Math 7330: Functional Analysis Fall 2002

Homework 5: Commutative Banach Algebras II A. Sengupta

We work with a complex commutative Banach algebra B.

It had been shown that the set $G(B)$ of all invertible elements in B is an open subset of B. A proper ideal I in B cannot contain any invertible elements (for if $x \in I$ is invertible then for any $y \in B$ we would have $y = (yx^{-1})x \in I$, which would mean $I = B$), i.e. is a subset of the closed set $G(B)^c$.

Zorn's lemma shows that every proper ideal of B is contained in a maximal ideal.

1. Let J be an ideal of B.
   (i) Check that the closure $\overline{J}$ is also an ideal.

   (ii) Show that if J is a proper ideal then so is its closure $\overline{J}$.

   (iii) Show that if J is a maximal ideal then J is closed. Hint: Consider the ideal $\overline{J}$. It is an ideal which contains J. Since J, being maximal, is proper, (ii) implies that $\overline{J}$ is a proper ideal.

A mapping $\phi : B \to \mathbf{C}$ is a *complex homomorphism* if f is linear and satisfies $f(xy) = f(x)f(y)$ for all $x, y \in B$. Note that then $f(x) = f(xe) = f(x)f(e)$ for every $x \in B$, and so either $f(e) = 1$ or $f(x) = 0$ for every $x \in B$. The set of all *non-zero* complex homomorphisms $B \to \mathbf{C}$ will be denoted Δ and is the *Gelfand spectrum* of the algebra B.

2. Let J be a maximal ideal of B. Show that there is a non-zero complex homomorphism $h : B \to \mathbf{C}$ such that $J = \ker h$. Hint: Consider B/J. This is a field because J is a maximal ideal, and, moreover, since J is a closed proper ideal in B, B/J is also a Banach algebra. Therefore, by Gelfand-Mazur, there is an isometric isomorphism $j : B/J \to \mathbf{C}$. Let $p : B \to B/J : x \mapsto p(x) = x + J$ be the usual projection homomorphism. Work with $h = j \circ p$.

3. Let $h_1, h_2 : B \to \mathbf{C}$ be complex homomorphisms such that $\ker h_1 = \ker h_2$. Show that $h_1 = h_2$. Hint: Write any $x \in B$ as $x = [x - h_1(x)e] + h_1(x)e$, and observe that $x - h_1(x)e \in \ker h_1$. Now calculate $h_2(x)$.

4. An element $y \in B$ is not invertible if and only if there is a non-zero complex homomorphism $h : B \to \mathbf{C}$ such that $h(y) = 0$.

   (Easy half) Suppose y is invertible. Then for any $h \in \Delta$ we have $h(y)h(y^{-1}) = h(yy^{-1}) = h(e) = 1$ and so $h(y)$ can't be 0.

   For the converse (harder half), suppose $y \in B$ is not invertible. Then the set $By = \{xy : x \in B\}$ is a *proper* ideal of B. Let J be a maximal ideal with $J \supset Bh$ (existence of J follows by an application of Zorn's lemma). By (2) there exists a non-zero complex homomorphism $h : B \to \mathbf{C}$ such that $J = \ker h$. Since $y \in By \subset J$ it follows that $h(y) = 0$, which is what we wished to prove.

5. Let $x \in B$. Prove that a complex number λ belongs to the spectrum $\sigma(x)$ if and only if there is a non-zero complex homomorphism $h : B \to \mathbf{C}$ such that $h(x) = \lambda$.

6. Let $h : B \to \mathbf{C}$ be a complex homomorphism. Show that h is continuous and, viewed as a linear functional on B, has norm $|h| \leq 1$, the norm being equal to 1 if $h \neq 0$. Hint: Combine the easy half of (5) with the fact that $\sigma(x) \subset \{\lambda \in \mathbf{C} : |\lambda| \leq 1\}$.

7. Let $h : B \to \mathbf{C}$ be a non-zero complex homomorphism. Then $\ker h$ is a maximal ideal in B.

   Since h is a non-zero homomorphism, $h(e) = 1$ and so $h(\lambda e) = \lambda$, which shows that h is surjective. So $B/\ker h \simeq \mathbf{C}$, and the latter is a field. So the ideal $\ker h$ must be maximal. This is a pure algebra result and uses nothing about the norm on B.

The preceding discussions establishes

a one-to-one correspondence $h \mapsto \ker h$ between the set Δ of all non-zero complex homomorphisms $B \to \mathbf{C}$ and the set of all maximal ideals of B.