Math 7330: Functional Analysis Fall 2002

Notes/Homework 6: Banach *-Algebras A. Sengupta

An *involution* $*$ on a complex algebra B is a map $* : B \to B$ for which

(i) $*(a + b) = *a + *b$ for all $a, b \in B$

(ii) $*(\lambda a) = \overline{\lambda} * a$ for all $\lambda \in \mathbf{C}$ and $a \in B$

(iii) $(xy) = y^* x^*$ for all $x, y \in B$

(iv) $(x^*)^* = x$ for all $x \in B$. An element $a \in B$ is *hermitian* if $a = a^*$.

On a complex *Banach* algebra we also require an involution $*$ to satisfy

(v) $|xy| \leq |x||y|$ for all $x, y \in B$.

Observe that for the identity e, we have $e^* = ee^*$ and so taking $*$ of this we get $(e^*)^* = (e^*)^* e^*$, which says $e = ee^*$. Thus

$$e = e^*$$

A *B^*-algebra* is a complex Banach algebra B on which there is an involution $*$ for which

$$|xx^*| = |x|^2 \qquad \text{for all } x \in B$$

1. Let B be a complex Banach algebra with involution.
   (i) Show that

   if B is a B*-algebra then $|x| = |x^*|$ for all $x \in B$

   (ii) Suppose $|y^* y| = |y|^2$ for all $y \in B$. Show that $|y| = |y^*|$ for all $y \in B$.

(iii) Suppose $|y^*y| = |y|^2$ for all $y \in B$. Show that $|xx^*| = |x|^2$ for all $x \in B$.

2. Let B be a B*-algebra.

(i) Show that if $y \in B$ is hermitian and s is any real number then

$$|se + iy|^2 = |s^2 e + y^2|$$

(ii) Show that $e + iy$ is invertible for every hermitian $y \in B$. Proceed as follows: Suppose $e + iy$ is not invertible. Then for every $\lambda \in \mathbf{R}$, $(\lambda + 1)e - (\lambda e - iy))$ is not invertible, i.e. $(\lambda + 1) \in \sigma(\lambda e - iy)$. So $|\lambda + 1| \leq |\lambda e - iy|$. By (i), this implies $(\lambda + 1)^2 \leq |\lambda^2 e + y^2|$ and the latter is $\leq \lambda^2 + |y^2|$. This would be true for every real number λ. Show that this is impossible.

3. Let B be a complex algebra with involution $*$.

(i) If $e + x^*x$ is invertible for every $x \in B$ then show that $e + iy$ is invertible for every hermitian $y \in B$. Hint: Note that $(e + iy)(e - iy) = e + y^2 = e + y^*y$.

(ii) If $e+iy$ is invertible for every hermitian $y \in B$ then $\sigma(a) \subset \mathbf{R}$ for every hermitian $a \in B$. Hint: Consider any complex number $\lambda = \alpha + i\beta$ with $\beta \neq 0$. Check that $\lambda e - a = i\beta(e+iy)$ for some *hermitian* element y. By (i) then $\lambda e - a$ is invertible and so $\lambda \notin \sigma(a)$.

(iii) If $e + x^*x$ is invertible for every $x \in B$ then $\sigma(y^*y) \subset [0, \infty)$ for every $y \in B$. Proceed as follows: Let $k > 0$ and show that $(-k)e - y^*y$ is invertible by writing it as $(-k)[e + x^*x]$ where $x = k^{-1/2}y$.

3. Let B be a complex commutative Banach algebra with an involution $*$. Show that the following are equivalent:
   (a) $e + x^*x$ is invertible for every $x \in B$
   (b) every hermitian element has real spectrum
   (c) $\hat{x^*} = \overline{\hat{x}}$ for every $x \in B$.
   (d) $J^* = J$ for every maximal ideal J of B.

   (a) implies (b) is from the previous problem.

   Now suppose (b) holds. Let $x \in B$. Then $a = x + x^*$ and $b = i(x - x^*)$ are hermitian. So their spectra are real. So $\hat{a}$ and $\hat{b}$ are real-valued. Thus $f = \hat{x} + \hat{x^*}$ and $g = i(\hat{x} - \hat{x^*})$ are *real-valued.* Now $\hat{x} = (f - ig)/2$ and $\hat{x^*} = (f + ig)/2$. It follows that $\hat{x^*} = \overline{\hat{x}}$.

   Assume (c). Let J be a maximal ideal. Then $J = \ker h$ for some $h \in \Delta$ (i.e. h is a non-zero complex homomorphism $B \to \mathbf{C}$). Let $x \in B$. Then

$$h(x^*) = \hat{x^*}(h) = \overline{\hat{x}(h)} = \overline{h(x)} = 0$$

   and so $x^* \in kerh = J$.

   Now suppose (d) holds. We prove (c). Let $x \in B$. Consider any $h \in \Delta$. Then $x - h(x)e \in \ker h$. Since $\ker h$ is a maximal ideal, (d) implies that $x^* - \overline{h(x)}e$ is also in $\ker h$. So $h(x^* - \overline{h(x)}e) = 0$ and this implies $h(x^*) = \overline{h(x)}$. This holds for all $h \in \Delta$. So (c) holds.

   Finally we show that (c) implies (a). Assume (c). Let $x \in B$. Then the Gelfand transform of $e + x^*x$ is $1 + |\hat{x}|^2$ which never has the value zero. So 0 is not in the spectrum of $e + x^*x$ and so $e + x^*x$ is invertible.