Math 7330: Functional Analysis Fall 2002

Notes/Homework 4: Commutative Banach Algebras I A. Sengupta

1. Let R be a commutative ring with multiplicative identity e. A subset $S \subset R$ is an *ideal* of R if :(a) $0 \in S$, (b) $x + y \in S$ for every $x, y \in S$, and (c) $rx \in S$ for every $r \in R$ and $x \in S$. The ideal S is a *proper* ideal if $S \neq R$. It is a *maximal ideal* if it is a proper ideal and if the only ideals containing S are S itself and the whole ring R. The ideal S is a *prime* ideal if for every $x, y \in S$ if $xy \in S$ then at least one of x and y must be in S.

   (i) Let I be an ideal of R. For any $x \in R$ we write $x + I$ be the set of all elements of the form $x + i$ with i running over I. Let R/I be the set of all sets of the form $x + I$ with x running over R:

$$R/I \stackrel{\text{def}}{=} \{x + I : x \in R\}$$

   Let

$$p : R \to R/I : x \mapsto x + I$$

   For any elements $a, b \in R$ we have

$$p(a) = p(b) \text{ if and only if } a - b \in I$$

   Show that if $x, x', y, y' \in R$ are such that $p(x) = p(x')$ and $p(y) = p(y')$ then $p(x + x') = p(y + y')$ and $p(xy) = p(yy')$.

   Thus there are well-defined operations of addition and multiplication on R/I given by

$$p(x) + p(y) \stackrel{\text{def}}{=} p(x + y), \qquad p(x)p(y) \stackrel{\text{def}}{=} p(xy)$$

   As is readily checked, these operations make R/I a ring and, of course, $p : R \to R/I$ is a ring homomorphism. Commutativity of R implies that R/I is commutative. If $e \in R$ is the identity of R then $p(e)$ is the multiplicative identity in R/I.

(ii) Suppose I is a maximal ideal of R. Show that then the commutative ring R/I is a *field*, i.e. every non-zero element has an inverse. Hint: Let $x \in R$ be such that $p(x)$ is a non-zero element of R/I, i.e. $x \in R$ is not in the ideal I. The set

$$Rx + I = \{rx + y : r \in R, y \in I\}$$

is clearly an ideal of R which contains I. Moreover, $Rx + I$ contains the element x which is not in I and so $Rx + I \neq I$. Since I is maximal, it follows then that $Rx + I$ equals the whole ring R. In particular, there is an element $y \in R$ and an element $a \in I$ such that $yx + a = e$. Apply p to this.

(iii) Let I be a ideal in R such that the quotient ring R/I is a field in which the multiplicative identity is not equal to 0. Show that I is maximal. Hint: Since $R/I \neq \{0\}$, the ideal I is proper. Let S be an ideal with $R \supset S \supset I$ and $S \neq I$. Choose $x \in S$ not in I. Then $p(x)$ is a non-zero element of R/I, where $p : R \to R/I : x \mapsto x + I$ is the projection map. So it has an inverse. Thus there is an element $y \in R$ such that $p(x)p(y) = p(e)$. This means $e - xy \in I$ and so $e - xy \in S$. But then $e = e - xy + xy \in S$.

In the following B is a complex Banach algebra which is assumed also to be *commutative*. An *ideal* in B is a subset $I \subset B$ which satisfies: (a) $x + y \in B$ for all $x, y \in I$, (b) $bx \in I$ for all $b \in B$ and $x \in I$. Note that taking $b = \lambda e$ for $\lambda \in \mathbf{C}$ in (b) shows, together with (a), that an ideal I is automatically a linear subspace of B. Recall the quotient

$$B/I = \{x + I : x \in B\}$$

and the projection map

$$p : B \to B/I : x \mapsto x + I$$

We have seen that B/I has a ring structure which makes p a ring homomorphism, and $p(e)$ is the identity element in B/I. Then the quotient B/I is also a complex vector space with multiplication by complex scalars λ defined by

$$\lambda p(x) \stackrel{\text{def}}{=} p(\lambda x)$$

This is well-defined because if $p(x) = p(y)$ then $x - y \in I$ and so $\lambda x - \lambda y = \lambda(x-y) \in I$ which means $p(\lambda x) = p(\lambda y)$. It is clear that B/I does become a vector space and indeed, together with the multiplication, B/I is a complex algebra and $p : B \to B/I$ a homomorphism of algebras (i.e. p is linear and $p(xy) = p(x)p(y)$ for all $x, y \in B$; $p(e)$ is the identity).

Any element B/I is of the form $p(x) = x + I$, for some $x \in B$. Thus it is a *translate* of the subspace I. Define

$$|p(x)| \stackrel{\text{def}}{=} \inf_{y \in p(x)} |y|,$$

the distance of $x + I$ from the origin. Since x itself belongs to $x + I$ it follows that

$$|p(x)| \leq |x|$$

2. We prove that if I is a closed proper ideal in B then $|\cdot|$ is a norm on B/I making it a complex Banach algebra.
   (i) For any $x, y \in B$,
   $$|p(x) + p(y)| \leq |p(x)| + |p(y)|$$
   Proceed as follows: Pick any $x' \in p(x) = x + I$ and $y' \in p(y) = y + I$. Then $p(x) = p(x')$ and $p(y) = p(y')$ and so $p(x + y) = p(x) + p(y) = p(x') + p(y') = p(x' + y')$. Therefore, $|p(x) + p(y)| = |p(x' + y')|$. So
   $$|p(x) + p(y)| \leq |x' + y'| \leq |x'| + |y'|$$
   Now take infimum over $x' \in p(x)$ and then over $y' \in p(y)$.

(ii) For any $x \in B$ and $\lambda \in \mathbf{C}$,

$$|\lambda p(x)| = |\lambda||p(x)|$$

Hint: Work as in (i), taking any $x' \in p(x)$ and showing that $|\lambda p(x)| = |p(\lambda x')| \leq |\lambda||x'|$ and taking inf over all $x' \in p(x) = x + I$. This shows $|\lambda p(x)| \leq |\lambda||p(x)|$. Now, for non-zero λ, write $p(x)$ on the right as $(1/\lambda)\lambda p(x)$.

(iii) Show that

$$|p(x)p(y)| \leq |p(x)||p(y)|$$

for every $x, y \in B$.

(iv) Show that if $I \neq B$ then $|p(e)| \neq 0$. Hint: Since I is a proper ideal it does not contain any invertible elements. The open ball of radius 1 around e consists entirely of invertible elements and so does not intersect I. So $e + I$ does not the open ball of radius 1 centered at 0. So $|p(e)| \geq ?$.

(v) Show that if $I \neq B$ then

$$|p(e)| = 1$$

Hint: Combine the observation obtained in proving (iv) with the inequality $|p(e)| \leq |e| = 1$. [Note also that if in (iii) we put $x = y = e$ then $|p(e)| \geq 1$ or $|p(e)| = 0$.]

(vi) Suppose that I is a *closed ideal* in B, i.e. suppose that I is an ideal and it is *closed* as a subset of B. If $|p(x)| = 0$ show that $p(x) = 0$. (Hint: If $|p(x)| = 0$ then every neighborhood of 0 contains a point of $x + I$, and so every neighborhood of x contains a point of I.)

The preceding parts show that if I is a closed ideal in B then the definition of $|p(x)|$ establishes a *norm* on the complex algebra B/I, and the map $p : B \to B/I$ is continuous.

(vii) Let $\epsilon > 0$ and $a, b \in B$. Suppose $|p(a) - p(b)| < \epsilon$. Then there is a $b' \in B$ such that $p(b') = p(b)$ and $|a - b'| < \epsilon$. Hint: Since $|p(a - b)| < \epsilon$, there is an element $x \in p(a - b) = a - b + I$ such that $|x| < \epsilon$. Since $x \in a - b + I$ there is an element $y \in I$ such that $x = a - b + y = a - (b - y)$.

(viii) Let I be a closed proper ideal in B. Suppose $a_1, a_2, ...$ is a Cauchy sequence in B/I. Then there is a subsequence $a_{j_1}, a_{j_2}, ...$ such that $|a_{j_r} - a_{j_{r+1}}| < 2^{-r}$ for every $r \in \{1, 2, 3, ...\}$. Pick $x_1, x_2, ... \in B$ such that $p(x_i) = a_i$ for all i. Check that by (vii) we can choose $x'_{j_1}, x'_{j_2}, ...$ such that $p(x'_{j_r}) = p(x_{j_r})$ for all $r \in \{1, 2, 3, ...\}$ and such that

$$|x'_{j_{r+1}} - x'_{j_r}| < 2^{-r}$$

Since B is a Banach space, the sequence $(x'_{j_r})_r$ converges. Since $p : B \to B/I$ is continuous it follows then that the sequence $(p(x'_{j_r}))_r$ is convergent in B/I. Note that $p(x'_{j_r}) = a_{j_r}$ and so we have proven that the original Cauchy sequence (a_j) in B/I has a convergent subsequence. Since (a_j) is Cauchy and has a convergent subsequence it follows that (a_j) is itself convergent. Thus *B/I is a Banach space*, i.e. *B/I is a complex Banach algebra*.