Math 7330: Functional Analysis Fall 2002
Notes 7: The Spectral Theorem A. Sengupta

Let B be a complex, commutative B^* algebra, with Δ its Gelfand spectrum. Then, as we have seen in class,

(i) the Gelfand transform $B \to C(\Delta) : x \mapsto \hat{x}$ satisfies

$$\hat{x^*} = \overline{\hat{x}}$$

for every $x \in B$;

(ii) the spectral radius $\rho(x)$ equals the norm $|x|$ for every $x \in B$.

Fact (ii) was proven first for hermitian elements in any B^* algebra and then, using the Gelfand transform, for all elements in a commutative B^* algebra. If $a \in B$ is hermitian then

$$\rho(a) = \lim_{n\to\infty} |a^n|^{1/n}$$

while $|a^2| = |aa^*| = |a|^2$ which implies $|a^{2^k}| = |a|^{2^k}$, and so, letting $n \to \infty$ through powers of 2 we get

$$\rho(a) = |a|$$

for every hermitian a in any B^* algebra. For a commutative B^* algebra B we have for a general $x \in B$,

$$\rho(xx^*) = |\hat{xx^*})|_{\sup} \leq |\hat{x}|_{\sup}|\hat{x}^*|_{\sup} = \rho(x)\rho(x^*) \leq \rho(x)|x^*|$$

Since xx^* is hermitian, $\rho(xx^*) = |xx^*|$, which is equal to $|x||x^*|$. So we have

$$|x| \leq \rho(x)$$

But we already know the opposite inequality. So $\rho(x) = |x|$.

By (i) and (ii) and other properties we have studied before, the Gelfand transform is a $*$–algebra homomorphism and is also an isometry. Its image $\hat{B}$ in $C(\Delta)$ is therefore a subalgebra of $C(\Delta)$ which is preserved under conjugation. Moreover, since the Gelfand transform is an isometry it follows that $\hat{B}$ is a *closed* subset of $C(\Delta)$: for if $x_n \in B$ are such that $\hat{x}_n \to f$ for some $f \in C(\Delta)$ then $(\hat{x}_n)_n$ is Cauchy in $C(\Delta)$ and so, by isometricity, $(x_n)_n$ is Cauchy in B and so is convergent, say to x and then by continuity of $\hat{}$ it follows that $f = \hat{x}$, and so f is in the image of the Gelfand transform. Finally, $\hat{B}$ separates points of Δ because if h_1 and h_2 are distinct elements of Δ, then, by definition of Δ, there must be some $x \in B$ for which $h_1(x) \neq h_2(x)$, i.e. $\hat{x}(h_1) \neq \hat{h}(x_2)$.

The Stone-Weierstrass theorem now implies that

$$\hat{B} = C(\Delta)$$

This proves the **Gelfand-Naimark** theorem:

Theorem. For a complex commutative B^*–algebra B, the Gelfand transform is an isometric isomorphism of B onto $C(\Delta)$, where Δ is the Gelfand spectrum of B.

1. Let H be a complex vector space and $F : H \times H \to \mathbf{C}$ a mapping such that $F(x,y)$ is linear in x and conjugate-linear in y.

   (i) Prove the polarization formula

$$F(x,y) = \frac{1}{4}F(x+y,x+y) - \frac{1}{4}F(x-y,x-y) + \frac{i}{4}F(x+iy,x+iy) - \frac{i}{4}F(x-iy,x-iy) \tag{1}$$

   (ii) Use this to prove that

$$\sup_{x,y\in H,|x|,|y|\le 1} |F(x,y)| \le 4 \sup_{v\in H,|v|\le 1} |F(v,v)| \tag{2}$$

[Hint: In (1), the first term equals $F(a,a)$ with $a = (x+y)/2$ and $|a| \le 1$ if $|x|, |y| \le 1$. Similarly for the other terms.]

(iii) If $y \in H$ then show that

$$\sup_{v \in H, |v| \leq 1} |(y, v)| = |y|$$

(iv) If $T : H \to H$ is a linear map for which $\sup_{v \in H, |v| \leq 1} |(Tv, v)| < \infty$, show that T is a bounded linear map and

$$|T| \leq 4 \sup_{v \in H, |v| \leq 1} |(Tv, v)|$$

(Recall that the norm of T is $|T| = \sup_{x \in H, |x| \leq 1} |Tx|$.)

2. Let H be a complex Hilbert space and $F : H \times H \to \mathbf{C}$ a map such that $F(x, y)$ is linear in x, conjugate linear in y, and $\sup_{x,y \in H, |x|,|y| \leq 1} |F(x, x)| < \infty$.

(i) Fix $x \in H$, and consider

$$\phi_x : H \to \mathbf{C} : y \mapsto \overline{F(x, y)}.$$

Show that this is a bounded linear functional. Consequently, there exists a *unique* element $Tx \in H$ such that $\phi_x(y) = (Tx, y)$ for every $y \in H$. Thus for each $x \in H$ there exists a unique element $Tx \in H$ such that

$$F(x, y) = (Tx, y) \qquad \text{for all } y \in H$$

(ii) Let $x, x' \in H$ and $a, b \in \mathbf{C}$. Show that

$$(aTx + bTx', y) = F(ax + bx', y) \qquad \text{for all } y \in H$$

Then by the uniqueness property noted in (i) it follows that

$$T(ax + bx') = aTx + bTx'$$

Thus $T : H \to H$ is *linear*.

(iii) Show that the map $T : H \to H$ is a *bounded* linear map. [Hint: Use 1(iii) and (ii).]

3. Let X be a non-empty set and $\mathcal{B}$ a σ–algebra of subsets of X.

(i) Suppose $\lambda_1, ..., \lambda_n$ and $\lambda'_1, ..., \lambda'_m$ are finite measures on $\mathcal{B}$ and $a_1, ..., a_n, a'_1, ..., a'_m$ are complex numbers such that

$$\sum_{j=1}^{n} a_j \lambda_j = \sum_{j=1}^{m} a'_j \lambda'_j$$

Then show that for any bounded $\mathcal{B}$–measurable function $f : X \to \mathbf{C}$,

$$\sum_{j=1}^{n} a_j \int_X f \, d\lambda_j = \sum_{j=1}^{m} a'_j \int_X f \, d\lambda'_j$$

[Hint: There is a sequence of measurable simple functions s_N such that $s_N(x) \to f(x)$ uniformly for $x \in X$ as $N \to \infty$.] If μ is the complex measure given by

$$\mu = \sum_{j=1}^{n} a_j \lambda_j$$

then we define

$$\int f \, d\mu \stackrel{\text{def}}{=} \sum_{j=1}^{n} a_j \int_X f \, d\lambda_j$$

for all bounded measurable functions f on X. The fact proven above says that this definition is independent of the particular choice of a_j and λ_j used to express μ.

(ii) If $b_1, ..., b_k$ are complex numbers and $\mu_1, ..., \mu_k$ are complex measures, each of the type described in (i), and μ is the complex measure given by

$$\mu = \sum_{j=1}^{n} b_j \, \mu_j$$

then show that

$$\int f \, d\mu = \sum_{j=1}^{n} b_j \int f \, d\mu_j$$

for all bounded measurable functions f on X.

(iii) Suppose now that X is a compact Hausdorff space and $\mathcal{B}$ is the Borel σ–algebra. Let μ_1, μ_2 be complex measures on $\mathcal{B}$, each μ_i being a complex linear combination of finite regular Borel measures λ_{ij} on $\mathcal{B}$. Show that if

$$\int f \, d\mu_1 = \int f \, d\mu_2 \qquad \text{for all } f \in C(X)$$

then

$$\mu_1 = \mu_2$$

Hint: Write $\mu_1 = \sum_j a_j \lambda_j$ and $\mu_2 = \sum_i a'_i \lambda'_i$, where the a_i, a'_j are complex numbers and λ_i, λ'_j are finite regular Borel measures. The $\lambda = \sum_i \lambda_i + \sum_j \lambda'_j$ is a finite regular Borel measure. Let g be any bounded Borel function. Then there is a sequence of continuous functions $g_n \in C(X)$ such that $g_n(x) \to g(x)$ for λ–a.e. x and $|g_n|_{\sup} \leq |g|_{\sup}$. Then the same holds a.e. for each λ_i and each λ'_j. Now use the dominated convergence theorem. Finally, set $g = 1_A$ for any Borel set $A \subset X$.

4. Let H be a complex Hilbert space, X a compact Hausdorff space, $\mathcal{B}$ its Borel $\sigma-$ algebra. Suppose that for each x we have a finite regular Borel measure $\mu_{x,x}$ on $\mathcal{B}$. Define, for every $x, y \in H$,

$$\mu_{x,y} = \frac{1}{4}\mu_{x+y,x+y} - \frac{1}{4}\mu_{x-y,x-y} + \frac{i}{4}\mu_{x+iy,x+iy} - \frac{i}{4}\mu_{x-iy,x-iy} \tag{3}$$

This is a complex measure which is a linear combination of *finite* regular Borel measures. *Assume that* $\int f\, d\mu_{x,y}$ *is linear in* x *and conjugate linear in* y *for every* $f \in C(X)$.

(i) Show that $\mu_{x,y}$ is linear in x and conjugate linear in y.
Hint: Let $x, x', y \in H$ and $a \in \mathbf{C}$. Then, by hypothesis, $\int f\, d\mu_{ax+x',y}$ equals $a \int f\, d\mu_{x,y} + \int f\, d\mu_{x',y}$, for every $f \in C(X)$, i.e. $\int f\, d\mu_{ax+x',y} = \int f\, d\big(a\mu_{x,y} + \mu_{x',y}\big)$ for every $f \in C(X)$. Now use 3(iii).

(ii) Show that

$$\sup_{x,y \in H, |x|, |y| \leq 1} \left| \int g\, d\mu_{x,y} \right| \leq 4|g|_{\text{sup}} \sup_{v \in H, |v| \leq 1} \mu_{v,v}(X)$$

for every bounded Borel function g on X.

(iii) Assume that $\sup_{v\in H,|v|\le 1} \mu_{v,v}(X) < \infty$. Show that for every bounded Borel function g on X there is a *unique* bounded linear operator $\Phi(g) : H \to H$ such that

$$\big(\Phi(g)x, y\big) = \int_X g\, d\mu_{x,y}$$

(iv) Assume that $\sup_{v\in H,|v|\le 1} \mu_{v,v}(X) < \infty$. Show that the mapping $g \mapsto \Phi(g)$ is linear. Hint: Let g, h be bounded Borel functions and a any complex number. Show that $\big(\Phi(ag + h)x, y\big)$ equals $a\big(\Phi(g)x, y\big) + \big(\Phi(h)x, y\big)$, i.e. is equal to $\big([a\Phi(g) + \Phi(h)]x, y\big)$. Now use the uniqueness of $\Phi(ag + h)$.

(v) Assume that $\sup_{v\in H,|v|\le 1} \mu_{v,v}(X) < \infty$. Assume also that $\Phi(\overline{f}) = \Phi(f)^*$ and $\Phi(fg) = \Phi(f)\Phi(g)$ hold for all $f, g \in C(X)$. Show that for any $x \in H$, the linear mapping

$$C(X) \to H : f \mapsto \Phi(f)x$$

satisfies

$$|\Phi(f)x| = |f|_{L^2(\mu_{x,x})}$$

for all $f \in C(X)$. Hint: $|\Phi(f)x|^2 = \big(\Phi(f)x, \Phi(f)x\big) = \big(\Phi(f)^*\Phi(f)x, x\big)$.

(vi) Assume the hypotheses of (v). Since $C(X)$ is a dense subspace of $L^2(\mu_{x,x})$, it follows from (v) that Φ extends to a linear isometry

$$L^2(X, \mu_{x,x}) \to H : g \mapsto \Phi(g)x$$

(vii) Assume the hypotheses of (v) and assume also that $\Phi(fg) = \Phi(f)\Phi(g)$ for all $f, g \in C(X)$. Now let h, k be bounded Borel functions on X. Let $x \in H$. Then $h, k \in L^2(\mu_{x,x})$ and so there exist sequences of functions $h_n, k_n \in C(X)$ converging pointwise $\mu_{x,x}$–a.e. to h, k, respectively, and within $|h_n|_{\text{sup}} \le |h|_{\text{sup}}$ and $|k_n|_{\text{sup}} \le |k|_{\text{sup}}$. Then, by dominated convergence, h_n, k_n converge in $L^2(\mu_{x,x})$ to h, k, respectively. Moreover, $h_n k_n$ also converges $\mu_{x,x}$–a.e. to hk and $|h_n k_k|_{\text{sup}} \le |h|_{\text{sup}}|k|_{\text{sup}}$. Then, by dominated convergence, $h_n, k_n, h_n k_n$ converge in $L^2(\mu_{x,x})$ to h, k, hk, respectively. Similarly, $\overline{h}_n$ converges to h. Consider

$$(\Phi(h_n)x, x) = (x, \Phi(h_n)^* x) = \big(x, \Phi(\overline{h}_n)x\big)$$

and

$$(\Phi(h_n k_n)x, x) = (\Phi(h_n)\Phi(k_n)x, x) = \big(\Phi(k_n)x, \Phi(\overline{h_n})x\big)$$

Let $n \to \infty$ to show that

$$\Phi(\overline{h}) = \Phi(h)^*$$

and

$$\Phi(hk) = \Phi(h)\Phi(k)$$

for all bounded Borel functions h, k on X.

(viii) All hypotheses as before. For any Borel set $A \subset X$ show that the operator

$$E(A) \stackrel{\text{def}}{=} \Phi(1_A)$$

is an orthogonal projection. From the isometry property in (vi) it follows that E is a *projection-valued measure* on the Borel σ–algebra of X.

(ix) All hypotheses as before. Now (iii) shows that

$$\mu_{x,y}(A) = (E(A)x, y)$$

By definition, if g is a bounded Borel function on X then $\int g\, dE$ is the unique operator on H for which $\big((\int g\, dE)x, x\big)$ equals $\int g\, dE_{x,x}$. Therefore, by (iii),

$$\int g\, dE = \Phi(g)$$

5. Let H be a complex Hilbert space, $B(H)$ the algebra of bounded linear operators on H, X a compact Hausdorff space, $\mathcal{B}$ its Borel σ–algebra, and suppose that

$$\Phi : C(X) \to B(H) : f \mapsto \Phi(f)$$

is an algebra homomorphism with $\Phi(\overline{f}) = \Phi(f)^*$ and $|\Phi(f)| = |f|_{\sup}$ for all $f \in C(X)$. For each $x \in H$, let $L_{x,x} : C(X) \to \mathbf{C}$ the mapping given by

$$L_{x,x} : C(X) \to \mathbf{C} : f \mapsto_{x,x} f \stackrel{\text{def}}{=} \big(\Phi(f)x, x\big)$$

Clearly, $L_{x,x}$ is a linear functional.

(i) Check that

$$L_{x,x}(\overline{f}) = \overline{L_{x,x}f}$$

for all $x \in H$ and $f \in C(X)$. Thus if f is real-valued then $L_{x,x}f$ is a real number, and so $L_{x,x}$ restricts to a real-linear map $C^{\text{real}}(X) \to \mathbf{R}$.

(ii) Show that if $f \in C(X)$ is non-negative then $L_{x,x}f \geq 0$ for all $x \in H$. Hint: Show that $L_{x,x}f = |\Phi(f^{1/2})|^2$.

(iii) From the observations noted above it follows by the Riesz-Markov theorem that for each $x \in H$ there is a *unique* regular Borel measure $\mu_{x,x}$ on X such that

$$\int f\, d\mu_{x,x} = \big(\Phi(f)x, x\big) \tag{4}$$

for every $f \in C^{\mathrm{real}}(X)$. Because both sides of (4) are complex-linear in f it follows that (4) holds for all $f \in C(X)$.
Now for any $x, y \in H$ let $\mu_{x,y}$ be the complex measure on $\mathcal{B}$ given by

$$\mu_{x,y} = \frac{1}{4}\mu_{x+y,x+y} - \frac{1}{4}\mu_{x-y,x-y} + \frac{i}{4}\mu_{x+iy,x+iy} - \frac{i}{4}\mu_{x-iy,x-iy} \tag{5}$$

Show that then

$$\int f\, d\mu_{x,y} = \big(\Phi(f)x, y\big) \tag{6}$$

for every $f \in C(X)$.

6. Let H be a complex Hilbert space and B a commutative subalgebra of $B(H)$ such that $T^* \in B$ for every $T \in B$ and B is a closed subset of $B(H)$ (in the norm topology). Then B is itself a commutative B^*–algebra. Let Δ be its Gelfand spectrum. By Gelfand-Naimark, the Gelfand transform

$$B \to C(\Delta) : T \mapsto \hat{T}$$

is an isometric $*$–isomorphism. Let

$$\Phi : C(\Delta) \to B : f \mapsto \Phi(f)$$

be its inverse. Applying the preceding results to this situation we see that there is a projection valued measure E on the Borel σ–algebra of Δ such that

$$\Phi(f) = \int f \, dE$$

for every continuous function f on Δ. Thus

$$T = \int_{\Delta} \hat{T} \, dE$$

for every $T \in B$. This is the *spectral resolution* of the operator T. Note that since T and T^* both belong to the commutative algebra B, the operator T must be normal. Conversely, for any bounded normal operator T on H we can take B to be the closure of the set of all operators which can be expressed as polynomials $p(T, T^*)$ in T and T^*.

7. Let the setting be as in Problem 6. Suppose E' is also a projection valued measure on the Borel sigma-algebra of Δ such that

$$\Phi(f) = \int f \, dE'$$

for every continuous function f on Δ. Assume that E' is regular in the sense that $E'_{x,x}$ is a regular Borel measure for each $x \in H$. Show that $E' = E$. [Hint: Show that $E'_{x,x} = E_{x,x}$ for every $x \in H$, and then see what this says about $\big(E'(A)x, x\big)$.]