1. Let \( R \) be a commutative ring with multiplicative identity \( e \). A subset \( S \subset R \) is an ideal of \( R \) if :
(a) \( 0 \in S \),
(b) \( x + y \in S \) for every \( x, y \in S \), and
(c) \( rx \in S \) for every \( r \in R \) and \( x \in S \). The ideal \( S \) is a proper ideal if \( S \neq R \). It is a maximal ideal if it is a proper ideal and if the only ideals containing \( S \) are \( S \) itself and the whole ring \( R \). The ideal \( S \) is a prime ideal if for every \( x, y \in S \) if \( xy \in S \) then at least one of \( x \) and \( y \) must be in \( S \).

(i) Let \( I \) be an ideal of \( R \). For any \( x \in R \) we write \( x + I \) be the set of all elements of the form \( x + i \) with \( i \) running over \( I \). Let \( R/I \) be the set of all sets of the form \( x + I \) with \( x \) running over \( R \):

\[
R/I \overset{\text{def}}{=} \{ x + I : x \in R \}
\]

Let

\[
p : R \to R/I : x \mapsto x + I
\]

For any elements \( a, b \in R \) we have

\[
p(a) = p(b) \text{ if and only if } a - b \in I
\]

Show that if \( x, x', y, y' \in R \) are such that \( p(x) = p(x') \) and \( p(y) = p(y') \) then \( p(x + x') = p(y + y') \) and \( p(xy) = p(yy') \).

Thus there are well-defined operations of addition and multiplication on \( R/I \) given by

\[
p(x) + p(y) \overset{\text{def}}{=} p(x + y), \quad p(x)p(y) \overset{\text{def}}{=} p(xy)
\]

As is readily checked, these operations make \( R/I \) a ring and, of course, \( p : R \to R/I \) is a ring homomorphism. Commutativity of \( R \) implies that \( R/I \) is commutative. If \( e \in R \) is the identity of \( R \) then \( p(e) \) is the multiplicative identity in \( R/I \).
(ii) Suppose $I$ is a maximal ideal of $R$. Show that then the commutative ring $R/I$ is a field, i.e. every non-zero element has an inverse. Hint: Let $x \in R$ be such that $p(x)$ is a non-zero element of $R/I$, i.e. $x \in R$ is not in the ideal $I$. The set

$$Rx + I = \{rx + y : r \in R, y \in I\}$$

is clearly an ideal of $R$ which contains $I$. Moreover, $Rx + I$ contains the element $x$ which is not in $I$ and so $Rx + I \neq I$. Since $I$ is maximal, it follows then that $Rx + I$ equals the whole ring $R$. In particular, there is an element $y \in R$ and an element $a \in I$ such that $yx + a = e$. Apply $p$ to this.

(iii) Let $I$ be a ideal in $R$ such that the quotient ring $R/I$ is a field in which the multiplicative identity is not equal to 0. Show that $I$ is maximal. Hint: Since $R/I \neq \{0\}$, the ideal $I$ is proper. Let $S$ be an ideal with $R \supset S \supset I$ and $S \neq I$. Choose $x \in S$ not in $I$. Then $p(x)$ is a non-zero element of $R/I$, where $p : R \to R/I : x \mapsto x + I$ is the projection map. So it has an inverse. Thus there is an element $y \in R$ such that $p(x)p(y) = p(e)$. This means $e - xy \in I$ and so $e - xy \in S$. But then $e = e - xy + xy \in S$. 


In the following $B$ is a complex Banach algebra which is assumed also to be commutative. An ideal in $B$ is a subset $I \subset B$ which satisfies: (a) $x + y \in B$ for all $x, y \in I$, (b) $bx \in I$ for all $b \in B$ and $x \in I$. Note that taking $b = \lambda e$ for $\lambda \in \mathbb{C}$ in (b) shows, together with (a), that an ideal $I$ is automatically a linear subspace of $B$. Recall the quotient

$$B/I = \{x + I : x \in B\}$$

and the projection map

$$p : B \to B/I : x \mapsto x + I$$

We have seen that $B/I$ has a ring structure which makes $p$ a ring homomorphism, and $p(e)$ is the identity element in $B/I$. Then the quotient $B/I$ is also a complex vector space with multiplication by complex scalars $\lambda$ defined by

$$\lambda p(x) \overset{\text{def}}{=} p(\lambda x)$$

This is well-defined because if $p(x) = p(y)$ then $x-y \in I$ and so $\lambda x - \lambda y = \lambda(x-y) \in I$ which means $p(\lambda x) = p(\lambda y)$. It is clear that $B/I$ does become a vector space and indeed, together with the multiplication, $B/I$ is a complex algebra and $p : B \to B/I$ a homomorphism of algebras (i.e. $p$ is linear and $p(xy) = p(x)p(y)$ for all $x, y \in B$; $p(e)$ is the identity).

Any element $B/I$ is of the form $p(x) = x + I$, for some $x \in B$. Thus it is a translate of the subspace $I$. Define

$$|p(x)| \overset{\text{def}}{=} \inf_{y \in p(x)} |y|,$$

the distance of $x + I$ from the origin. Since $x$ itself belongs to $x + I$ it follows that

$$|p(x)| \leq |x|$$

2. We prove that if $I$ is a closed proper ideal in $B$ then $\cdot$ is a norm on $B/I$ making it a complex Banach algebra.

(i) For any $x, y \in B$,

$$|p(x) + p(y)| \leq |p(x)| + |p(y)|$$

Proceed as follows: Pick any $x' \in p(x) = x + I$ and $y' \in p(y) = y + I$. Then $p(x) = p(x')$ and $p(y) = p(y')$ and so $p(x + y) = p(x) + p(y) = p(x') + p(y') = p(x' + y')$. Therefore, $|p(x) + p(y)| = |p(x' + y')|$. So

$$|p(x) + p(y)| \leq |x' + y'| \leq |x'| + |y'|$$

Now take infimum over $x' \in p(x)$ and then over $y' \in p(y)$.  

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(ii) For any \( x \in B \) and \( \lambda \in C \),
\[
|\lambda p(x)| = |\lambda||p(x)|
\]
Hint: Work as in (i), taking any \( x' \in p(x) \) and showing that \( |\lambda p(x)| = |p(\lambda x')| \leq |\lambda||x'| \) and taking inf over all \( x' \in p(x) = x + I \). This shows \( |\lambda p(x)| \leq |\lambda||p(x)| \).
Now, for non-zero \( \lambda \), write \( p(x) \) on the right as \((1/\lambda)p(x)\).

(iii) Show that
\[
|p(x)p(y)| \leq |p(x)||p(y)|
\]
for every \( x, y \in B \).

(iv) Show that if \( I \neq B \) then \( |p(e)| \neq 0 \). Hint: Since \( I \) is a proper ideal it does not contain any invertible elements. The open ball of radius 1 around \( e \) consists entirely of invertible elements and so does not intersect \( I \). So \( e + I \) does not the open ball of radius 1 centered at 0. So \( |p(e)| \geq ? \).
(v) Show that if \( I \neq B \) then \(|p(e)| = 1\)

Hint: Combine the observation obtained in proving (iv) with the inequality \(|p(e)| \leq |e| = 1\). [Note also that if in (iii) we put \( x = y = e \) then \(|p(e)| \geq 1\) or \(|p(e)| = 0\).]

(vi) Suppose that \( I \) is a closed ideal in \( B \), i.e. suppose that \( I \) is an ideal and it is closed as a subset of \( B \). If \(|p(x)| = 0\) show that \( p(x) = 0\). (Hint: If \(|p(x)| = 0\) then every neighborhood of 0 contains a point of \( x + I \), and so every neighborhood of \( x \) contains a point of \( I \).)

The preceding parts show that if \( I \) is a closed ideal in \( B \) then the definition of \(|p(x)|\) establishes a norm on the complex algebra \( B/I \), and the map \( p : B \to B/I \) is continuous.

(vii) Let \( \epsilon > 0 \) and \( a, b \in B \). Suppose \(|p(a) - p(b)| < \epsilon\). Then there is a \( b' \in B \) such that \( p(b') = p(b) \) and \(|a - b'| < \epsilon\). Hint: Since \(|p(a - b)| < \epsilon\), there is an element \( x \in p(a - b) = a - b + I \) such that \(|x| < \epsilon\). Since \( x \in a - b + I \) there is an element \( y \in I \) such that \( x = a - b + y = a - (b - y)\).

(viii) Let \( I \) be a closed proper ideal in \( B \). Suppose \( a_1, a_2, ... \) is a Cauchy sequence in \( B/I \). Then there is a subsequence \( a_{j_1}, a_{j_2}, ... \) such that \(|a_{j_r} - a_{j_{r+1}}| < 2^{-r}\) for every \( r \in \{1, 2, 3, ...\} \). Pick \( x_1, x_2, ... \in B \) such that \( p(x_i) = a_i \) for all \( i \). Check that by (vii) we can choose \( x'_{j_1}, x'_{j_2}, ... \) such that \( p(x'_{j_r}) = p(x_{j_r}) \) for all \( r \in \{1, 2, 3, ...\} \) and such that

\[ |x'_{j_{r+1}} - x'_{j_r}| < 2^{-r} \]
Since $B$ is a Banach space, the sequence $(x'_j)_r$ converges. Since $p : B \to B/I$ is continuous it follows then that the sequence $(p(x'_j))_r$ is convergent in $B/I$. Note that $p(x'_j) = a_j$, and so we have proven that the original Cauchy sequence $(a_j)$ in $B/I$ has a convergent subsequence. Since $(a_j)$ is Cauchy and has a convergent subsequence it follows that $(a_j)$ is itself convergent. Thus $B/I$ is a Banach space, i.e. $B/I$ is a complex Banach algebra.