{1}{|\lambda|} \left( \frac{1}{1 - |x||\lambda|} \right) = \frac{1}{|\lambda| - |x|}$$

So as $|\lambda| \to \infty$ the norm of $(\lambda e - x)^{-1}$ goes to 0. Since the linear functional $f$ is continuous on $B$ it follows that

$$\lim_{|\lambda| \to \infty} h(\lambda) = 0$$

Since $h$ is also continuous (and hence bounded on any compact set) it follows that $h$ is bounded. Then by Liouville’s theorem it follows that $h$ is constant. Since $\lim_{|\lambda| \to \infty} h(\lambda) = 0$, the constant value of $h$ is actually 0. Looking back at the definition of $h$, this says that $f ((\lambda e - x)^{-1})$ is 0 for every $f \in B^*$ (and every $\lambda \in \mathbb{C}$). By the Hahn-Banach theorem it follows that $(\lambda e - x)^{-1}$ must be 0. But this is absurd since $(\lambda e - x)^{-1}(\lambda e - x) = e$.

6. The Gelfand-Mazur theorem. A complex Banach algebra in which every non-zero element is invertible is isometrically isomorphic to the Banach algebra $\mathbb{C}$.

Assume that $B$ is a complex Banach algebra in which every non-zero element is invertible. Consider the map

$$F : \mathbb{C} \to B : \lambda \mapsto \lambda e$$

It is clear that this is a homomorphism of complex algebras and that it is an isometry. The substance of the result lies in the surjectivity of $B$. For this consider any element $x \in B$. We know that $\sigma(x)$. Take $\lambda \in \sigma(x)$. This means $\lambda e - x$ is not invertible. So $\lambda e - x$ must be 0. So $x = \lambda e$, i.e. $x = F(\lambda)$. Thus $F$ is surjective.