Let $B$ be a complex, commutative $B^*$ algebra, with $\Delta$ its Gelfand spectrum. Then, as we have seen in class,

(i) the Gelfand transform $B \to \mathcal{C}(\Delta) : x \mapsto \hat{x}$ satisfies

$$\hat{x}^* = \overline{\hat{x}}$$

for every $x \in B$;

(ii) the spectral radius $\rho(x)$ equals the norm $\|x\|$ for every $x \in B$.

Fact (ii) was proven first for hermitian elements in any $B^*$ algebra and then, using the Gelfand transform, for all elements in a commutative $B^*$ algebra. If $a \in B$ is hermitian then

$$\rho(a) = \lim_{n \to \infty} |a^n|^{1/n}$$

while $|a^2| = |aa^*| = |a|^2$ which implies $|a^{2k}| = |a|^{2k}$, and so, letting $n \to \infty$ through powers of 2 we get

$$\rho(a) = |a|$$

for every hermitian $a$ in any $B^*$ algebra. For a commutative $B^*$ algebra $B$ we have for a general $x \in B$,

$$\rho(xx^*) = |xx^*|_{\text{sup}} \leq |\hat{x}|_{\text{sup}} |\hat{x}^*|_{\text{sup}} = \rho(x)\rho(x^*) \leq \rho(x)|x^*|$$

Since $xx^*$ is hermitian, $\rho(xx^*) = |xx^*|$, which is equal to $|x||x^*|$. So we have

$$|x| \leq \rho(x)$$

But we already know the opposite inequality. So $\rho(x) = |x|$.

By (i) and (ii) and other properties we have studied before, the Gelfand transform is a $\ast$–algebra homomorphism and is also an isometry. Its image $\hat{B}$ in $\mathcal{C}(\Delta)$ is therefore a subalgebra of $\mathcal{C}(\Delta)$ which is preserved under conjugation. Moreover, since the Gelfand transform is an isometry it follows that $\hat{B}$ is a closed subset of $\mathcal{C}(\Delta)$: for if $x_n \in B$ are such that $\hat{x}_n \to f$ for some $f \in \mathcal{C}(\Delta)$ then $(\hat{x}_n)_n$ is Cauchy in $\mathcal{C}(\Delta)$ and so, by isometricity, $(x_n)_n$ is Cauchy in $B$ and so is convergent, say to $x$ and then by continuity of $\hat{\cdot}$ it follows that $f = \hat{x}$, and so $f$ is in the image of the Gelfand transform. Finally, $\hat{B}$ separates points of $\Delta$ because if $h_1$ and $h_2$ are distinct elements of $\Delta$, then, by definition of $\Delta$, there must be some $x \in B$ for which $h_1(x) \neq h_2(x)$, i.e. $\hat{x}(h_1) \neq \hat{h}(x_2)$.

The Stone-Weierstrass theorem now implies that

$$\hat{B} = C(\Delta)$$

This proves the Gelfand-Naimark theorem:

**Theorem.** For a complex commutative $B^*$–algebra $B$, the Gelfand transform is an isometric isomorphism of $B$ onto $\mathcal{C}(\Delta)$, where $\Delta$ is the Gelfand spectrum of $B$. 

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1. Let $H$ be a complex vector space and $F : H \times H \rightarrow \mathbb{C}$ a mapping such that $F(x, y)$ is linear in $x$ and conjugate-linear in $y$.

(i) Prove the polarization formula

$$F(x, y) = \frac{1}{4}F(x+y, x+y) - \frac{1}{4}F(x-y, x-y) + \frac{i}{4}F(x+iy, x+iy) - \frac{i}{4}F(x-iy, x-iy)$$

(1)

(ii) Use this to prove that

$$\sup_{x, y \in H, |x|, |y| \leq 1} |F(x, y)| \leq 4 \sup_{v \in H, |v| \leq 1} |F(v, v)|$$

(2)

[Hint: In (1), the first term equals $F(a, a)$ with $a = (x + y)/2$ and $|a| \leq 1$ if $|x|, |y| \leq 1$. Similarly for the other terms.]
(iii) If $y \in H$ then show that

$$\sup_{v \in H, |v| \leq 1} |(y, v)| = |y|$$

(iv) If $T : H \to H$ is a linear map for which $\sup_{v \in H, |v| \leq 1} |(Tv, v)| < \infty$, show that $T$ is a bounded linear map and

$$|T| \leq 4 \sup_{v \in H, |v| \leq 1} |(Tv, v)|$$

(Recall that the norm of $T$ is $|T| = \sup_{x \in H, |x| \leq 1} |Tx|$.)
2. Let $H$ be a complex Hilbert space and $F : H \times H \to \mathbb{C}$ a map such that $F(x, y)$ is linear in $x$, conjugate linear in $y$, and $\sup_{x, y \in H, |x|, |y| \leq 1} |F(x, x)| < \infty$.

(i) Fix $x \in H$, and consider

$$\phi_x : H \to \mathbb{C} : y \mapsto F(x, y).$$

Show that this is a bounded linear functional. Consequently, there exists a unique element $Tx \in H$ such that $\phi_x(y) = (Tx, y)$ for every $y \in H$. Thus for each $x \in H$ there exists a unique element $Tx \in H$ such that

$$F(x, y) = (Tx, y) \quad \text{for all } y \in H$$

(ii) Let $x, x' \in H$ and $a, b \in \mathbb{C}$. Show that

$$(aTx + bTx', y) = F(ax + bx', y) \quad \text{for all } y \in H$$

Then by the uniqueness property noted in (i) it follows that

$$T(ax + bx') = aTx + bTx'$$

Thus $T : H \to H$ is linear.

(iii) Show that the map $T : H \to H$ is a bounded linear map. [Hint: Use 1(iii) and (ii).]
3. Let $X$ be a non-empty set and $\mathcal{B}$ a $\sigma$–algebra of subsets of $X$.

(i) Suppose $\lambda_1, \ldots, \lambda_n$ and $\lambda'_1, \ldots, \lambda'_m$ are finite measures on $\mathcal{B}$ and $a_1, \ldots, a_n, a'_1, \ldots, a'_m$ are complex numbers such that

$$\sum_{j=1}^{n} a_j \lambda_j = \sum_{j=1}^{m} a'_j \lambda'_j$$

Then show that for any bounded $\mathcal{B}$–measurable function $f : X \to \mathbb{C}$,

$$\sum_{j=1}^{n} a_j \int_X f \, d\lambda_j = \sum_{j=1}^{m} a'_j \int_X f \, d\lambda'_j$$

[Hint: There is a sequence of measurable simple functions $s_N$ such that $s_N(x) \to f(x)$ uniformly for $x \in X$ as $N \to \infty$.] If $\mu$ is the complex measure given by

$$\mu = \sum_{j=1}^{n} a_j \lambda_j$$

then we define

$$\int f \, d\mu \overset{\text{def}}{=} \sum_{j=1}^{n} a_j \int_X f \, d\lambda_j$$

for all bounded measurable functions $f$ on $X$. The fact proven above says that this definition is independent of the particular choice of $a_j$ and $\lambda_j$ used to express $\mu$. 
(ii) If \( b_1, \ldots, b_k \) are complex numbers and \( \mu_1, \ldots, \mu_k \) are complex measures, each of the type described in (i), and \( \mu \) is the complex measure given by

\[
\mu = \sum_{j=1}^{n} b_j \mu_j
\]

then show that

\[
\int f \, d\mu = \sum_{j=1}^{n} b_j \int f \, d\mu_j
\]

for all bounded measurable functions \( f \) on \( X \).

(iii) Suppose now that \( X \) is a compact Hausdorff space and \( \mathcal{B} \) is the Borel \( \sigma \)-algebra. Let \( \mu_1, \mu_2 \) be complex measures on \( \mathcal{B} \), each \( \mu_i \) being a complex linear combination of finite regular Borel measures \( \lambda_{ij} \) on \( \mathcal{B} \). Show that if

\[
\int f \, d\mu_1 = \int f \, d\mu_2
\]

for all \( f \in C(X) \)

then

\[
\mu_1 = \mu_2
\]

Hint: Write \( \mu_1 = \sum_j a_j \lambda_j \) and \( \mu_2 = \sum_i a'_i \lambda'_i \), where the \( a_i, a'_j \) are complex numbers and \( \lambda_i, \lambda'_j \) are finite regular Borel measures. The \( \lambda = \sum_i a_i \lambda_i + \sum_j a'_j \lambda'_j \) is a finite regular Borel measure. Let \( g \) be any bounded Borel function. Then there is a sequence of continuous functions \( g_n \in C(X) \) such that \( g_n(x) \to g(x) \) for \( \lambda \)-a.e. \( x \) and \( |g_n|_{\text{sup}} \leq |g|_{\text{sup}} \). Then the same holds a.e. for each \( \lambda_i \) and each \( \lambda'_j \). Now use the dominated convergence theorem. Finally, set \( g = 1_A \) for any Borel set \( A \subset X \).
4. Let $H$ be a complex Hilbert space, $X$ a compact Hausdorff space, $\mathcal{B}$ its Borel $\sigma$-algebra. Suppose that for each $x$ we have a finite regular Borel measure $\mu_{x,x}$ on $\mathcal{B}$. Define, for every $x, y \in H$, \begin{equation}
abla_{x,y} = \frac{1}{4} \mu_{x+y,x+y} - \frac{1}{4} \mu_{x-y,x-y} + \frac{i}{4} \mu_{x+iy,x+iy} - \frac{i}{4} \mu_{x-iy,x-iy}
abla \tag{3}
abla
abla
\end{equation}
This is a complex measure which is a linear combination of finite regular Borel measures. Assume that $\int f d\mu_{x,y}$ is linear in $x$ and conjugate linear in $y$ for every $f \in C(X)$.

(i) Show that $\mu_{x,y}$ is linear in $x$ and conjugate linear in $y$.

Hint: Let $x, x', y \in H$ and $a \in \mathbb{C}$. Then, by hypothesis, $\int f d\mu_{ax+x',y}$ equals $a \int f d\mu_{x,y} + \int f d\mu_{x',y}$, for every $f \in C(X)$, i.e. $\int f d\mu_{ax+x',y} = \int f d(a\mu_{x,y} + \mu_{x',y})$ for every $f \in C(X)$. Now use 3(iii).

(ii) Show that
\[ \sup_{x,y \in H, |x|, |y| \leq 1} \left| \int g d\mu_{x,y} \right| \leq 4 \|g\|_{\sup} \sup_{v \in H, |v| \leq 1} \mu_{v,v}(X) \]
for every bounded Borel function $g$ on $X$. 

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(iii) Assume that $\sup_{v \in H, |v| \leq 1} \mu_{v,v}(X) < \infty$. Show that for every bounded Borel function $g$ on $X$ there is a unique bounded linear operator $\Phi(g) : H \to H$ such that
\[
(\Phi(g)x, y) = \int_X g d\mu_{x,y}
\]

(iv) Assume that $\sup_{v \in H, |v| \leq 1} \mu_{v,v}(X) < \infty$. Show that the mapping $g \mapsto \Phi(g)$ is linear. Hint: Let $g, h$ be bounded Borel functions and $a$ any complex number. Show that $(\Phi(ag + h)x, y)$ equals $a(\Phi(g)x, y) + (\Phi(h)x, y)$, i.e. is equal to $(a\Phi(g) + \Phi(h)]x, y)$. Now use the uniqueness of $\Phi(ag + h)$.

(v) Assume that $\sup_{v \in H, |v| \leq 1} \mu_{v,v}(X) < \infty$. Assume also that $\Phi(f) = \Phi(f)^*$ and $\Phi(fg) = \Phi(f)\Phi(g)$ hold for all $f, g \in C(X)$. Show that for any $x \in H$, the linear mapping
\[
C(X) \to H : f \mapsto \Phi(f)x
\]
satisfies
\[
|\Phi(f)x| = |f|_{L^2(\mu_{x,x})}
\]
for all $f \in C(X)$. Hint: $|\Phi(f)x|^2 = (\Phi(f)x, \Phi(f)x) = (\Phi(f)^*\Phi(f)x, x)$.

(vi) Assume the hypotheses of (v). Since $C(X)$ is a dense subspace of $L^2(\mu_{x,x})$, it follows from (v) that $\Phi$ extends to a linear isometry
\[
L^2(X, \mu_{x,x}) \to H : g \mapsto \Phi(g)x
\]
(vii) Assume the hypotheses of (v) and assume also that \( \Phi(fg) = \Phi(f)\Phi(g) \) for all \( f, g \in C(X) \). Now let \( h, k \) be bounded Borel functions on \( X \). Let \( x \in H \). Then \( h, k \in L^2(\mu_{x,x}) \) and so there exist sequences of functions \( h_n, k_n \in C(X) \) converging pointwise \( \mu_{x,x} \)-a.e. to \( h, k \), respectively, and within \( |h_n|_{\text{sup}} \leq |h|_{\text{sup}} \) and \( |k_n|_{\text{sup}} \leq |k|_{\text{sup}} \). Then, by dominated convergence, \( h_n, k_n \) converge in \( L^2(\mu_{x,x}) \) to \( h, k \), respectively. Moreover, \( h_nk_n \) also converges \( \mu_{x,x} \)-a.e. to \( hk \) and \( |h_nk_n|_{\text{sup}} \leq |h|_{\text{sup}}|k|_{\text{sup}} \). Then, by dominated convergence, \( h_n, k_n, h_nk_n \) converge in \( L^2(\mu_{x,x}) \) to \( h, k, hk \), respectively. Similarly, \( \overline{h_n} \) converges to \( h \). Consider

\[
(\Phi(h_n)x, x) = (x, \Phi(h_n)^*x) = (x, \Phi(\overline{h_n})x)
\]

and

\[
(\Phi(h_nk_n)x, x) = (\Phi(h_n)\Phi(k_n)x, x) = (\Phi(k_n)x, \Phi(\overline{h_n})x)
\]

Let \( n \to \infty \) to show that

\[
\Phi(\overline{h}) = \Phi(h)^*
\]

and

\[
\Phi(hk) = \Phi(h)\Phi(k)
\]

for all bounded Borel functions \( h, k \) on \( X \).

(viii) All hypotheses as before. For any Borel set \( A \subset X \) show that the operator

\[
E(A) \overset{\text{def}}{=} \Phi(1_A)
\]

is an orthogonal projection. From the isometry property in (vi) it follows that \( E \) is a projection-valued measure on the Borel \( \sigma \)-algebra of \( X \).

(ix) All hypotheses as before. Now (iii) shows that

\[
\mu_{x,y}(A) = (E(A)x, y)
\]

By definition, if \( g \) is a bounded Borel function on \( X \) then \( \int g \, dE \) is the unique operator on \( H \) for which \( ((\int g \, dE)x, x) \) equals \( \int g \, dE_{x,x} \). Therefore, by (iii),

\[
\int g \, dE = \Phi(g)
\]
5. Let $H$ be a complex Hilbert space, $B(H)$ the algebra of bounded linear operators on $H$, $X$ a compact Hausdorff space, $\mathcal{B}$ its Borel $\sigma$–algebra, and suppose that 

$$\Phi : C(X) \to B(H) : f \mapsto \Phi(f)$$

is an algebra homomorphism with $\Phi(\overline{f}) = \Phi(f)^*$ and $|\Phi(f)| = |f|_{\text{sup}}$ for all $f \in C(X)$. For each $x \in H$, let $L_{x,x} : C(X) \to \mathbb{C}$ the mapping given by

$$L_{x,x} : C(X) \to \mathbb{C} : f \mapsto_{x,x} f \overset{\text{def}}{=} (\Phi(f)x,x)$$

Clearly, $L_{x,x}$ is a linear functional.

(i) Check that

$$L_{x,x}(\overline{f}) = \overline{L_{x,x}f}$$

for all $x \in H$ and $f \in C(X)$. Thus if $f$ is real-valued then $L_{x,x}f$ is a real number, and so $L_{x,x}$ restricts to a real-linear map $C^{\text{real}}(X) \to \mathbb{R}$.

(ii) Show that if $f \in C(X)$ is non-negative then $L_{x,x}f \geq 0$ for all $x \in H$. Hint: Show that $L_{x,x}f = |\Phi(f^{1/2})|^2$. 


(iii) From the observations noted above it follows by the Riesz-Markov theorem that for each \( x \in H \) there is a \textit{unique} regular Borel measure \( \mu_{x,x} \) on \( X \) such that

\[
\int f \, d\mu_{x,x} = (\Phi(f)x, x) \tag{4}
\]

for every \( f \in C^{\text{real}}(X) \). Because both sides of (4) are complex-linear in \( f \) it follows that (4) holds for all \( f \in C(X) \).

Now for any \( x, y \in H \) let \( \mu_{x,y} \) be the complex measure on \( B \) given by

\[
\mu_{x,y} = \frac{1}{4} \mu_{x+y,x+y} - \frac{1}{4} \mu_{x-y,x-y} + \frac{i}{4} \mu_{x+iy,x+iy} - \frac{i}{4} \mu_{x-iy,x-iy} \tag{5}
\]

Show that then

\[
\int f \, d\mu_{x,y} = (\Phi(f)x, y) \tag{6}
\]

for every \( f \in C(X) \).
6. Let $H$ be a complex Hilbert space and $B$ a commutative subalgebra of $B(H)$ such that $T^* \in B$ for every $T \in B$ and $B$ is a closed subset of $B(H)$ (in the norm topology). Then $B$ is itself a commutative $B^*$–algebra. Let $\Delta$ be its Gelfand spectrum. By Gelfand-Naimark, the Gelfand transform

$$B \to C(\Delta) : T \mapsto \hat{T}$$

is an isometric $*$–isomorphism. Let

$$\Phi : C(\Delta) \to B : f \mapsto \Phi(f)$$

be its inverse. Applying the preceding results to this situation we see that there is a projection valued measure $E$ on the Borel $\sigma$–algebra of $\Delta$ such that

$$\Phi(f) = \int f \, dE$$

for every continuous function $f$ on $\Delta$. Thus

$$T = \int_\Delta \hat{T} \, dE$$

for every $T \in B$. This is the spectral resolution of the operator $T$. Note that since $T$ and $T^*$ both belong to the commutative algebra $B$, the operator $T$ must be normal. Conversely, for any bounded normal operator $T$ on $H$ we can take $B$ to be the closure of the set of all operators which can be expressed as polynomials $p(T, T^*)$ in $T$ and $T^*$.

7. Let the setting be as in Problem 6. Suppose $E'$ is also a projection valued measure on the Borel sigma-algebra of $\Delta$ such that

$$\Phi(f) = \int f \, dE'$$

for every continuous function $f$ on $\Delta$. Assume that $E'$ is regular in the sense that $E'_x, x$ is a regular Borel measure for each $x \in H$. Show that $E' = E$. [Hint: Show that $E'_x, x = E_x, x$ for every $x \in H$, and then see what this says about $(E'(A)x, x)$.]